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GRAPH-LINKS: NONREALIZABILITY, ORIENTATION, AND JONES POLYNOMIAL

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Abstract. The present paper is devoted to graph-links with many components and consists of two parts. In the first part of the paper we classify vertices of a labeled graph according to the component they belong to. Using this classification, we construct an invariant of graph-links. This invariant shows that the labeled second Bouchet graph generates a nonrealizable graph-link.

In the second part of the work we introduce the notion of an oriented graph-link. We define a writhe number for the oriented graph-link and we get an invariant of oriented graph-links, the Jones polynomial, by normalizing the Kauffman bracket with the writhe number.

CONTENTS

1. Introduction

Let us first consider the following question. Let a virtual knot diagram be given, i.e., we have a 4-valent graph embedded (the embedding is fixed) into the plane and each vertex of it is endowed with a cross structure and over/undercrossing structure. The question is: How much information about the diagram must we have in order to define the Kauffman bracket and the Jones polynomial? It turns out that these polynomials can be obtained by knowing only the intersection graph of the Gauss diagram (all the definitions are given below) and the writhe number of each classical crossing. Thus the right-left information which is given by arrows on the Gauss diagram is unnecessary for defining the Kauffman bracket and the Jones polynomial. Note that if we forget about the writhe number and have just the cross structure (the structure of opposite edges), then we get nontrivial objects (modulo Reidemeister moves), see [19].

Probably, the simplest evidence that one can get some information out of the intersection graph is the formula allowing one to count the number of circles in Kauffman's states out of the intersection graph. More precisely, this formula allows one to get this number from the adjacency matrix of the intersection graph [1, 5, 22, 26, 29] (the number of circles is equal to the corank or nullity of the adjacency matrix plus one, see Fig. 1). In particular, this means that graphs not necessarily corresponding to knots (these graphs are called *nonrealizable*, see examples below) admit a way of generalizing the Kauffman bracket polynomial, which coincides with the usual Kauffman bracket polynomial when the graph is realizable by a chord diagram. This was the initial point of investigation for Traldi and Zulli [30] (*looped interlacement graphs*): They constructed a self-contained theory of "nonrealizable graphs" possessing lots of interesting knot theoretic properties. These objects are equivalence classes of (decorated) graphs modulo "Reidemeister moves" (translated into the language

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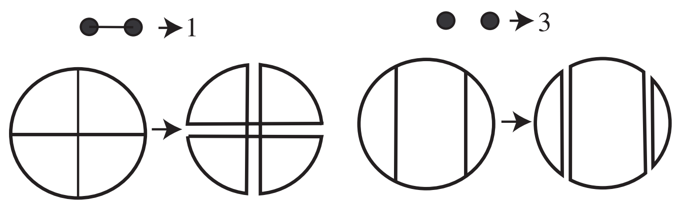

Fig. 1. Resmoothing along two chords yields one or three circles

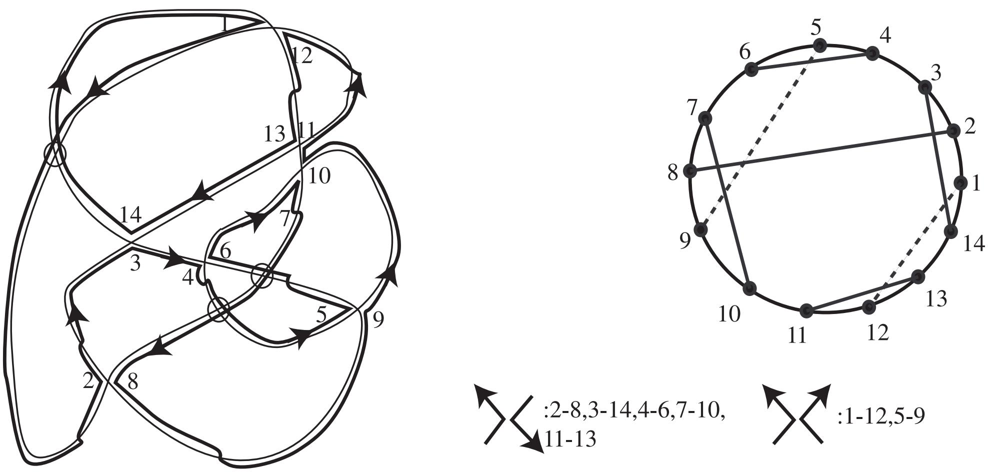

Fig. 2. Rotating circuit shown by a thick line; chord diagram

of intersection graphs). A significant disadvantage of this approach was that it had applications only to knots, not links: In order to encode a link, one has to use a more complicated object rather than just a Gauss diagram, namely a Gauss diagram on many circles. This approach was further developed in Traldi's works [27, 28], and it allowed one to encode not only knots but also links with any number of components by decorated graphs.

In [9, 10] we suggested another way of looking at knots and links and generalizing them: when a Gauss diagram corresponds to a transverse passage along a knot, one may consider a rotating circuit [7, 8, 12, 16, 17]. Moreover, one can also encode the type of smoothing (Kauffman's A-smoothing or Kauffman's B-smoothing) corresponding to the crossing where the circuit turns right or left and never goes straight; see Fig. 2. We note that each vertex has a label depending on the orientation of opposite edges (framing 0 or framing 1). Due to Gauss diagrams we can define moves on intersection graphs; these moves correspond to the moves on chord diagrams, and we can extend the moves for the case of all simple graphs. As a result, we have a new object: a *graph-link*.

Thus, an analogy arises: the passage from realizable Gauss diagrams (classical knots) to arbitrary chord diagrams leads to the concept of a virtual knot, and the passage from realizable (by means of chord diagrams) graphs to arbitrary graphs leads to the concept of a new object, graph-link.

Note that passing from knots to intersection graphs we lose some information about a knot, for instance, sometimes a chord diagram can be obtained from the intersection graph in a nonunique way; see Fig. 3. But it turns out that one can obtain much information about a knot and its invariants from the graph-link generated by the knot. For example, we can define the *number of components of the graph-link*, and this number coincides with the actual number of components of a link in the realizable case, i.e., in the case where the graph-link can be realized by a link, we can construct an analogue of the Kauffman bracket polynomial. Having defined the number of components of a graph-link, we can single out the class of graph-links having one component: *graph-knots*. Graph-knots, in some sense,

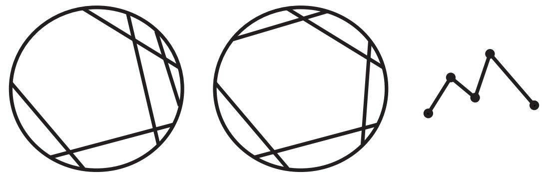

Fig. 3. A graph not uniquely represented by chord diagrams

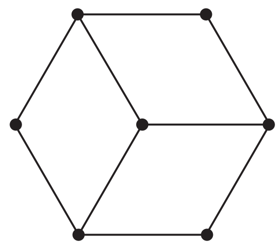

Fig. 4. The second Bouchet graph BW_3

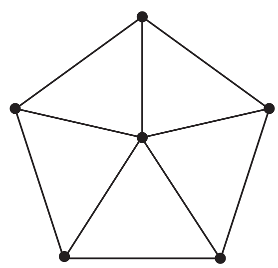

Fig. 5. The first Bouchet graph W_5

are analogues of knots in the set of links. For a graph-knot we have constructed the *writhe number* and an invariant: the *Jones polynomial* [9].

One of the simplest invariants allowing one to show in some cases that a virtual link is not equivalent to a classical link is the linking number analogous to the usual linking number for classical links [16, 23]. If a virtual link is given, for each pair of components we can find the parity of the number of classical crossings belonging to both components. If this number is odd for any pair, then this virtual link is not equivalent to a classical link. In the first part of the paper we rewrite the question whether a vertex belongs to one or two components of the link in the language of adjacency matrices of intersection graphs. It turns out that these conditions are invariant under the graph-moves. Thus we can classify vertices of the graph-link: whether a vertex belongs to one component or to two. Further we classify vertices belonging to two components. Namely, each class consists of only those vertices that belong to two components, and these components are the same for the vertices from this class. It turns out that the parity of the number of classes consisting of an odd number of vertices is invariant under the graph-moves. Using this invariant, we show that the graph-link generated by the second Bouchet graph BW_3 (see Fig. 4 and [9]) where each vertex has the framing 0, is nonrealizable, i.e., each graph of the class cannot be realized by a chord diagram. Note that the nonrealizability of the graph-link generated by the first Bouchet graph W_5 (see Fig. 5 and [8]) where each vertex has the framing 0, was proved with the help of parity; see [18–20].

The second part of the paper is devoted to the Jones polynomial for links with many components. It is well known that in the case of knots the Jones polynomial does not depend on an orientation of knots, but in the case of oriented links it does, and we should construct the Jones polynomial

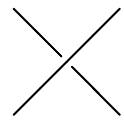

Fig. 6. The local structure of a crossing

for oriented links. In [9] the Jones polynomial was constructed for graph-knots. To construct the Jones polynomial for graph-links we should first introduce the notion of an *oriented* graph-link. In the second part of the paper we firstly define an oriented graph-link and secondly we construct the writhe number for it, which generalizes the usual writhe number of a link. Using the latter number, we define a polynomial analogous to the Jones polynomial. This polynomial is an invariant of oriented graph-links.

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2. Main Definitions and Notions: a Graph-Link

2.1. Classical and virtual knots. A (*classical*) *knot* or *link* is the image of a smooth embedding of the circle S^1 or disjoint union of several circles $S^1 \sqcup \ldots \sqcup S^1$ in the 3-dimensional sphere S^3 (in the case of links the image of each circle, a knot, is called a *component* of the link). We call such a smooth embedding of a circle (a disjoint union of circles) also a *knot* (a *link*). The main question of knot theory is the following: Which two knots (links) are isotopic and which are not? Here two knots $f_1, f_2\colon S^1 \to S^3$ (links $f_1, f_2\colon S^1 \sqcup \ldots \sqcup S^1 \to S^3$) are called *isotopic* if there is an isotopy $F_t\colon S^3 \to S^3$, $t \in [0,1]$, such that $F_0 = \mathrm{id}$ (the identity map) and $F_1(f_1(S^1)) = f_2(S^1)$ ($F_1(f_1(S^1 \sqcup \ldots \sqcup S^1)) = f_2(S^1 \sqcup \ldots \sqcup S^1)$). Note that a classification of knots in S^3 is equivalent to a classification of knots in the space $\mathbb{R}^3$.

In the case of fixing an orientation of the circle S^1, we have an *oriented knot* (respectively, in the case of an *oriented link* we require orientations of the circles, i.e., the preimages of components of links); in the case of an isotopy of oriented links we require that the diffeomorphism of the ambient space preserves both the orientation of S^3 (or $\mathbb{R}^3$) and the orientations of all the components.

Usually knots (links) are encoded as follows. Fix a knot (link) and consider a plane and the projection of the knot (link) onto it. Without loss of generality, we can assume that the projection of the knot (link) is a finite embedded 4-valent graph, being the image of a smooth immersion of the circle (disjoint union of circles) in the plane. Each vertex of this graph, also called a *crossing of the diagram of the link*, is endowed with over/undercrossing structure; see Fig. 6 (the branch going above forms an *overcrossing*, and the branch going below forms an *undercrossing*). Edges of overcrossings are depicted by continuous lines, and edges of undercrossings are depicted by lines having a break at the crossing. This image of a knot (link) on the plane is called a *plane knot* (*link*) *diagram* or a *knot* (*link*) *diagram*.

Reidemeister [25] proved that any two planar diagrams give the same link if and only if they can be obtained from each other by a finite sequence of some moves (later called *Reidemeister moves*) and planar isotopies (see Fig. 7). The Reidemeister theorem allows one to consider isotopy classes of links as combinatorial objects: they represent equivalence classes of planar diagrams modulo Reidemeister moves.

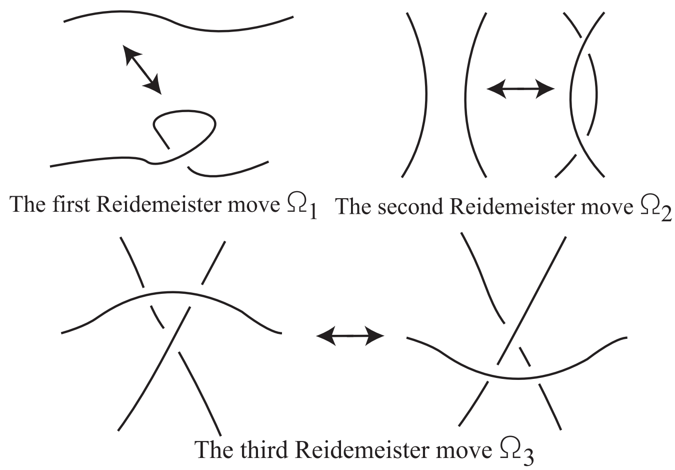

Fig. 7. Reidemeister moves $\Omega_1, \Omega_2, \Omega_3$

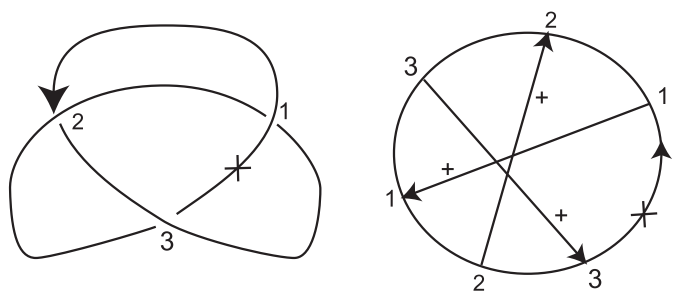

Fig. 8. The Gauss diagram of the right-handed trefoil

On the other hand, each planar diagram of a knot can be represented by the *Gauss diagram*. Recall that the *Gauss diagram* corresponding to a planar diagram is the diagram consisting of an oriented circle (with a fixed point, which is not the preimage of a crossing) on which the preimages of the overcrossing and the undercrossing for each crossing are connected by an arrow oriented from the preimage of the overcrossing to the preimage of the undercrossing. Moreover, each arrow is endowed with a sign equal to the sign of the crossing, i.e., the sign is equal to 1 for a crossing and -1 for a crossing .

The Gauss diagram of the right-handed trefoil is shown in Fig. 8. There are circles with chords (*chord diagrams*) which are not Gauss diagrams of planar diagrams of classical knots.

It is not difficult to rewrite the Reidemeister moves in the language of Gauss diagrams. Let us extend all these new moves to the case of all chord diagrams (chords are oriented and endowed with a sign) and consider equivalence classes of chord diagrams modulo the moves. The list of the Reidemeister moves for Gauss diagrams can be found in [4]. As a result we get a new theory: virtual knot theory [6, 14]. Note that all information about a knot and its invariants can be read from any Gauss diagram encoding this knots [15].

Since Gauss diagrams (on one circle) can encode only knot diagrams, we can define only a virtual knot, not a link, with the help of Gauss diagrams. Let us give another definition of a virtual link.

Let G be a graph with the set of vertices $V(G)$ and the set of edges $E(G)$ (we consider only finite graphs, i.e., graphs with finite sets of vertices and edges). We think of an *edge* as an equivalence class of the two half-edges constituting the edge. We say that a vertex $v \in V(G)$ has the *degree* k if v is incident to k half-edges. A graph whose vertices have the same degree k is called k-*valent* or a k-*graph*.

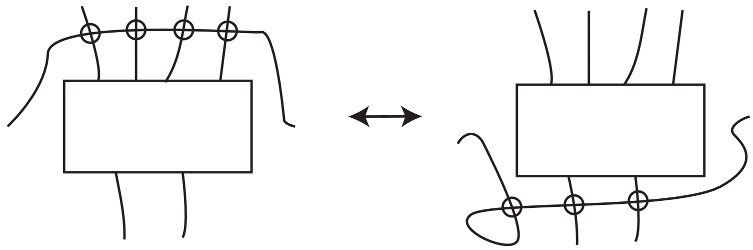

Fig. 9. Detour move

The *free loop*, i.e., the graph without vertices but with one *cyclic edge*, is considered as k-valent graph for any k.

Definition 2.1. A 4-graph is called *framed* or a *graph with a cross structure* if for every vertex of this graph the four emanating half-edges are split into two pairs of (formally) opposite edges, i.e., we have the structure of opposite edges. The edges from one pair are called *opposite to each other*.

Definition 2.2. By an *isomorphism* between 4-graphs with a cross structure we assume a framing-preserving isomorphism. All 4-graphs with a cross structure are considered up to isomorphism. Denote by G_0 the 4-graph with a cross structure isomorphic to the circle (the free loop).

When drawing graphs with a cross structure on the plane, we always assume the cross structure to be induced from the plane $\mathbb{R}^2$.

Definition 2.3. A *virtual diagram* (or a *diagram of a virtual link*) is the image of an immersion of a 4-graph with a cross structure in $\mathbb{R}^2$ with a finite number of intersections of edges. Moreover, each intersection is a transverse double point which we call a *virtual crossing* and mark by a small circle, and each vertex of the graph is endowed with the classical crossing structure (with a choice for underpass and overpass specified). The vertices of the graph with that additional structure are called *classical crossings* or just *crossings*.

A virtual diagram is called *connected* if the corresponding 4-graph is connected.

A *virtual link* is an equivalence class of virtual diagrams modulo the generalized Reidemeister moves. The latter consist of the usual Reidemeister moves referring to classical crossings and the detour move (see Fig. 9).

Thereby, in order to define a virtual link we need know only the position of classical crossings and their connections with each other. Moreover, positions of paths connecting classical crossings, their intersections and self-intersections, are not important for us.

Remark 2.1. Note that this approach, the standard moves inside a local Euclidean plane and the detour move, was used by N. Kamada and S. Kamada [13] for constructing a formal theory of multi-dimensional "virtual knots" and their invariants.

Like a classical link, a virtual link has a number of *components* (*unicursal components*). The components of a virtual link can be described combinatorially by using virtual diagrams. By a *unicursal component* of a diagram of a virtual link we mean the following. Consider a virtual diagram K as a one-dimensional cell-complex on the plane. Some of the connected components of this complex are circles; we call each such component a (*unicursal*) *component* of a link. The remaining part of the cell-complex represents a 4-graph Γ with vertices which are classical or virtual crossings. *Unicursal components* of a diagram are (besides circles) equivalence classes on the set of edges of Γ: two edges e, e' are *equivalent* if there exists a collection of edges $e = e_1, \ldots, e_k = e'$ and a collection of vertices $v_1, \ldots, v_{k-1}$ (some of them may coincide) of Γ such that edges e_i, e_{i+1} pass to the vertex v_i from the opposite sides. It is easy to see that the number of components of a virtual diagram is invariant under

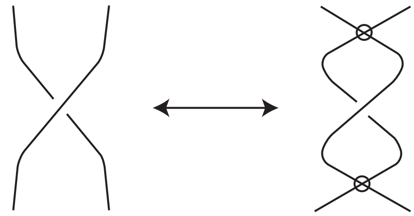

Fig. 10. Virtualization

the generalized Reidemeister moves. In the classical case this definition coincides with the definition given before.

A *virtual knot* is a virtual link with one *unicursal component.*

The *writhe number* $w(K)$ of a virtual diagram K is the number equal to the number of positive crossings minus the number of negative crossings.

2.2. Graph-link. Since any two equivalent (in the class of all virtual diagrams) connected (see Definition 2.3) virtual diagrams are equivalent in the class of connected virtual diagrams [9], without loss of generality, all virtual diagrams are assumed to be connected and to contain at least one classical crossing [9, 10].

We will construct chord diagrams for links, not only for knots. Therefore, the approach with Gauss diagrams is not suitable: instead of Gauss diagrams we consider chord diagrams corresponding to *rotating circuits*, and at each (classical) crossing we rotate from an edge to an adjacent edge. Further all chord diagrams correspond to rotating circuits unless we state otherwise.

Definition 2.4. By a *chord diagram* we mean a cubic graph consisting of one selected nonoriented Hamiltonian cycle (a cycle passing through all vertices of the graph, *core circle* or *circle*) and a set of nonoriented edges (*chords*), connecting points on the cycle. Moreover, distinct chords have no common points on the cycle.

We say that two chords of a chord diagram are *linked* if the ends of one chord belong to two different connected components of the complement to the ends of the other chord in the core circle. Otherwise, we say that the chords are *unlinked.*

Remark 2.2. As a rule, a chord diagram is depicted on the plane as the Euclidean circle with a collection of chords connecting end points of chords (intersection points of chords which appear as artefacts of drawing chords do not count as vertices).

Definition 2.5. A chord diagram is *labeled* if every chord is endowed with a label (a, α), where $a \in \{0, 1\}$ is the *framing* of the chord, and $\alpha \in \{\pm\}$ is the *sign* of the chord. If labels of a chord diagram are not pointed, then we consider them to be equal to $(0, +)$.

We will also consider chord diagrams whose chords have only one bit of information, a label 0 or 1. We call such diagrams *framed.*

Let D be a labeled chord diagram. One can construct a virtual link diagram $K(D)$ (up to *virtualization*, also called *Z-move*, see Fig. 10) in such a way that the chord diagram D coincides with the chord diagram of a rotating circuit on $K(D)$. Let us immerse the diagram D in $\mathbb{R}^2$ by taking an embedding of the core circle and placing some chords inside the circle and the others outside the circle. After that we remove neighborhoods of each of the chord ends and replace the chord with a pair of lines connecting four points on the circle which are obtained after removing neighborhoods. The new chords lead to a classical crossing only (with each other) if the chord is framed by 0, and form classical and virtual crossings if the chord is framed by 1; see Fig. 11 (intersections of chords from different pair form virtual crossings). We also require that the initial piece of the circle corresponds to the A-smoothing $A\colon \to$ if the chord is positive and to the B-smoothing $B\colon \to$ if it is negative.

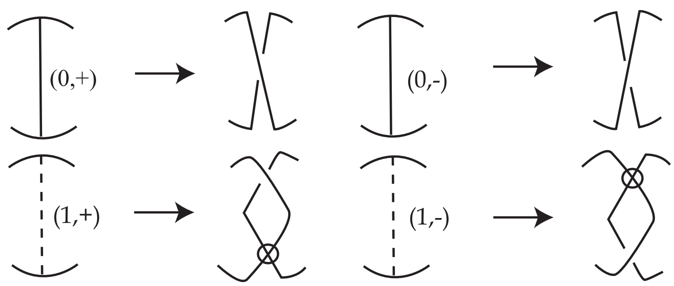

Fig. 11. Replacing a chord with a pair of lines

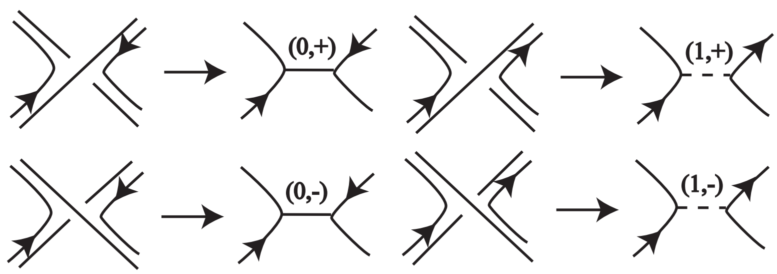

Fig. 12. Replacing a classical crossing with the labeled chord

Conversely, having a connected virtual diagram K, one can get a labeled chord diagram $D_C(K)$. Indeed, one takes a rotating circuit C on K (more precisely, on the underlying graph of K) and constructs the labeled chord diagram. The sign of the chord is $+$ (respectively, $-$) if the circuit locally agrees with the A-smoothing (respectively, the B-smoothing), and the framing of a chord is 0 (respectively, 1) if two opposite half-edges have the opposite (respectively, the same) orientations; see Fig. 12. It can be easily checked that this operation is indeed the inverse operation to the operation of constructing a virtual link diagram out of a chord diagram: If we take a chord diagram D and construct a virtual diagram $K(D)$ out of it, then for some circuit C the chord diagram $D_C(K(D))$ will coincide with D. This proves the following theorem.

Theorem 2.1 (see [17])**.** *For any connected virtual diagram L there is a certain labeled chord diagram D such that $L = K(D)$.*

Now we are describing moves on graphs obtained from the Reidemeister moves on virtual diagrams by using rotating circuits [9, 10]. These moves correspond to the "real" Reidemeister moves when applied to realizable diagrams. Then we will extend these moves to all graphs (not only realizable ones). As a result, we get a new object: a *graph-link*.

Definition 2.6. The *intersection graph* (see [3]) $G(D)$ of a chord diagram D is a simple graph (without loops and multiple edges); its vertices are in one-to-one correspondence with chords of D, and two vertices are connected by an edge if and only if the corresponding chords are linked.

A graph is *labeled* if every vertex v of it is endowed with a pair (a, α), where $a \in \{0, 1\}$ is the *framing* of v, and $\alpha \in \{\pm\}$ is the *sign* of v.

Let D be a labeled chord diagram. The *labeled intersection graph* (cf. [3, 24]) $G(D)$ of D is the intersection graph whose vertices are endowed with the corresponding labels. A simple graph H is called *realizable* if there is a chord diagram D such that $H = G(D)$. Otherwise, a graph is called *nonrealizable*.

Remark 2.3. We will also consider simple graphs whose vertices have only one label, 0 or 1. We call these graphs *framed*. Thus we have two types of framed graphs: 4-valent graphs and simple graphs. In the first case we will often refer to these graphs as graphs with a cross structure.

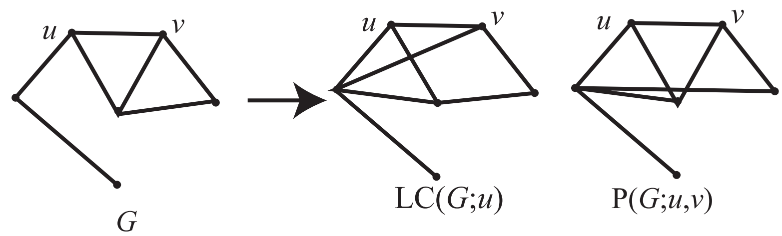

Fig. 13. Local complementation and pivoting operation

In the realizable case framed graphs are intersection graphs of framed chord diagrams.

The following lemma is evident.

Lemma 2.1. *A simple graph is realizable if and only if each of its connected components is realizable.*

Definition 2.7. Let G be a graph with the set of vertices $V(G)$ and let $v \in V(G)$. The set of all vertices adjacent to v is called the *neighborhood* of v and denoted by $N(v)$ or $N_G(v)$.

Let us define two operations on simple unlabeled graphs.

Definition 2.8 (Local complementation)**.** Let G be a graph. The *local complementation* of G at $v \in V(G)$ is the operation which toggles adjacencies between $a, b \in N(v)$, $a \neq b$, and does not change the rest of G. Denote the graph obtained from G by the local complementation at a vertex v by $\mathrm{LC}(G; v)$.

Definition 2.9 (Pivoting operation)**.** Let G be a graph with distinct vertices u and v. The *pivoting operation* of a graph G at u and v is the operation which toggles adjacencies between x, y such that $x, y \notin \{u, v\}$, $x \in N(u)$, $y \in N(v)$ and either $x \notin N(v)$ or $y \notin N(u)$, and does not change the rest of G. Denote the graph obtained from G by the pivoting operation at the vertices u and v by $\mathrm{piv}(G; u, v)$.

Example 2.1. In Fig. 13 we depict graphs G, $\mathrm{LC}(G; u)$ and $\mathrm{piv}(G; u, v)$.

It is not difficult to prove the following lemma.

Lemma 2.2. *If u and v are adjacent, then there is an isomorphism*

$$\mathrm{piv}(G; u, v) \cong \mathrm{LC}(\mathrm{LC}(\mathrm{LC}(G; u); v); u) \cong \mathrm{LC}(\mathrm{LC}(\mathrm{LC}(G; v); u); v).$$

Let us define *graph-moves*, i.e., moves on labeled graphs. We consider labeled chord diagrams constructed by using rotating circuits and moves on them which originate from the "real" Reidemeister moves on virtual diagrams. Then we extend these moves to arbitrary labeled graphs by using intersection graphs of chord diagrams. As a result, we get a new object: an equivalence class of labeled graphs modulo formal moves [9, 10].

Definition 2.10.

Ω_g1. *The first graph-move* is an addition/removal of an isolated vertex labeled $(0, \alpha)$, $\alpha \in \{\pm\}$.

Ω_g2. *The second graph-move* is an addition/removal of two nonadjacent (respectively, adjacent) vertices labeled $(0, \pm\alpha)$ (respectively, $(1, \pm\alpha)$) and having the same adjacencies with other vertices.

Ω_g3. *The third graph-move* is defined as follows. Let u, v, w be three vertices of G all having the label $(0, -)$ so that u is adjacent only to v and w in G, and v and w are not adjacent to each other. Then we only change the adjacencies of u with the vertices v, w and $t \in \big(N(v) \setminus N(w)\big) \cup \big(N(w) \setminus N(v)\big)$ (for other pairs of vertices we do not change their adjacencies). In addition, we switch the sign of v and w to $+$. The inverse operation is also called the third Reidemeister graph-move.

Ω_g4. *The fourth graph-move* for G is defined as follows. We take two adjacent vertices u and v labeled $(0,\alpha)$ and $(0,\beta)$, respectively. Replace G with $\mathrm{piv}(G;u,v)$ and change the signs of u and v so that the sign of u becomes $-\beta$ and the sign of v becomes $-\alpha$.

Ω_g4'. In this fourth graph-move we take a vertex v with the label $(1,\alpha)$. Replace G with $\mathrm{LC}(G;v)$ and change the sign of v and the framing for each $u \in N(v)$.

Remark 2.4. The fourth graph-moves Ω_g4 and Ω_g4' in the realizable case correspond to a rotating circuit change on a virtual diagram. Sometimes, applying these graph-moves we just say that we change the circuit.

Remark 2.5. If a labeled graph G_2 is obtained from a labeled graph G_1 by applying one graph-move, then there is a one-to-one correspondence between vertices not taking part in this move.

In the case of the third graph-move we will always assume that the vertex u of G_1 corresponds to the vertex u of G_2, and the vertex v (respectively, w) of G_1 corresponds to the vertex w (respectively, v) of G_2.

In the case of the fourth graph-move Ω_g4 we will always assume that the vertex u of G_1 corresponds to the vertex v of G_2, and the vertex v of G_1 corresponds to the vertex u of G_2.

Unless we state otherwise, vertices of G_i are enumerated as follows: vertices of graphs not taking part in the move have the same number; additional vertices under the first two graph-moves have the last numbers, the vertices u, v, w for the third graph-move have the numbers 1, 2, 3 for G_1 and 1, 3, 2 for G_2, and the vertices u, v (respectively, the vertex u) for the fourth graph-move Ω_g4 (respectively, Ω_g4') have the numbers 1, 2 (respectively, 1) for G_1 and 2, 1 (respectively, 1) for G_2 (i.e., the corresponding vertices under the moves have the same numbers).

Remark 2.6. We have defined the graph-moves for labeled graphs. If we consider framed graphs, then graph-moves for them are obtained from the graph-moves Ω_g1–Ω_g4' by neglecting the sign, i.e., the second component of the label. In this case we use the same notation.

A comparison of the graph-moves with the Reidemeister moves yields the following theorem.

Theorem 2.2. *Let G_1 and G_2 be two labeled intersection graphs corresponding to virtual diagrams K_1 and K_2, respectively. If K_1 and K_2 are equivalent in the class of connected diagrams, then G_1 and G_2 are obtained from one another by a sequence of Ω_g1–Ω_g4'.*

Definition 2.11. A *graph-link* is an equivalence class of simple labeled graphs modulo Ω_g1–Ω_g4' graph-moves. A graph from a graph-link $\{G\}$ is called a *representative* for $\{G\}$.

Definition 2.12. The *adjacency matrix* $A(G)$ of a labeled graph G is the matrix over $\mathbb{Z}_2$ defined as follows: a_{ii} is equal to the framing of v_i, $a_{ij} = 1$, $i \neq j$, if and only if v_i is adjacent to v_j and $a_{ij} = 0$ otherwise.

Note that $\operatorname{corank}(A(G_1) + E) = \operatorname{corank}(A(G_2) + E)$, where $G_1, G_2 \in \{G\}$ (see [9]), and $\operatorname{corank} A$ is the corank or nullity of the matrix A, i.e., the difference between the size of the matrix and its rank (the rank is calculated over $\mathbb{Z}_2$). Define the *number of components* of a graph-link $\{G\}$ as $\operatorname{corank}(A(G)+E)+1$, where E is the identity matrix. A *graph-knot* is a graph-link with one component. A graph-link is called *nonrealizable* if each of its representative is nonrealizable graph.

Note that in the realizable case the number of components of a graph-link coincides with the number of components of the corresponding links.

3. Intersection of Different Components

Let G be a labeled graph with the set of vertices $V(G) = \{v_1, \ldots, v_n\}$ and $B(G) = A(G) + E$, $B_i(G) = A(G) + E + E_{ii}$, where E_{ii} is the matrix with the only one nonzero element equal to 1 in the ith column and ith row.

3.1. Auxiliary results. Let us prove some lemmas which we need later.

Lemma 3.1. *The equality*

$$\operatorname{corank} \begin{pmatrix} a & \mathbf{b}^\top \\ \mathbf{b} & C \end{pmatrix} = \operatorname{corank} C,$$

where C is a symmetric matrix, is true if and only if the vector $\begin{pmatrix} a+1 \\ \mathbf{b} \end{pmatrix}$ is a linear combination of columns of the matrix $\begin{pmatrix} \mathbf{b}^\top \\ C \end{pmatrix}$, but the vector $\begin{pmatrix} a \\ \mathbf{b} \end{pmatrix}$ is not (linear combinations are considered over $\mathbb{Z}_2$, and the case where the empty set of columns take part is possible, i.e., the vector $\begin{pmatrix} a+1 \\ \mathbf{b} \end{pmatrix}$ is zero vector).

Remark 3.1. Bold characters indicate column vectors.

Proof. The implication $\Longleftarrow$ is evident.

Consider the implication $\Longrightarrow$. We have

$$\operatorname{rank} \begin{pmatrix} a & \mathbf{b}^\top \\ \mathbf{b} & C \end{pmatrix} = \operatorname{rank} C + 1.$$

If the vector $\mathbf{b}$ is not a linear combination of columns of the matrix C, then

$$\operatorname{rank} \begin{pmatrix} \mathbf{b} & C \end{pmatrix} = \operatorname{rank} C + 1 \quad \text{and} \quad \operatorname{rank} \begin{pmatrix} a & \mathbf{b}^\top \\ \mathbf{b} & C \end{pmatrix} = \operatorname{rank} C + 2,$$

and we get a contradiction.

Therefore, one of the two vectors $\begin{pmatrix} a+1 \\ \mathbf{b} \end{pmatrix}$ or $\begin{pmatrix} a \\ \mathbf{b} \end{pmatrix}$ is a linear combination of columns of the matrix $\begin{pmatrix} \mathbf{b}^\top \\ C \end{pmatrix}$. In the second case we get

$$\operatorname{corank} \begin{pmatrix} a & \mathbf{b}^\top \\ \mathbf{b} & C \end{pmatrix} = \operatorname{corank} \begin{pmatrix} 0 & \mathbf{0}^\top \\ \mathbf{0} & C \end{pmatrix} = \operatorname{corank} C + 1,$$

where $\mathbf{0}$ is the column vector consisting of zeros. □

Denote by $G \backslash \{v_i\}$ the labeled graph obtained from G by deleting the vertex v_i and all edges incident to it (the labels of other vertices are preserved).

Corollary 3.1. *If* $\operatorname{corank} B_i(G) = \operatorname{corank} B(G \setminus \{v_i\})$, *then* $\operatorname{corank} B_i(G) = \operatorname{corank} B(G) - 1$.

Corollary 3.2. *If* $\operatorname{corank} B_i(G) \neq \operatorname{corank} B(G \setminus \{v_i\})$, *then either*

$$\operatorname{corank} B(G) = \operatorname{corank} B(G \setminus \{v_i\}) = \operatorname{corank} B_i(G) - 1,$$

or

$$\operatorname{corank} B(G) = \operatorname{corank} B_i(G) = \operatorname{corank} B(G \setminus \{v_i\}) - 1.$$

Proof. Without loss of generality, we assume that $i = 1$ and

$$B(G) = \begin{pmatrix} a & \mathbf{b}^\top \\ \mathbf{b} & C \end{pmatrix}.$$

Then

$$B_1(G) = \begin{pmatrix} a+1 & \mathbf{b}^\top \\ \mathbf{b} & C \end{pmatrix} \quad \text{and} \quad B(G \setminus \{v_1\}) = C.$$

Using Lemma 3.1, we have two cases.

(1) The vector $\begin{pmatrix} a+1 \\ \mathbf{b} \end{pmatrix}$ is a linear combination of columns of the matrix $\begin{pmatrix} \mathbf{b}^\top \\ C \end{pmatrix}$. Then

$$\operatorname{corank} B(G) = \operatorname{corank} B(G \setminus \{v_i\}) = \operatorname{corank} B_i(G) - 1.$$

(2) The vector $\mathbf{b}$ is not a linear combination of columns of the matrix C. Then

$$\operatorname{corank} B(G) = \operatorname{corank} B_i(G) = \operatorname{corank} B(G \setminus \{v_i\}) - 1.$$

□

3.2. Vertices and components of graph-links. In this section, we first rewrite the condition stating when a classical crossing of a link belongs to one component of the link into the language of intersection graphs, more precisely, into the language of adjacency matrices of intersection graphs, and use this new condition for introducing the notion of *lying on one component*. Then we do the same for a pair of crossings where each of two crossings belong to different components, but they both belong to the same two components.

Let v be a classical crossing of a virtual diagram K. Let us consider an arbitrary rotating circuit on K and construct the labeled chord diagram and its intersection graph G, see Fig. 14 (the upper picture). It is evident that the crossing v belongs to the same component if and only if the virtual diagrams K_1 and K_2 (see the middle and lower pictures in Fig. 14) have a different number of components. It is easy to see that the number of components of the diagram K_1 equals $\operatorname{corank} B(G \setminus \{v\}) + 1$, and for K_2 we have $\operatorname{corank} B(G') + 1 = \operatorname{corank} B_i(G) + 1$, where i is the number of the vertex v. Using these two numbers, we give the following definition.

Let G be a labeled graph with the set of vertices $V(G) = \{v_1, \dots, v_n\}$.

Definition 3.1. We say that a vertex $v_i \in V(G)$ *lies on one component* of G if

$$\operatorname{corank} B_i(G) \neq \operatorname{corank} B(G \setminus \{v_i\}).$$

Otherwise, we say that v_i *belongs to different components* of G.

Let us investigate the behavior of vertices of a labeled graph after applying graph-moves to G.

Lemma 3.2. *Let a labeled graph G_2 be obtained from a labeled graph G_1 by applying a first graph-move which adds a vertex with the label $(0, \alpha)$. Then*

(1) *the additional vertex lies on one component of G_2;*
(2) *a vertex of G_1 lies on one component if and only if the corresponding vertex of G_2 lies on one component.*

Proof. Let $V(G_1) = \{v_1, \dots, v_n\}$ and $V(G_2) = \{v'_1, \dots, v'_{n+1}\}$, and

$$A(G_2) = \begin{pmatrix} A(G_1) & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{pmatrix}, \qquad B(G_2) = \begin{pmatrix} B(G_1) & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{pmatrix}.$$

(1) From the definition it immediately follows that

$$\operatorname{corank} B_{n+1}(G_2) \neq \operatorname{corank} B(G_2 \setminus \{v'_{n+1}\}).$$

From here we have the validity of the first assertion.

(2) It is easy to see that the two equalities

$$\operatorname{corank} B_i(G_1) = \operatorname{corank} B(G_1 \setminus \{v_i\}),$$
$$\operatorname{corank} B_i(G_2) = \operatorname{corank} B(G_2 \setminus \{v'_i\})$$

are true or not true simultaneously if $i < n + 1$. From here we have the validity of the second assertion.

□

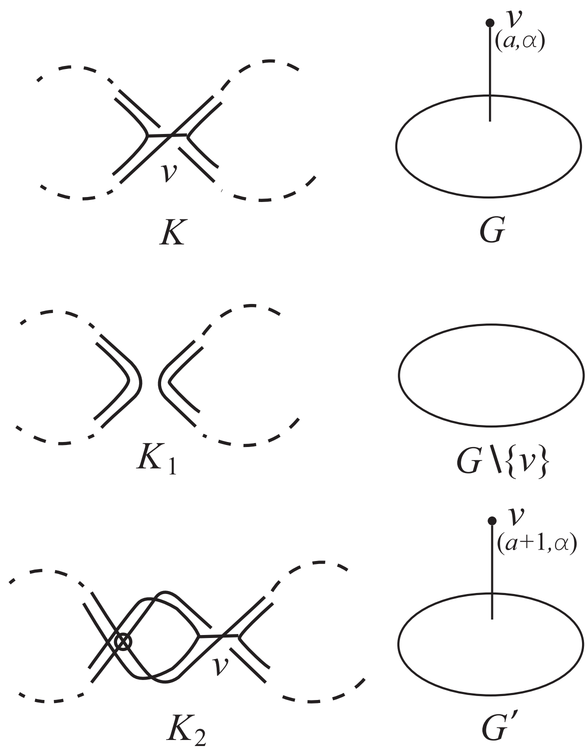

Fig. 14. A crossing of a link

Lemma 3.3. *Let a labeled graph G_2 be obtained from a labeled graph G_1 by applying a second graph-move which adds two vertices. Then:*

(1) *the additional vertices of G_2 simultaneously either lie on one component or belong to different components of the graph G_2;*
(2) *a vertex of G_1 lies on one component if and only if the corresponding vertex of G_2 lies on one component.*

Proof. Let $V(G_1) = \{v_1, \dots, v_n\}$ and $V(G_2) = \{v'_1, \dots, v'_{n+2}\}$. Then

$$A(G_2) = \begin{pmatrix} A(G_1) & \mathbf{c} & \mathbf{c} \\ \mathbf{c}^\top & a & a \\ \mathbf{c}^\top & a & a \end{pmatrix}, \qquad B(G_2) = \begin{pmatrix} B(G_1) & \mathbf{c} & \mathbf{c} \\ \mathbf{c}^\top & a+1 & a \\ \mathbf{c}^\top & a & a+1 \end{pmatrix}.$$

The first assertion is evident. Let us consider the second one.

Using elementary manipulations (see also [9, Lemma 5.1]), we get

$$B(G_2) = \begin{pmatrix} B(G_1) & \mathbf{c} & \mathbf{c} \\ \mathbf{c}^\top & a+1 & a \\ \mathbf{c}^\top & a & a+1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} B(G_1) & \mathbf{c} & \mathbf{c} \\ \mathbf{c}^\top & a+1 & a \\ \mathbf{0}^\top & 1 & 1 \end{pmatrix}$$
$$\rightsquigarrow \begin{pmatrix} B(G_1) & \mathbf{0} & \mathbf{0} \\ \mathbf{c}^\top & a+1 & a \\ \mathbf{0}^\top & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} B(G_1) & \mathbf{0} & \mathbf{0} \\ \mathbf{c}^\top & 1 & a \\ \mathbf{0}^\top & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} B(G_1) & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & 1 & 0 \\ \mathbf{0}^\top & 0 & 1 \end{pmatrix}.$$

Therefore, if $i < n+1$ the equalities

$$\operatorname{corank} B_i(G_1) = \operatorname{corank} B(G_1 \setminus \{v_i\})$$

and

$$\operatorname{corank} B_i(G_2) = \operatorname{corank} B(G_2 \setminus \{v'_i\})$$

are true or not true simultaneously. ◻

Lemma 3.4. *Let a labeled graph G_2 be obtained from a labeled graph G_1 by applying a third graph-move. Then a vertex of G_1 lies on one component if and only if the corresponding vertex of G_2 lies on one component.*

Proof. We have

$$B(G_1) = \begin{pmatrix} 1 & 1 & 1 & \mathbf{0}^\top \\ 1 & 1 & 0 & \mathbf{a}^\top \\ 1 & 0 & 1 & \mathbf{b}^\top \\ \mathbf{0} & \mathbf{a} & \mathbf{b} & C \end{pmatrix}, \qquad B(G_2) = \begin{pmatrix} 1 & 0 & 0 & (\mathbf{a}+\mathbf{b})^\top \\ 0 & 1 & 0 & \mathbf{b}^\top \\ 0 & 0 & 1 & \mathbf{a}^\top \\ \mathbf{a}+\mathbf{b} & \mathbf{b} & \mathbf{a} & C \end{pmatrix}.$$

Since for any $\mathbf{a}$, $\mathbf{b}$, and C there exists a chain of elementary manipulations transforming the matrix $B(G_2)$ to the matrix $B(G_1)$ (see the proof of [9, Lemma 5.1]), we have the validity of the assertion for any vertex with the number $i > 3$.

Let us consider the remaining three cases.

For the first vertex we have

$$B_1(G_2) = \begin{pmatrix} 0 & 0 & 0 & (\mathbf{a}+\mathbf{b})^\top \\ 0 & 1 & 0 & \mathbf{b}^\top \\ 0 & 0 & 1 & \mathbf{a}^\top \\ \mathbf{a}+\mathbf{b} & \mathbf{b} & \mathbf{a} & C \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 1 & \mathbf{0}^\top \\ 1 & 1 & 0 & \mathbf{a}^\top \\ 1 & 0 & 1 & \mathbf{b}^\top \\ \mathbf{0} & \mathbf{a} & \mathbf{b} & C \end{pmatrix} = B_1(G_1),$$

i.e., $\operatorname{corank} B_1(G_1) = \operatorname{corank} B_1(G_2)$.

Further, the matrices $B(G_1 \setminus \{v_1\})$ and $B(G_2 \setminus \{v_1'\})$ are equal to each other up to a permutation of rows and columns, i.e., $\operatorname{corank} B(G_1 \setminus \{v_1\}) = \operatorname{corank} B(G_2 \setminus \{v_1'\})$. Hence we have the validity of the assertion for the first vertex.

For the second vertex we have

$$B_2(G_1) = \begin{pmatrix} 1 & 1 & 1 & \mathbf{0}^\top \\ 1 & 0 & 0 & \mathbf{a}^\top \\ 1 & 0 & 1 & \mathbf{b}^\top \\ \mathbf{0} & \mathbf{a} & \mathbf{b} & C \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & \mathbf{0}^\top \\ 0 & 1 & 1 & \mathbf{a}^\top \\ 0 & 1 & 0 & \mathbf{b}^\top \\ \mathbf{0} & \mathbf{a} & \mathbf{b} & C \end{pmatrix},$$

$$B_2(G_2) = \begin{pmatrix} 1 & 0 & 0 & (\mathbf{a}+\mathbf{b})^\top \\ 0 & 0 & 0 & \mathbf{b}^\top \\ 0 & 0 & 1 & \mathbf{a}^\top \\ \mathbf{a}+\mathbf{b} & \mathbf{b} & \mathbf{a} & C \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 & \mathbf{0}^\top \\ 0 & 0 & 0 & \mathbf{b}^\top \\ 1 & 0 & 1 & \mathbf{a}^\top \\ \mathbf{0} & \mathbf{b} & \mathbf{a} & C \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 0 & 0 & 1 & \mathbf{0}^\top \\ 0 & 0 & 0 & \mathbf{b}^\top \\ 1 & 0 & 0 & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{b} & \mathbf{0} & C \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & \mathbf{0}^\top \\ 0 & 1 & 0 & \mathbf{0}^\top \\ 0 & 0 & 0 & \mathbf{b}^\top \\ \mathbf{0} & \mathbf{0} & \mathbf{b} & C \end{pmatrix},$$

i.e.,

$$\operatorname{corank} B_2(G_1) = \operatorname{corank} \begin{pmatrix} 1 & 1 & \mathbf{a}^\top \\ 1 & 0 & \mathbf{b}^\top \\ \mathbf{a} & \mathbf{b} & C \end{pmatrix}, \quad \operatorname{corank} B_2(G_2) = \operatorname{corank} \begin{pmatrix} 0 & \mathbf{b}^\top \\ \mathbf{b} & C \end{pmatrix}.$$

Further,

$$B(G_1 \setminus \{v_2\}) = \begin{pmatrix} 1 & 1 & \mathbf{0}^\top \\ 1 & 1 & \mathbf{b}^\top \\ \mathbf{0} & \mathbf{b} & C \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & \mathbf{0}^\top \\ 0 & 0 & \mathbf{b}^\top \\ \mathbf{0} & \mathbf{b} & C \end{pmatrix},$$

$$B(G_2 \setminus \{v_2'\}) = \begin{pmatrix} 1 & 0 & (\mathbf{a}+\mathbf{b})^\top \\ 0 & 1 & \mathbf{a}^\top \\ \mathbf{a}+\mathbf{b} & \mathbf{a} & C \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & \mathbf{a}^\top \\ 1 & 0 & \mathbf{b}^\top \\ \mathbf{a} & \mathbf{b} & C \end{pmatrix}.$$

i.e.,

$$\operatorname{corank} B_2(G_1) = \operatorname{corank} B(G_2 \setminus \{v_2'\}), \quad \operatorname{corank} B_2(G_2) = \operatorname{corank} B(G_1 \setminus \{v_2\}).$$

Hence we have the validity of the assertion for the second vertex.

The third vertex is considered analogously to the second one. □

Lemma 3.5. *Let a labeled graph G_2 be obtained from a labeled graph G_1 by applying a fourth graph-move. Then a vertex of G_1 lies on one component if and only if the corresponding vertex of G_2 lies on one component.*

Proof.

1. Let us consider the graph-move $\Omega_g 4$. We have

$$B(G_1) = \begin{pmatrix} 1 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ 1 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{1} & \mathbf{0} & A_1^\top & B_4 & A_5 & A_6 \\ \mathbf{0} & \mathbf{1} & A_2^\top & A_5^\top & B_7 & A_8 \\ \mathbf{1} & \mathbf{1} & A_3^\top & A_6^\top & A_8^\top & B_9 \end{pmatrix},$$

$$B(G_2) = \begin{pmatrix} 1 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ 1 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{0} & \mathbf{1} & A_1^\top & B_4 & A_5+(1) & A_6+(1) \\ \mathbf{1} & \mathbf{0} & A_2^\top & A_5^\top+(1) & B_7 & A_8+(1) \\ \mathbf{1} & \mathbf{1} & A_3^\top & A_6^\top+(1) & A_8^\top+(1) & B_9 \end{pmatrix},$$

where (1) is the matrix of the corresponding size consisting of 1.

Since there exists a chain of elementary manipulations transforming the matrix $B(G_2)$ to the matrix $B(G_1)$ (see [9, Lemma 5.1]), we have the validity of the assertion for any vertex with the number $i > 2$.

Let us consider the remaining two vertices. We consider only the first vertex (the other is treated analogously).

We get

$$B_1(G_1) = \begin{pmatrix} 0 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ 1 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{1} & \mathbf{0} & A_1^\top & B_4 & A_5 & A_6 \\ \mathbf{0} & \mathbf{1} & A_2^\top & A_5^\top & B_7 & A_8 \\ \mathbf{1} & \mathbf{1} & A_3^\top & A_6^\top & A_8^\top & B_9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top \\ 1 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{1} & \mathbf{1} & A_1^\top & B_4 & A_5+(1) & A_6+(1) \\ \mathbf{0} & \mathbf{0} & A_2^\top & A_5^\top+(1) & B_7 & A_8+(1) \\ \mathbf{1} & \mathbf{1} & A_3^\top & A_6^\top+(1) & A_8^\top+(1) & B_9 \end{pmatrix},$$

$$B_1(G_2) = \begin{pmatrix} 0 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ 1 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{0} & \mathbf{1} & A_1^\top & B_4 & A_5+(1) & A_6+(1) \\ \mathbf{1} & \mathbf{0} & A_2^\top & A_5^\top+(1) & B_7 & A_8+(1) \\ \mathbf{1} & \mathbf{1} & A_3^\top & A_6^\top+(1) & A_8^\top+(1) & B_9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top \\ 1 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{0} & \mathbf{1} & A_1^\top & B_4 & A_5 & A_6 \\ \mathbf{1} & \mathbf{0} & A_2^\top & A_5^\top & B_7 & A_8 \\ \mathbf{1} & \mathbf{1} & A_3^\top & A_6^\top & A_8^\top & B_9 \end{pmatrix},$$

$$B(G_1 \setminus \{v_1\}) = \begin{pmatrix} 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{0} & A_1^\top & B_4 & A_5 & A_6 \\ \mathbf{1} & A_2^\top & A_5^\top & B_7 & A_8 \\ \mathbf{1} & A_3^\top & A_6^\top & A_8^\top & B_9 \end{pmatrix},$$

$$B(G_2\setminus\{v_1'\}) = \begin{pmatrix} 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{1} & A_1^\top & B_4 & A_5+(1) & A_6+(1) \\ \mathbf{0} & A_2^\top & A_5^\top+(1) & B_7 & A_8+(1) \\ \mathbf{1} & A_3^\top & A_6^\top+(1) & A_8^\top+(1) & B_9 \end{pmatrix},$$

i.e.,

$$\operatorname{corank} B_1(G_1) = \operatorname{corank} B(G_2\setminus\{v_1'\}), \quad \operatorname{corank} B_1(G_2) = \operatorname{corank} B(G_1\setminus\{v_1\}).$$

Hence we have the validity of the assertion for the first vertex.

2. Let us consider the graph-move $\Omega_g 4'$. We have

$$B(G_1) = \begin{pmatrix} 0 & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & A_0 & A_1 \\ \mathbf{1} & A_1^\top & A_2 \end{pmatrix}, \qquad B(G_2) = \begin{pmatrix} 0 & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & A_0 & A_1 \\ \mathbf{1} & A_1^\top & A_2+(1) \end{pmatrix}.$$

The validity of the assertion for vertices with numbers $i > 1$ is evident. Let us consider the first vertex. We have

$$B_1(G_1) = \begin{pmatrix} 1 & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & A_0 & A_1 \\ \mathbf{1} & A_1^\top & A_2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & \mathbf{0}^\top & \mathbf{0}^\top \\ \mathbf{0} & A_0 & A_1 \\ \mathbf{1} & A_1^\top & A_2+(1) \end{pmatrix},$$

$$B_1(G_2) = \begin{pmatrix} 1 & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & A_0 & A_1 \\ \mathbf{1} & A_1^\top & A_2+(1) \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & \mathbf{0}^\top & \mathbf{0}^\top \\ \mathbf{0} & A_0 & A_1 \\ \mathbf{1} & A_1^\top & A_2 \end{pmatrix},$$

$$B(G_1\setminus\{v_1\}) = \begin{pmatrix} A_0 & A_1 \\ A_1^\top & A_2 \end{pmatrix}, \qquad B(G_2\setminus\{v_1'\}) = \begin{pmatrix} A_0 & A_1 \\ A_1^\top & A_2+(1) \end{pmatrix},$$

i.e.,

$$\operatorname{corank} B_1(G_1) = \operatorname{corank} B(G_2\setminus\{v_1'\}), \quad \operatorname{corank} B_1(G_2) = \operatorname{corank} B(G_1\setminus\{v_1\}).$$

Hence we have the validity of the assertion. □

Let us now consider the mutual position of two vertices of a labeled graph.

Definition 3.2. Let v_i and v_j be two vertices from $V(G)$ belonging to different components. We say that *two components meet* at these vertices if either $v_i = v_j$, or v_i lies on one component of the labeled graph $G\setminus\{v_j\}$, i.e., either

$$\operatorname{corank} B_i(G\setminus\{v_j\}) \neq \operatorname{corank} B(G\setminus\{v_j, v_i\})$$

if $i < j$, or

$$\operatorname{corank} B_{i-1}(G\setminus\{v_j\}) \neq \operatorname{corank} B(G\setminus\{v_j, v_i\})$$

if $i > j$. Otherwise, we say that *different components meet* at these vertices.

Denote by $\widehat{C}^{i,j,\dots,k}$ (respectively, $\widehat{C}_{i,j,\dots,k}$) the matrix obtained from the matrix C by deleting the columns (respectively, rows) with numbers $i, j, \dots, k$. Instead of $\widehat{B(G)}$ we write $\widehat{B}(G)$.

Lemma 3.6. *Let* $B(G) = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & C \end{pmatrix}$. *Different components meet at the vertices* v_1 *and* v_2 *if and only if the vectors* $\mathbf{b}_1$ *and* $\mathbf{b}_2$ *are linear combinations of columns of the matrix* C.

Proof. Let

$$B(G) = \begin{pmatrix} a & b & \mathbf{d}^\top \\ b & c & \mathbf{e}^\top \\ \mathbf{d} & \mathbf{e} & F \end{pmatrix},$$

and the vertices v_1 and v_2 belong to different components each, i.e., by using Lemma 3.1, the vector $\begin{pmatrix} a \\ b \\ \mathbf{d} \end{pmatrix}$ is a linear combination of columns of the matrix $\begin{pmatrix} b & \mathbf{d}^\top \\ c & \mathbf{e}^\top \\ \mathbf{e} & F \end{pmatrix}$, but the vector $\begin{pmatrix} a+1 \\ b \\ \mathbf{d} \end{pmatrix}$ is not, and the vector $\begin{pmatrix} b \\ c \\ \mathbf{e} \end{pmatrix}$ is a linear combination of columns of the matrix $\begin{pmatrix} a & \mathbf{d}^\top \\ b & \mathbf{e}^\top \\ \mathbf{d} & F \end{pmatrix}$, but the vector $\begin{pmatrix} b \\ c+1 \\ \mathbf{e} \end{pmatrix}$ is not (all calculations are done over $\mathbb{Z}_2$).

($\Longrightarrow$) Let different components meet at the vertices v_1 and v_2. Then

$$\operatorname{corank} B_1(G \setminus \{v_2\}) = \operatorname{corank} B(G \setminus \{v_1, v_2\}),$$

and, by Lemma 3.1, the vector $\begin{pmatrix} a \\ \mathbf{d} \end{pmatrix}$ is a linear combination of columns of the matrix $\begin{pmatrix} \mathbf{d}^\top \\ F \end{pmatrix}$. Then the vector $\begin{pmatrix} b \\ \mathbf{e} \end{pmatrix}$ is a linear combination of columns of the matrix $\begin{pmatrix} \mathbf{d}^\top \\ F \end{pmatrix}$. Since the vector $\begin{pmatrix} b \\ c+1 \\ \mathbf{e} \end{pmatrix}$ is not a linear combination of columns of the matrix $\begin{pmatrix} \mathbf{d}^\top \\ \mathbf{e}^\top \\ F \end{pmatrix}$, we get the needed assertion.

($\Longleftarrow$) It immediately follows from Lemma 3.1. □

Lemma 3.7. *Let two components meet at vertices v_i and v_j, $i < j$. Then*

$$\operatorname{corank} B_{j-1}(G \setminus \{v_i\}) = \operatorname{corank} B_i(G \setminus \{v_j\}).$$

Proof. Without loss of generality, we assume that $i = 1$ and $j = 2$. We have

$$B(G) = \begin{pmatrix} a & b & \mathbf{d}^\top \\ b & c & \mathbf{e}^\top \\ \mathbf{d} & \mathbf{e} & F \end{pmatrix}.$$

Using Lemmas 3.1 and 3.6, the sum of the vectors $\begin{pmatrix} a \\ b \\ \mathbf{d} \end{pmatrix}$ and $\begin{pmatrix} b \\ c \\ \mathbf{e} \end{pmatrix}$ is a linear combination of columns of the matrix $\begin{pmatrix} \mathbf{d}^\top \\ \mathbf{e}^\top \\ F \end{pmatrix}$. Performing elementary manipulations, we get

$$\begin{aligned} \operatorname{corank} B_{j-1}(G \setminus \{v_i\}) &= \operatorname{corank} \begin{pmatrix} c+1 & \mathbf{e}^\top \\ \mathbf{e} & F \end{pmatrix} = \operatorname{corank} \begin{pmatrix} b+1 & \mathbf{e}^\top \\ \mathbf{d} & F \end{pmatrix} \\ &= \operatorname{corank} \begin{pmatrix} a+1 & \mathbf{d}^\top \\ \mathbf{d} & F \end{pmatrix} = \operatorname{corank} B_i(G \setminus \{v_j\}). \end{aligned}$$

□

Lemma 3.8. *The relation from Definition* 3.2 *is an equivalence relation on the set of vertices belonging to different components.*

Proof.

1. Reflexivity, i.e., when two vertices coincide, is evident.

2. Symmetry (if two components meet at the vertices v_i and v_j, then two components meet at the vertices v_j and v_i) immediately follows from Lemma 3.7.

3. Transitivity. Let us show that if two components meet at the vertices v_i and v_j and two components meet at the vertices v_j and v_k; then two components meet at the vertices v_i and v_k.

Without loss of generality, we assume that $(i,j,k)=(1,2,3)$. Let

$$B(G)=\begin{pmatrix} a & b & c & \mathbf{k}^\top \\ b & d & e & \mathbf{l}^\top \\ c & e & f & \mathbf{m}^\top \\ \mathbf{k} & \mathbf{l} & \mathbf{m} & Q \end{pmatrix}=\begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \widetilde{Q} \end{pmatrix}.$$

Since the vertices v_s, $s=1,2,3$, belong to different components each, then the vectors $\mathbf{p}_s$ are linear combinations of columns of the matrix $\widehat{B}^s(G)$.

Let two different components meet at the vertices v_1 and v_3. Then, by Lemma 3.6, the vectors $\mathbf{p}_1$ and $\mathbf{p}_3$ are linear combinations of columns of the matrix $\widehat{B}^{1,3}(G)$.

If the vector $\mathbf{p}_1$ (respectively, $\mathbf{p}_3$) is a linear combination of columns of the matrix $\widehat{B}^{1,2,3}(G)$, then the vector $\mathbf{p}_2$ is a linear combination of columns of the matrix $\widehat{B}^{1,2}(G)$ (respectively, $\widehat{B}^{2,3}(G)$). Thus different components meet at the vertices v_1 and v_2 (respectively, at the vertices v_2 and v_3). We get a contradiction with the fact that two components meet at the vertices v_1 and v_2 (respectively, at the vertices v_2 and v_3).

Suppose that the vectors $\mathbf{p}_1$ and $\mathbf{p}_3$ are not linear combinations of columns of the matrix $\widehat{B}^{1,2,3}(G)$. Then $\mathbf{p}_1+\mathbf{p}_3$ is a linear combination of columns of the matrix $\widehat{B}^{1,2,3}(G)$ (linear combinations are considered over $\mathbb{Z}_2$), but the vector $\mathbf{p}_2$ is a linear combination of columns of the matrix $\widehat{B}^{1,2}(G)$ or $\widehat{B}^{2,3}(G)$, but not a linear combination of columns of the matrix $\widehat{B}^{1,2,3}(G)$. In the first case, i.e., the vector $\mathbf{p}_2$ is a linear combination of columns of the matrix $\widehat{B}^{1,2}(G)$, we have that different components meet at the vertices v_1 and v_2, and in the second case we have the same situation at the vertices v_2 and v_3. □

Let us investigate the behavior of a pair of vertices of a labeled graph after applying graph-moves.

Lemma 3.9. *Let a labeled graph G_2 be obtained from a labeled graph G_1 by applying the first graph-move which adds a vertex with the label $(0,\alpha)$. Then at vertices of G_1 belonging to different components two components meet if and only if two components meet at the corresponding vertices of G_2.*

Proof. We know (see Lemma 3.2) that under the first graph-move the additional vertex lies on one component, and the corresponding vertices of G_1 and G_2 simultaneously either lie on one component or belong to different components.

Let $V(G_1)=\{v_1,\dots,v_n\}$, $V(G_2)=\{v'_1,\dots,v'_{n+1}\}$, and

$$A(G_2)=\begin{pmatrix} A(G_1) & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{pmatrix},\qquad B(G_2)=\begin{pmatrix} B(G_1) & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{pmatrix}.$$

Assume that the vertices v_i, v_j, $i>j$, belong to different components. Then the equality

$$\operatorname{corank} B_j(G_1\setminus\{v_i\})=\operatorname{corank} B(G_1\setminus\{v_i,v_j\})$$

is equivalent to the equality

$$\operatorname{corank} B_j(G_2\setminus\{v'_i\})=\operatorname{corank} B(G_2\setminus\{v'_i,v'_j\}).$$

Hence we have the assertion of the lemma. □

Lemma 3.10. *Let a labeled graph G_2 be obtained from a labeled graph G_1 by applying the second graph-move which adds two vertices. Then at vertices of G_1 belonging to different components two components meet if and only if two components meet at the corresponding vertices of G_2. Moreover, if two additional vertices of G_2 belong to different components, then two components meet at them.*

Proof. Let $V(G_1)=\{v_1,\dots,v_n\}$ and $V(G_2)=\{v'_1,\dots,v'_{n+2}\}$. Then

$$A(G_2)=\begin{pmatrix} A(G_1) & \mathbf{c} & \mathbf{c} \\ \mathbf{c}^\top & a & a \\ \mathbf{c}^\top & a & a \end{pmatrix},\qquad B(G_2)=\begin{pmatrix} B(G_1) & \mathbf{c} & \mathbf{c} \\ \mathbf{c}^\top & a+1 & a \\ \mathbf{c}^\top & a & a+1 \end{pmatrix}.$$

We know (see Lemma 3.3) that under the second graph-move the additional vertices simultaneously either lie on one component or belong to different components, and the same is true for the corresponding vertices of G_1 and G_2.

The validity of the assertion, i.e., the validity of the equalities

$$\operatorname{corank} B_j(G_1 \setminus \{v_i\}) = \operatorname{corank} B(G_1 \setminus \{v_i, v_j\}) \quad \text{and} \quad \operatorname{corank} B_j(G_2 \setminus \{v'_i\}) = \operatorname{corank} B(G_2 \setminus \{v'_i, v'_j\}),$$

where v_i, v_j, $i < j < n+1$, belong to different components each, easily follows from manipulations given in Lemma 3.3.

Let us prove the second assertion (about the additional vertices). We have

$$\operatorname{corank} B_{n+1}(G_2) = \operatorname{corank} B(G_2 \setminus \{v'_{n+1}\}) \Longleftrightarrow \operatorname{corank} B_{n+2}(G_2) = \operatorname{corank} B(G_2 \setminus \{v'_{n+2}\})$$

$$\Longleftrightarrow \operatorname{corank} \begin{pmatrix} B(G_1) & \mathbf{c} & \mathbf{c} \\ \mathbf{c}^\top & a+1 & a \\ \mathbf{c}^\top & a & a \end{pmatrix} = \operatorname{corank} \begin{pmatrix} B(G_1) & \mathbf{c} \\ \mathbf{c}^\top & a+1 \end{pmatrix},$$

since the vertices v'_{n+1}, v'_{n+2} belong to different components each. Since

$$\begin{pmatrix} B(G_1) & \mathbf{c} & \mathbf{c} \\ \mathbf{c}^\top & a+1 & a \\ \mathbf{c}^\top & a & a \end{pmatrix} \rightsquigarrow \begin{pmatrix} B(G_1) & \mathbf{0} & \mathbf{c} \\ \mathbf{0}^\top & 1 & 0 \\ \mathbf{c}^\top & 0 & a \end{pmatrix},$$

we have

$$\operatorname{corank} \begin{pmatrix} B(G_1) & \mathbf{c} \\ \mathbf{c}^\top & a \end{pmatrix} = \operatorname{corank} \begin{pmatrix} B(G_1) & \mathbf{c} \\ \mathbf{c}^\top & a+1 \end{pmatrix}.$$

Therefore, the column-vector $\mathbf{c}$ is not a linear combination of columns of the matrix $B(G_1)$. If it is a linear combination, then the first matrix should be equivalent to the matrix

$$\begin{pmatrix} B(G_1) & \mathbf{0} \\ \mathbf{0}^\top & d \end{pmatrix},$$

and the second one should be equivalent to

$$\begin{pmatrix} B(G_1) & \mathbf{0} \\ \mathbf{0}^\top & d+1 \end{pmatrix}.$$

We get a contradiction. Therefore, since matrices are symmetric, the rank of the matrix

$$\begin{pmatrix} B(G_1) & \mathbf{c} \\ \mathbf{c}^\top & a \end{pmatrix}$$

is greater than the rank of the matrix $B(G_1)$ by at least 2, i.e.,

$$\operatorname{corank} \begin{pmatrix} B(G_1) & \mathbf{c} \\ \mathbf{c}^\top & a \end{pmatrix} \neq \operatorname{corank} B(G_1)$$

$$\Longleftrightarrow \operatorname{corank} B_{n+1}(G_2 \setminus \{v'_{n+2}\}) \neq \operatorname{corank} B(G_2 \setminus \{v'_{n+1}, v'_{n+2}\}).$$

□

Lemma 3.11. *Let a labeled graph G_2 be obtained from a labeled graph G_1 by applying the third graph-move. Then at vertices of G_1 belonging to different components two components meet if and only if two components meet at the corresponding vertices of G_2.*

Proof. Let $V(G_1) = \{v_1, \ldots, v_n\}$ and $V(G_2) = \{v'_1, \ldots, v'_n\}$. We have

$$B(G_1) = \begin{pmatrix} 1 & 1 & 1 & \mathbf{0}^\top \\ 1 & 1 & 0 & \mathbf{a}^\top \\ 1 & 0 & 1 & \mathbf{b}^\top \\ \mathbf{0} & \mathbf{a} & \mathbf{b} & C \end{pmatrix}, \qquad B(G_2) = \begin{pmatrix} 1 & 0 & 0 & (\mathbf{a}+\mathbf{b})^\top \\ 0 & 1 & 0 & \mathbf{b}^\top \\ 0 & 0 & 1 & \mathbf{a}^\top \\ \mathbf{a}+\mathbf{b} & \mathbf{b} & \mathbf{a} & C \end{pmatrix}.$$

By using Lemma 3.8, the assertion of the lemma for pairs of vertices (v_1, v_i), (v_2, v_i), (v_3, v_i), (v_i, v_j), $i, j > 3$, can be proved analogously as is done in Lemma 3.4.

Let us consider the remaining cases. Due to symmetry it is sufficient to consider the pairs (v_1, v_2) and (v_2, v_3). We consider only one pair; the second one is considered analogously.

Let us consider the pair (v_1, v_2). We have to show that the equalities (we should watch the numeration of vertices in the new graph)

$$\operatorname{corank} B_1(G_1 \setminus \{v_1\}) = \operatorname{corank} B(G_1 \setminus \{v_1, v_2\}) \quad \text{and} \quad \operatorname{corank} B_1(G_2 \setminus \{v_1'\}) = \operatorname{corank} B(G_2 \setminus \{v_1', v_2'\})$$

are true or not true simultaneously.

Using Lemma 3.6, we see that the first equality is true if and only if the vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ \mathbf{0} \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \\ \mathbf{a} \end{pmatrix}$ are linear combinations of columns of the matrix $\begin{pmatrix} 1 & \mathbf{0}^\top \\ 0 & \mathbf{a}^\top \\ 1 & \mathbf{b}^\top \\ \mathbf{b} & C \end{pmatrix}$. Adding the second and third rows to the first one, we get that the vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ \mathbf{0} \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \\ \mathbf{a} \end{pmatrix}$ are linear combinations of columns of the matrix $\begin{pmatrix} 0 & (\mathbf{a}+\mathbf{b})^\top \\ 0 & \mathbf{a}^\top \\ 1 & \mathbf{b}^\top \\ \mathbf{b} & C \end{pmatrix}$. Adding the vectors $\begin{pmatrix} 0 \\ 1 \\ 0 \\ \mathbf{a} \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \\ \mathbf{b} \end{pmatrix}$ to the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \\ \mathbf{0} \end{pmatrix}$ and transposing the second and third rows, we get that the vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ \mathbf{a}+\mathbf{b} \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \\ \mathbf{a} \end{pmatrix}$ are linear combinations of columns of the matrix $\begin{pmatrix} 0 & (\mathbf{a}+\mathbf{b})^\top \\ 1 & \mathbf{b}^\top \\ 0 & \mathbf{a}^\top \\ \mathbf{b} & C \end{pmatrix}$. The latter condition, by Lemma 3.6, is equivalent to the second equality. Since in all cases we had equivalent transformations, we get the validity of the assertion. □

Lemma 3.12. *Let a labeled graph G_2 be obtained from a labeled graph G_1 by applying the fourth graph-move. Then at vertices of G_1 belonging to different components two components meet if and only if two components meet at the corresponding vertices of G_2.*

Proof. Let $V(G_1) = \{v_1, \dots, v_n\}$ and $V(G_2) = \{v_1, \dots, v_n'\}$.

1. Let us consider the graph-move $\Omega_g 4$. We have

$$B(G_1) = \begin{pmatrix} 1 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ 1 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{1} & \mathbf{0} & A_1^\top & B_4 & A_5 & A_6 \\ \mathbf{0} & \mathbf{1} & A_2^\top & A_5^\top & B_7 & A_8 \\ \mathbf{1} & \mathbf{1} & A_3^\top & A_6^\top & A_8^\top & B_9 \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & C \end{pmatrix},$$

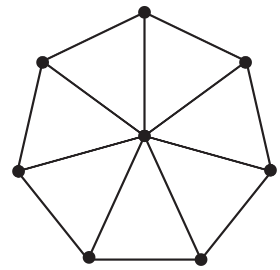

Fig. 15. The third Bouchet graph W_7

$$B(G_2) = \begin{pmatrix} 1 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ 1 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{0} & \mathbf{1} & A_1^\top & B_4 & A_5 + (1) & A_6 + (1) \\ \mathbf{1} & \mathbf{0} & A_2^\top & A_5^\top + (1) & B_7 & A_8 + (1) \\ \mathbf{1} & \mathbf{1} & A_3^\top & A_6^\top + (1) & A_8^\top + (1) & B_9 \end{pmatrix} = \begin{pmatrix} \mathbf{b}_2 & \mathbf{b}_1 & \widetilde{C} \end{pmatrix},$$

where $\mathbf{1}$ is the column-vector consisting of 1.

The validity of the assertion for pairs of vertices (v_1, v_i), (v_2, v_i), (v_i, v_j), $i, j > 3$, can be proved analogously to the case of Lemma 3.5 (we use the fact that the relation from Definition 3.2 is an equivalence relation; see Lemma 3.8).

Let us consider the pair (v_1, v_2). Let the vertices v_1 and v_2 belong to different components each. Using Lemma 3.1 and elementary manipulations of rows, it is easy to show that the vector $\mathbf{b}_1$ is a linear combination of columns of the matrix C if and only if the vector $\mathbf{b}_2$ is a linear combination of columns of the matrix $\widetilde{C}$.

2. Let us consider the graph-move $\Omega_g 4'$. The proof of this item is analogous to the proof of 2) from Lemma 3.5. □

3.3. Invariant and Bouchet graphs. Let G be a labeled graph. Let us consider equivalence classes in the set of vertices belonging to different components modulo the relation from Definition 3.8. Define the number $\vartheta(G)$ to be the number of equivalence classes having an odd number of vertices. From Lemmas 3.9, 3.10, 3.11 and 3.12 the theorem follows.

Theorem 3.1. *The number $\vartheta(G)$ is invariant under graph-moves, i.e., it is an invariant of graph-links.*

Example 3.1. Let us consider the second Bouchet graph BW_3 (see Fig. 4) and endow each its vertex with the framing 0 (the sign is not important for us). Consider the graph-link generated by this graph. It is easy to show that this graph-link consists of four components and $\vartheta(BW_3) = 7$.

It is easy to check that for any virtual link with four components the invariant ϑ is strictly less than 7. Thus, it follows that the graph-link is nonrealizable.

Remark 3.2. Note that the nonrealizability of the graph-link generated by the third Bouchet graph W_7 (see Fig. 15), where all vertices except the center have the framing 0, can be proved by using parity [18–20] and the equivalence from [8].

4. An Orientation on Graph-Links and the Jones Polynomial

In this section, we give a definition of an *oriented graph-link*. For an oriented graph-link we construct the writhe number corresponding to the real writhe number in the realizable case. With the help of this number we normalize the Kauffman bracket polynomial (see [9, 10]) to get the Jones polynomial.

4.1. Deleting vertices from a graph. Let us formulate lemmas showing how the rank and corank of an adjacency matrix change under the deletion of a vertex from the graph. Later on, these lemmas help us to define the writhe number for graph-links.

Let G be a labeled graph with the set of vertices $V(G) = \{v_1, \dots, v_n\}$.

Lemma 4.1. *Let a vertex v_i of G belong to different components. Then there is a graph $G_{cs(v_i)} \in \{G\}$ obtained from G by fourth graph-moves such that the vertex v'_i of $G_{cs(v_i)}$ corresponding to the vertex v_i has sign opposite to the sign $\operatorname{sgn}(v_i)$ of the vertex v_i. Moreover, if the vertex v_i belongs to different components, then the same is true for the vertex v'_i.*

Proof. If a vertex belongs to different components, then by Lemma 3.2 we cannot apply the first graph-move to it. Let us consider four cases.

1. Let the framing of v_i equal 1. Then we can apply the fourth graph-move $\Omega_g 4'$ to this vertex and we immediately get the vertex v'_i.
2. Let the framing of v_i equal 0 and let there exist a vertex adjacent to it and having framing 0. Then we can apply the fourth graph-move $\Omega_g 4$ to this vertex and immediately get the vertex v'_i.
3. Let the framing of v_i equal 0 and let there exist a vertex v_j adjacent to it and having framing 0. Then we first apply the $\Omega_g 4'$ to the vertex v_j, and then we apply the same graph-move to v_i in order to get the vertex v'_i.
4. Let v_i be an isolated vertex with framing 0. In this case the vertex v_i lies on one component and we have a contradiction with the condition of the lemma.

The second assertion of the lemma follows straightforwardly from Lemma 3.5. ☐

Lemma 4.2. *Let a vertex $v_k \in V(G)$ belong to different components.*

1. *If two components of the graph G meet at vertices v_i and v_j, $i, j \neq k$, then the corresponding vertices from the graph $G \setminus \{v_k\}$ either lie on one component or two components of the graph $G \setminus \{v_k\}$ meet at them.*
2. *Let a vertex v_i, $i < k$, lie on one component of G. Then the vertex of the graph $G \setminus \{v_k\}$ corresponding to v_i also lies on one component of the graph $G \setminus \{v_k\}$ and, moreover,*

$$\operatorname{corank} B_i(G) - \operatorname{corank} B(G) = \operatorname{corank} B_i(G \setminus \{v_k\}) - \operatorname{corank} B(G \setminus \{v_k\}).$$

Proof.

(1) The first assertion of the lemma follows from Lemma 3.6. Let us only prove the second one.

(2) Let $i = n-1$, $k = n$ and $V(G \setminus \{v_n\}) = \{v_1, \dots, v_{n-1}\}$. We have

$$B(G) = \begin{pmatrix} F & \mathbf{d} & \mathbf{e} \\ \mathbf{d}^\top & a & c \\ \mathbf{e}^\top & c & b \end{pmatrix}, \qquad B(G \setminus \{v_n\}) = \begin{pmatrix} F & \mathbf{d} \\ \mathbf{d}^\top & a \end{pmatrix}$$

and

$$\operatorname{corank} B_n(G) = \operatorname{corank} B(G \setminus \{v_n\}), \quad \operatorname{corank} B_{n-1}(G) \neq \operatorname{corank} B(G \setminus \{v_{n-1}\}).$$

By Lemma 3.1 the vector $\begin{pmatrix} \mathbf{e} \\ c \\ b \end{pmatrix}$ is a linear combination of the vector $\begin{pmatrix} \mathbf{d} \\ a \\ c \end{pmatrix}$ and columns of the matrix $\begin{pmatrix} F \\ \mathbf{d}^\top \\ \mathbf{e}^\top \end{pmatrix}$, but the vector $\begin{pmatrix} \mathbf{d} \\ a \\ c \end{pmatrix}$ is not a linear combination of the vectors $\begin{pmatrix} \mathbf{e} \\ c \\ b \end{pmatrix}$ and columns of the matrix $\begin{pmatrix} F \\ \mathbf{d}^\top \\ \mathbf{e}^\top \end{pmatrix}$. Therefore, the vector $\begin{pmatrix} \mathbf{e} \\ c \\ b \end{pmatrix}$ is a linear combination of columns of the matrix $\begin{pmatrix} F \\ \mathbf{d}^\top \\ \mathbf{e}^\top \end{pmatrix}$.

We get

$$\operatorname{corank} B_{n-1}(G) = \operatorname{corank}\begin{pmatrix} F & \mathbf{d} & \mathbf{e} \\ \mathbf{d}^\top & a+1 & c \\ \mathbf{e}^\top & c & b \end{pmatrix} = \operatorname{corank}\begin{pmatrix} F & \mathbf{d} & \mathbf{0} \\ \mathbf{d}^\top & a+1 & 0 \\ \mathbf{0}^\top & 0 & 0 \end{pmatrix}$$
$$= \operatorname{corank}\begin{pmatrix} F & \mathbf{d} \\ \mathbf{d}^\top & a+1 \end{pmatrix} + 1 = \operatorname{corank} B_{n-1}(G \setminus \{v_n\}) + 1,$$
$$\operatorname{corank} B(G \setminus \{v_{n-1}\}) = \operatorname{corank}\begin{pmatrix} F & \mathbf{e} \\ \mathbf{e}^\top & b \end{pmatrix} = \operatorname{corank}\begin{pmatrix} F & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{pmatrix}$$
$$= \operatorname{corank} F + 1 = \operatorname{corank} B(G \setminus \{v_{n-1}, v_n\}) + 1.$$

Using Corollary 3.1 and the last equalities, we get the validity of the second assertion of the lemma. □

Lemma 4.3. *Let a vertex v_i lie on one component of a graph G. Then the number*

$$w_i = \operatorname{sgn}(v_i)(-1)^{\operatorname{corank} B_i(G) - \operatorname{corank} B(G)}$$

is invariant under the fourth graph-moves.

Proof.

1. Let a graph $\widetilde{G}$ with the set of vertices $V(\widetilde{G}) = \{v'_1, \dots, v'_n\}$ be obtained from the graph G with the first graph-move $\Omega_g 4'$ at a vertex v_j. We have two cases: $i = j$ and $i \neq j$.

Let us consider the first case. We assume $i = j = 1$. We have

$$B(G) = \begin{pmatrix} 0 & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & A_0 & A_1 \\ \mathbf{1} & A_1^\top & A_2 \end{pmatrix}, \qquad B(\widetilde{G}) = \begin{pmatrix} 0 & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & A_0 & A_1 \\ \mathbf{1} & A_1^\top & A_2 + (1) \end{pmatrix}.$$

Then $\operatorname{sgn}(v_i) = -\operatorname{sgn}(v'_i)$, $\operatorname{corank} B(G) = \operatorname{corank} B(\widetilde{G})$ and

$$\operatorname{corank} B_i(\widetilde{G}) = \operatorname{corank}\begin{pmatrix} 1 & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & A_0 & A_1 \\ \mathbf{1} & A_1^\top & A_2 + (1) \end{pmatrix} = \operatorname{corank}\begin{pmatrix} 1 & \mathbf{0}^\top & \mathbf{0}^\top \\ \mathbf{0} & A_0 & A_1 \\ \mathbf{0} & A_1^\top & A_2 \end{pmatrix} = \operatorname{corank} B(G \setminus \{v_i\}).$$

Since $\operatorname{corank} B_i(G) \neq \operatorname{corank} B(G \setminus \{v_i\})$, we get $\operatorname{corank} B_i(\widetilde{G}) = \operatorname{corank} B_i(G) \pm 1$ and $w_i = w'_i$.

Let us consider the second case. We assume $j = 1$ and $i = 2$. We have

$$B(G) = \begin{pmatrix} 0 & a & \mathbf{0}^\top & \mathbf{1}^\top \\ a & b & \mathbf{c}^\top & \mathbf{d}^\top \\ \mathbf{0} & \mathbf{c} & A_0 & A_1 \\ \mathbf{1} & \mathbf{d} & A_1^\top & A_2 \end{pmatrix}, \qquad B(\widetilde{G}) = \begin{pmatrix} 0 & a & \mathbf{0}^\top & \mathbf{1}^\top \\ a & b+a & \mathbf{c}^\top & \mathbf{d}^\top + (a)^\top \\ \mathbf{0} & \mathbf{c} & A_0 & A_1 \\ \mathbf{1} & \mathbf{d} + (a) & A_1^\top & A_2 + (1) \end{pmatrix}.$$

Then $\operatorname{sgn}(v_i) = \operatorname{sgn}(v'_i)$, $\operatorname{corank} B(G) = \operatorname{corank} B(\widetilde{G})$, and $\operatorname{corank} B_i(\widetilde{G}) = \operatorname{corank} B_i(G)$, i.e., $w_i = w'_i$.

2. Let a graph $\widetilde{G}$ with the set of vertices $V(\widetilde{G}) = \{v'_1, \dots, v'_n\}$ be obtained from the graph G by the fourth graph-move $\Omega_g 4$ at vertices v_j and v_k. We have two cases: $i \in \{j, k\}$ and $i \notin \{j, k\}$.

Let us consider the first case. We assume that $i = j = 1$ and $k = 2$. We have

$$B(G) = \begin{pmatrix} 1 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ 1 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{1} & \mathbf{0} & A_1^\top & B_4 & A_5 & A_6 \\ \mathbf{0} & \mathbf{1} & A_2^\top & A_5^\top & B_7 & A_8 \\ \mathbf{1} & \mathbf{1} & A_3^\top & A_6^\top & A_8^\top & B_9 \end{pmatrix},$$

$$B(\widetilde{G}) = \begin{pmatrix} 1 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ 1 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{0} & \mathbf{1} & A_1^\top & B_4 & A_5 + (1) & A_6 + (1) \\ \mathbf{1} & \mathbf{0} & A_2^\top & A_5^\top + (1) & B_7 & A_8 + (1) \\ \mathbf{1} & \mathbf{1} & A_3^\top & A_6^\top + (1) & A_8^\top + (1) & B_9 \end{pmatrix}.$$

Then $\operatorname{sgn}(v_i) = -\operatorname{sgn}(v'_i)$, $\operatorname{corank} B(G) = \operatorname{corank} B(\widetilde{G})$ and

$$\operatorname{corank} B_i(\widetilde{G}) = \operatorname{corank} \begin{pmatrix} 0 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ 1 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{0} & \mathbf{1} & A_1^\top & B_4 & A_5 + (1) & A_6 + (1) \\ \mathbf{1} & \mathbf{0} & A_2^\top & A_5^\top + (1) & B_7 & A_8 + (1) \\ \mathbf{1} & \mathbf{1} & A_3^\top & A_6^\top + (1) & A_8^\top + (1) & B_9 \end{pmatrix}$$

$$= \operatorname{corank} \begin{pmatrix} 0 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top \\ 1 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{0} & \mathbf{1} & A_1^\top & B_4 & A_5 & A_6 \\ \mathbf{1} & \mathbf{0} & A_2^\top & A_5^\top & B_7 & A_8 \\ \mathbf{1} & \mathbf{1} & A_3^\top & A_6^\top & A_8^\top & B_9 \end{pmatrix} = \operatorname{corank} B(G \setminus \{v_i\})$$

Since $\operatorname{corank} B_i(G) \neq \operatorname{corank} B(G \setminus \{v_i\})$, we get $\operatorname{corank} B_i(\widetilde{G}) = \operatorname{corank} B_i(G) \pm 1$ and $w_i = w'_i$.

Let us consider the second case. We assume that $j = 1$, $k = 2$ and $i = 3$. We have

$$B(G) = \begin{pmatrix} 1 & 1 & a & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ 1 & 1 & h & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ a & h & b & \mathbf{c}^\top & \mathbf{d}^\top & \mathbf{e}^\top & \mathbf{f}^\top \\ \mathbf{0} & \mathbf{0} & \mathbf{c} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{1} & \mathbf{0} & \mathbf{d} & A_1^\top & B_4 & A_5 & A_6 \\ \mathbf{0} & \mathbf{1} & \mathbf{e} & A_2^\top & A_5^\top & B_7 & A_8 \\ \mathbf{1} & \mathbf{1} & \mathbf{f} & A_3^\top & A_6^\top & A_8^\top & B_9 \end{pmatrix},$$

$$B(\widetilde{G}) = \begin{pmatrix} 1 & 1 & h & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ 1 & 1 & a & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ h & a & b & \mathbf{c}^\top & \mathbf{d}^\top + (h)^\top & \mathbf{e}^\top + (a)^\top & \mathbf{f}^\top + (a+h)^\top \\ \mathbf{0} & \mathbf{0} & \mathbf{c} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{0} & \mathbf{1} & \mathbf{d} + (h) & A_1^\top & B_4 & A_5 + (1) & A_6 + (1) \\ \mathbf{1} & \mathbf{0} & \mathbf{e} + (a) & A_2^\top & A_5^\top + (1) & B_7 & A_8 + (1) \\ \mathbf{1} & \mathbf{1} & \mathbf{f} + (a+h) & A_3^\top & A_6^\top + (1) & A_8^\top + (1) & B_9 \end{pmatrix}.$$

Then $\operatorname{sgn}(v_i) = \operatorname{sgn}(v'_i)$, $\operatorname{corank} B(G) = \operatorname{corank} B(\widetilde{G})$, and $\operatorname{corank} B_i(\widetilde{G}) = \operatorname{corank} B_i(G)$, i.e., $w_i = w'_i$. □

Lemma 4.4. *Let two components meet at vertices v_i and v_j, $i < j$. Then*

$$\operatorname{sgn}(v_i) \operatorname{sgn}(v_j) (-1)^{\operatorname{corank} B_{j-1}(G \setminus \{v_i\})} = \operatorname{sgn}(v'_i) \operatorname{sgn}(v'_j) (-1)^{\operatorname{corank} B_{j-1}(G_{cs(v_i)} \setminus \{v'_i\})},$$

where $V(G_{cs(v_i)}) = \{v'_1, \ldots, v'_n\}$, and the corresponding vertices of the graphs G and $G_{cs(v_i)}$ have the same numbers.

Proof. Without loss of generality, we assume that $i = 1$, $j = 2$. From Definition 3.2 and Lemma 3.7 it follows that

$$\operatorname{corank} B_{j-1}(G \setminus \{v_i\}) = \operatorname{corank} B_i(G \setminus \{v_j\}) \neq \operatorname{corank} B(G \setminus \{v_i, v_j\}).$$

According to Lemma 4.3 and the proof of Lemma 4.1, it is enough to consider three cases.

1. Let the framing of the vertex v_1 be equal to 1, and we apply the fourth graph-move $\Omega_g 4'$ to it. We have

$$B(G)=\begin{pmatrix} 0 & a & \mathbf{0}^\top & \mathbf{1}^\top \\ a & b & \mathbf{c}^\top & \mathbf{d}^\top \\ \mathbf{0} & \mathbf{c} & A_0 & A_1 \\ \mathbf{1} & \mathbf{d} & A_1^\top & A_2 \end{pmatrix}, \qquad B(G_{cs(v_i)})=\begin{pmatrix} 0 & a & \mathbf{0}^\top & \mathbf{1}^\top \\ a & b+a & \mathbf{c}^\top & \mathbf{d}^\top+(a)^\top \\ \mathbf{0} & \mathbf{c} & A_0 & A_1 \\ \mathbf{1} & \mathbf{d}+(a) & A_1^\top & A_2+(1) \end{pmatrix},$$

where (a) is the vector consisting of $a\in\{0,1\}$.

Applying Lemma 3.7 and elementary manipulations, we get

$$\begin{aligned}\operatorname{corank} B_{j-1}(G_{cs(v_i)}\setminus\{v_i'\}) &= \operatorname{corank}\begin{pmatrix} 1 & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & A_0 & A_1 \\ \mathbf{1} & A_1^\top & A_2+(1) \end{pmatrix} \\ &= \operatorname{corank}\begin{pmatrix} 1 & \mathbf{0}^\top & \mathbf{0}^\top \\ \mathbf{0} & A_0 & A_1 \\ \mathbf{0} & A_1^\top & A_2 \end{pmatrix} = \operatorname{corank} B(G\setminus\{v_i,v_j\}),\end{aligned}$$

i.e.,

$$\operatorname{corank} B_{j-1}(G_{cs(v_i)}\setminus\{v_i'\}) = \operatorname{corank} B_{j-1}(G\setminus\{v_i\})\pm 1.$$

Taking into account the equalities $\operatorname{sgn}(v_i)=-\operatorname{sgn}(v_i')$ and $\operatorname{sgn}(v_j)=\operatorname{sgn}(v_j')$, we get the assertion of the lemma.

2. Let the framing of the vertex v_1 be equal to 0 and let there exist a vertex v_k adjacent to v_1 and having the framing 0. In this case we apply the fourth graph-move $\Omega_g 4$ to them.

We have two cases: $v_k=v_2$ and $v_k\neq v_2$. In the first case we have

$$B(G)=\begin{pmatrix} 1 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ 1 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{1} & \mathbf{0} & A_1^\top & B_4 & A_5 & A_6 \\ \mathbf{0} & \mathbf{1} & A_2^\top & A_5^\top & B_7 & A_8 \\ \mathbf{1} & \mathbf{1} & A_3^\top & A_6^\top & A_8^\top & B_9 \end{pmatrix},$$

$$B(G_{cs(v_i)})=\begin{pmatrix} 1 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ 1 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{0} & \mathbf{1} & A_1^\top & B_4 & A_5+(1) & A_6+(1) \\ \mathbf{1} & \mathbf{0} & A_2^\top & A_5^\top+(1) & B_7 & A_8+(1) \\ \mathbf{1} & \mathbf{1} & A_3^\top & A_6^\top+(1) & A_8^\top+(1) & B_9 \end{pmatrix}.$$

Applying elementary manipulations, we get

$$\begin{aligned}\operatorname{corank} B_{j-1}(G_{cs(v_i)}\setminus\{v_i'\}) &= \operatorname{corank}\begin{pmatrix} 0 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{0} & A_1^\top & B_4 & A_5+(1) & A_6+(1) \\ \mathbf{1} & A_2^\top & A_5^\top+(1) & B_7 & A_8+(1) \\ \mathbf{1} & A_3^\top & A_6^\top+(1) & A_8^\top+(1) & B_9 \end{pmatrix} \\ &= \operatorname{corank}\begin{pmatrix} 0 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{0} & A_1^\top & B_4 & A_5 & A_6 \\ \mathbf{1} & A_2^\top & A_5^\top & B_7 & A_8 \\ \mathbf{1} & A_3^\top & A_6^\top & A_8^\top & B_9 \end{pmatrix} = \operatorname{corank} B_{j-1}(G\setminus\{v_i\}).\end{aligned}$$

Taking into account the equalities $\operatorname{sgn}(v_i) = -\operatorname{sgn}(v_i')$ and $\operatorname{sgn}(v_j) = -\operatorname{sgn}(v_j')$, we get the assertion of the lemma.

In the second case (let $k = 3$) we have

$$B(G) = \begin{pmatrix} 1 & a & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ a & b & h & \mathbf{c}^\top & \mathbf{d}^\top & \mathbf{e}^\top & \mathbf{f}^\top \\ 1 & h & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{c} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{1} & \mathbf{d} & \mathbf{0} & A_1^\top & B_4 & A_5 & A_6 \\ \mathbf{0} & \mathbf{e} & \mathbf{1} & A_2^\top & A_5^\top & B_7 & A_8 \\ \mathbf{1} & \mathbf{f} & \mathbf{1} & A_3^\top & A_6^\top & A_8^\top & B_9 \end{pmatrix},$$

$$B(G_{cs(v_i)}) = \begin{pmatrix} 1 & h & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ h & b & a & \mathbf{c}^\top & \mathbf{d}^\top + (h)^\top & \mathbf{e}^\top + (a)^\top & \mathbf{f}^\top + (a+h)^\top \\ 1 & a & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{c} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{0} & \mathbf{d} + (h) & \mathbf{1} & A_1^\top & B_4 & A_5 + (1) & A_6 + (1) \\ \mathbf{1} & \mathbf{e} + (a) & \mathbf{0} & A_2^\top & A_5^\top + (1) & B_7 & A_8 + (1) \\ \mathbf{1} & \mathbf{f} + (a+h) & \mathbf{1} & A_3^\top & A_6^\top + (1) & A_8^\top + (1) & B_9 \end{pmatrix}.$$

Applying Lemma 3.7 and elementary manipulations, we get

$$\operatorname{corank} B_{j-1}(G_{cs(v_i)} \setminus \{v_i'\}) = \operatorname{corank} \begin{pmatrix} 0 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ 1 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{0} & \mathbf{1} & A_1^\top & B_4 & A_5 + (1) & A_6 + (1) \\ \mathbf{1} & \mathbf{0} & A_2^\top & A_5^\top + (1) & B_7 & A_8 + (1) \\ \mathbf{1} & \mathbf{1} & A_3^\top & A_6^\top + (1) & A_8^\top + (1) & B_9 \end{pmatrix}$$

$$= \operatorname{corank} \begin{pmatrix} 0 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top \\ 1 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{0} & \mathbf{1} & A_1^\top & B_4 & A_5 & A_6 \\ \mathbf{1} & \mathbf{0} & A_2^\top & A_5^\top & B_7 & A_8 \\ \mathbf{1} & \mathbf{1} & A_3^\top & A_6^\top & A_8^\top & B_9 \end{pmatrix} = \operatorname{corank} \begin{pmatrix} 1 & 0 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top \\ 0 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{0} & \mathbf{0} & A_1^\top & B_4 & A_5 & A_6 \\ \mathbf{0} & \mathbf{1} & A_2^\top & A_5^\top & B_7 & A_8 \\ \mathbf{0} & \mathbf{1} & A_3^\top & A_6^\top & A_8^\top & B_9 \end{pmatrix}$$
$$= \operatorname{corank} B(G \setminus \{v_i, v_j\}),$$

i.e.,

$$\operatorname{corank} B_{j-1}(G_{cs(v_i)} \setminus \{v_i'\}) = \operatorname{corank} B_{j-1}(G \setminus \{v_i\}) \pm 1.$$

Taking into account the equalities $\operatorname{sgn}(v_i) = -\operatorname{sgn}(v_i')$ and $\operatorname{sgn}(v_j) = \operatorname{sgn}(v_j')$, we get the assertion of the lemma.

3. Let the framing of the vertex v_1 be equal to 0 and let there exist a vertex v_k adjacent to the vertex v_1 and having the framing 1. In this case we apply the fourth graph-move $\Omega_g 4'$ to the vertex v_k, and then we apply this move to the vertex in the new graph corresponding to the vertex v_1.

Two cases are possible: $v_k = v_2$ and $v_k \neq v_2$. In the first case we have

$$B(G) = \begin{pmatrix} 1 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ 1 & 0 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{1} & \mathbf{0} & A_1^\top & B_4 & A_5 & A_6 \\ \mathbf{0} & \mathbf{1} & A_2^\top & A_5^\top & B_7 & A_8 \\ \mathbf{1} & \mathbf{1} & A_3^\top & A_6^\top & A_8^\top & B_9 \end{pmatrix},$$

$$B(G_{cs(v_i)}) = \begin{pmatrix} 0 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top & \mathbf{0}^\top \\ 1 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{1} & \mathbf{1} & A_1^\top & B_4+(1) & A_5+(1) & A_6 \\ \mathbf{1} & \mathbf{0} & A_2^\top & A_5^\top+(1) & B_7 & A_8+(1) \\ \mathbf{0} & \mathbf{1} & A_3^\top & A_6^\top & A_8^\top+(1) & B_9+(1) \end{pmatrix}.$$

Applying elementary manipulations, we get

$$\operatorname{corank} B_{j-1}(G_{cs(v_i)} \setminus \{v_i'\}) = \operatorname{corank} \begin{pmatrix} 0 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{1} & A_1^\top & B_4+(1) & A_5+(1) & A_6 \\ \mathbf{0} & A_2^\top & A_5^\top+(1) & B_7 & A_8+(1) \\ \mathbf{1} & A_3^\top & A_6^\top & A_8^\top+(1) & B_9+(1) \end{pmatrix}$$
$$= \operatorname{corank} \begin{pmatrix} 0 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{1} & A_1^\top & B_4 & A_5 & A_6 \\ \mathbf{0} & A_2^\top & A_5^\top & B_7 & A_8 \\ \mathbf{1} & A_3^\top & A_6^\top & A_8^\top & B_9 \end{pmatrix} = \operatorname{corank} B_i(G \setminus \{v_j\}).$$

Taking into account the equalities $\operatorname{sgn}(v_i) = -\operatorname{sgn}(v_i')$ and $\operatorname{sgn}(v_j) = -\operatorname{sgn}(v_j')$, we get the needed assertion.

In the second case (let $k = 3$) we have

$$B(G) = \begin{pmatrix} 1 & a & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ a & b & h & \mathbf{c}^\top & \mathbf{d}^\top & \mathbf{e}^\top & \mathbf{f}^\top \\ 1 & h & 0 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{c} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{1} & \mathbf{d} & \mathbf{0} & A_1^\top & B_4 & A_5 & A_6 \\ \mathbf{0} & \mathbf{e} & \mathbf{1} & A_2^\top & A_5^\top & B_7 & A_8 \\ \mathbf{1} & \mathbf{f} & \mathbf{1} & A_3^\top & A_6^\top & A_8^\top & B_9 \end{pmatrix},$$

$$B(G_{cs(v_i)}) = \begin{pmatrix} 0 & a+h & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top & \mathbf{0}^\top \\ a+h & b & a & \mathbf{c}^\top & \mathbf{d}^\top+(a+h)^\top & \mathbf{e}^\top+(a)^\top & \mathbf{f}^\top+(h)^\top \\ 1 & a & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{c} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{1} & \mathbf{d}+(a+h) & \mathbf{1} & A_1^\top & B_4+(1) & A_5+(1) & A_6 \\ \mathbf{1} & \mathbf{e}+(a) & \mathbf{0} & A_2^\top & A_5^\top+(1) & B_7 & A_8+(1) \\ \mathbf{0} & \mathbf{f}+(h) & \mathbf{1} & A_3^\top & A_6^\top & A_8^\top+(1) & B_9+(1) \end{pmatrix}.$$

Applying Lemma 3.7 and elementary manipulations, we get

$$\operatorname{corank} B_{j-1}(G_{cs(v_i)} \setminus \{v_i'\}) = \operatorname{corank} \begin{pmatrix} 1 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top & \mathbf{0}^\top \\ 1 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{1} & \mathbf{1} & A_1^\top & B_4+(1) & A_5+(1) & A_6 \\ \mathbf{1} & \mathbf{0} & A_2^\top & A_5^\top+(1) & B_7 & A_8+(1) \\ \mathbf{0} & \mathbf{1} & A_3^\top & A_6^\top & A_8^\top+(1) & B_9+(1) \end{pmatrix}$$

$$= \operatorname{corank} \begin{pmatrix} 1 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top \\ 1 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{1} & \mathbf{1} & A_1^\top & B_4 & A_5 & A_6 \\ \mathbf{1} & \mathbf{0} & A_2^\top & A_5^\top & B_7 + (1) & A_8 + (1) \\ \mathbf{0} & \mathbf{1} & A_3^\top & A_6^\top & A_8^\top + (1) & B_9 + (1) \end{pmatrix}$$

$$= \operatorname{corank} \begin{pmatrix} 1 & 0 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top \\ 0 & 0 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & B_0 & A_1 & A_2 & A_3 \\ \mathbf{0} & \mathbf{0} & A_1^\top & B_4 & A_5 & A_6 \\ \mathbf{0} & \mathbf{1} & A_2^\top & A_5^\top & B_7 & A_8 \\ \mathbf{0} & \mathbf{1} & A_3^\top & A_6^\top & A_8^\top & B_9 \end{pmatrix} = \operatorname{corank} B(G \setminus \{v_i, v_j\}),$$

i.e.,

$$\operatorname{corank} B_{j-1}(G_{cs(v_i)} \setminus \{v_i'\}) = \operatorname{corank} B_{j-1}(G \setminus \{v_i\}) \pm 1.$$

Taking into account the equalities $\operatorname{sgn}(v_i) = -\operatorname{sgn}(v_i')$ and $\operatorname{sgn}(v_j) = \operatorname{sgn}(v_j')$, we get the assertion. □

4.2. Oriented graph-links and the Jones polynomial. Let G be a labeled graph with k components and $V(G) = \{v_1, \dots, v_n\}$. Since under the deletion of a vertex belonging to different components the number of components of a graph-link decreases by one (see Corollary 3.1), there exists a sequence of vertices $(v_{i_1}, \dots, v_{i_{k-1}})$ consisting of $k-1$ vertices $v_{i_1}, \dots, v_{i_{k-1}}$ such that the graph

$$(\dots((G \setminus \{v_{i_1}\}) \setminus \{v_{i_2}\}) \dots) \setminus \{v_{i_{k-1}}\}$$

obtained from G by deleting vertices of the sequence in consecutive order has only one component.

Lemma 4.5. *For any sequence $(\alpha_{i_1}, \dots, \alpha_{i_{k-1}})$ of signs there exists a labeled graph $G(v_{i_1}, \dots, v_{i_{k-1}})$ obtained from G by the fourth graph-moves such that the labeled graph $G' = G(v_{i_1}, \dots, v_{i_{k-1}}) \setminus \{v_{i_1}, \dots, v_{i_{k-1}}\}$ has one component and the signs of the vertices of $G(v_{i_1}, \dots, v_{i_{k-1}})$, which correspond to the vertices v_{i_j}, $j = 1, \dots, k-1$, of G, coincide with α_{i_j}.*

Proof. First note that the fourth graph-moves change signs of only those vertices these moves are applied to. Further, since deleting a vertex from a graph has no influence on signs and adjacencies of the remaining vertices, the lemma can be proved by the induction with the help of Lemma 4.1. □

Since G' has one component, we can define the *writhe number* w_i of each v_i' of its vertices by putting

$$w_i = (-1)^{\operatorname{corank} B_i(G')} \operatorname{sgn}(v_i')$$

and the *writhe number* of G' itself by the formula $w(G') = \sum\limits_{i=1}^{n-k+1} w_i$.

Definition 4.1. The *writhe number* $w_i(G)$ of the graph G at the vertex $v_i \in V(G)$ with respect to a sequence of vertices $(v_{i_1}, \dots, v_{i_{k-1}})$ with signs $(\alpha_{i_1}, \dots, \alpha_{i_{k-1}})$ is the writhe number of the vertex of G', which corresponds to the vertex v_i if $i \neq i_j$, and $w_{i_j}(G) = \alpha_{i_j}$ otherwise. The *writhe number* of G with respect to the sequence of vertices $(v_{i_1}, \dots, v_{i_{k-1}})$ with signs $(\alpha_{i_1}, \dots, \alpha_{i_{k-1}})$ is

$$w(G) = \sum_{i=1}^{n} w_i(G).$$

Remark 4.1. From Lemmas 4.3 and 4.4 it follows that the writhe number is well defined, i.e., it does not depend on a graph $G(v_{i_1}, \dots, v_{i_{k-1}})$ and does only on the signs of the vertices being deleted. Indeed, from these lemmas it follows that a vertex lying on one component has always the same writhe number for each equivalence class of vertices, and by the relation from Definition 3.2, the writhe number of each vertex from this class is defined by the writhe number of any vertex from this class.

Definition 4.2. Consider two sequences of vertices $(v_{i_1}, \dots, v_{i_{k-1}})$ and $(v_{j_1}, \dots, v_{j_{k-1}})$ with signs $(\alpha_{i_1}, \dots, \alpha_{i_{k-1}})$ and $(\alpha_{j_1}, \dots, \alpha_{j_{k-1}})$, respectively. Assume that after the deletion of these vertices from the corresponding graphs $G(v_{i_1}, \dots, v_{i_{k-1}})$ and $G(v_{j_1}, \dots, v_{j_{k-1}})$ we have two graphs with one component each. We say that these sequences are *equivalent* if the writhe numbers at vertices v_{i_p} (respectively, v_{j_p}), $p = 1, \dots, k-1$, with respect to these sequences, coincide.

Definition 4.3. We say that a labeled graph G is *oriented* if an equivalence class of sequences of vertices with signs after the deletion of which we get a labeled graph with one component, is fixed.

Lemma 4.6. *Let a labeled graph G have two components. Assume that vertices v_i and v_j, $i \neq j$, belong to different components and we can apply a second decreasing graph-move to them. Then there exists a vertex v_k, $k \neq i, j$, also belonging to different components.*

Proof. Let $i = 1$ and $j = 2$. Then

$$B(G) = \begin{pmatrix} a+1 & a & \mathbf{b}^\top \\ a & a+1 & \mathbf{b}^\top \\ \mathbf{b} & \mathbf{b} & C \end{pmatrix}$$

and $\det B(G) = 0$. Since the first and second columns of the matrix $B(G)$ are not zero columns and are not linearly dependent, there exists a number k such that the kth column is a linear combination of the other columns of the matrix $B(G)$. Therefore, the vertex v_k belongs to different components. $\square$

If we have a graph G with an orientation, i.e., a sequence of vertices $(v_{i_1}, \dots, v_{i_{k-1}})$ of G with signs $(\alpha_{i_1}, \dots, \alpha_{i_{k-1}})$ is fixed, we can define an orientation on any graph $\widetilde{G}$ obtained from G by applying a single graph-move. Note that after applying the first, third, and fourth graph-move or the second graph-move increasing the number of vertices of G, we can define the orientation on $\widetilde{G}$ by the sequence of vertices corresponding to the sequence $(v_{i_1}, \dots, v_{i_{k-1}})$ with the same signs. But if we apply the second graph-move decreasing the number of vertices of G, then, by Lemma 4.6, we first choose a sequence of vertices $(v_{j_1}, \dots, v_{j_{k-1}})$ with signs $(\alpha_{j_1}, \dots, \alpha_{j_{k-1}})$ equivalent to the sequence $(v_{i_1}, \dots, v_{i_{k-1}})$ with signs $(\alpha_{i_1}, \dots, \alpha_{i_{k-1}})$ such that the second graph-move has no effect on $v_{j_1}, \dots, v_{j_{k-1}}$, and then the orientation on $\widetilde{G}$ is generated by the sequence of vertices corresponding to $(v_{j_1}, \dots, v_{j_{k-1}})$ with signs $(\alpha_{j_1}, \dots, \alpha_{j_{k-1}})$. We say that the graphs G and $\widetilde{G}$ *have the same orientation.*

Lemma 4.7. *Let a vertex v_i of G belong to different components. Then there exists a graph $\widetilde{G}$ obtained from G by applying the second increasing graph-move such that two components meet at the additional vertices and at the vertex corresponding to the vertex v_i.*

Proof. We can consider the graph $\widetilde{G}$ obtained from G be adding two vertices with framing 0 and adjacent to v_i. $\square$

Lemma 4.8. *The writhe number of an oriented labeled graph is changed by ± 1 under the first graph-move $\Omega_g 1$. More precisely, it is changed by -1 if we add a vertex with the positive sign, and by $+1$ if we add a vertex with the negative sign.*

The writhe number of an oriented labeled graph is invariant under graph-moves $\Omega_g 2$–$\Omega_g 4'$.

Proof.

1. The changing by ± 1 under the first graph-move was proved in [9], and the invariance under the fourth graph-moves follows from Lemmas 4.3 and 4.4.

2. The invariance under the second graph-move follows from the possibility of taking a sequence of vertices, giving the orientation, such that each vertex of the sequence does not take part in our second graph-move, and also from Lemmas 4.2 and 4.3 and the invariance of the writhe number for labeled graphs with one component; see [9, Lemma 5.4].

3. Let us consider the third graph-move. Using Lemma 4.7 and the invariance under the second graph-move, we can assume that the orientation on the graph is given with a sequence of vertices

not taking part in our graph-move. In this case the validity of the assertion of the lemma follows from Lemmas 4.2 and 4.3 and from the invariance for labeled graphs with one component; see [9, Lemma 5.4]. □

Definition 4.4 (see [9, 10])**.** We call a subset of $V(G)$ a *state* of G.

The *Kauffman bracket polynomial* of G is

$$\langle G\rangle(a) = \sum_s a^{\alpha(s)-\beta(s)}(-a^2 - a^{-2})^{\operatorname{corank} A(G(s))},$$

where the sum is taken over all states s of G, and $\alpha(s)$ is equal to the sum of the vertices labeled $(a,-)$ from s and the vertices labeled $(b,+)$ from $V(G)\setminus s$, $\beta(s) = |V(G)| - \alpha(s)$.

Theorem 4.1 (see [9, 10])**.** *The Kauffman bracket polynomial of a labeled graph is invariant under* Ω_g2–Ω_g4' *and gets multiplied by* $(-a^{\pm 3})$ *under* Ω_g1.

Definition 4.5. We say that a graph-link is *oriented* if all its representatives are oriented and for any two representatives G' and G'' of it there exists a sequence $G_1 = G', G_2, \ldots, G_s = G''$ such that the graphs G_p and G_{p+1}, $p = 1, \ldots, s-1$, are obtained from each other by one graph-move and have the same orientation.

Remark 4.2. It is easy to see that to define an orientation on a graph-link it is sufficient to define it for any representative.

Definition 4.6. Let $\mathfrak{F}$ be an oriented graph-link. Define the *Jones polynomial* as

$$X(G)(t) = (-a)^{-3w(G)}\langle G\rangle(t),$$

where G is any representative of $\mathfrak{F}$.

Remark 4.3. Indeed, to obtain the "real" Jones polynomial we should replace the variable of the polynomial $X(G)$ by putting $a = q^{-1/4}$.

From Lemma 4.8 and Theorem 4.1 we get the following theorem.

Theorem 4.2. *The Jones polynomial is an invariant of oriented graph-links.*

4.3. Examples.

1. The easiest example of a nontrivial graph-link having two components is the graph-link generated by the labeled graph G consisting of one isolated vertex v with the framing 1 (the sign of this vertex is not important, since the labeled graphs with different signs are equivalent to each other). For G we have two states $s_1 = \varnothing$ and $s_2 = \{v\}$. Let the sign of v be equal to $+$. Then $\alpha(s_1) = 1$, $\beta(s_1) = 0$, $\operatorname{corank} A(G(s_1)) = 0$ and $\alpha(s_2) = 0$, $\beta(s_2) = 1$, $\operatorname{corank} A(G(s_2)) = \operatorname{corank}(1) = 0$. We have $\langle G\rangle(a) = a + a^{-1}$. We can define two orientations on this graph: in the first case the writhe number is equal to $+1$, and in the second case it equals -1. We have two Jones polynomials: $-a^{-2} - a^{-4}$ and $-a^4 - a^2$.

2. Let us consider the graph-link generated by the labeled graph G depicted in Fig. 16 left. This graph is obtained from the Hopf link, see Fig. 16 right. For G we have four states $s_1 = \varnothing$, $s_2 = \{v_1\}$, $s_3 = \{v_2\}$, $s_4 = \{v_1, v_2\}$. Then $\alpha(s_1) = 1$, $\beta(s_1) = 1$, $\operatorname{corank} A(G(s_1)) = 0$; $\alpha(s_2) = 2$, $\beta(s_2) = 0$, $\operatorname{corank} A(G(s_2)) = \operatorname{corank}(0) = 1$; $\alpha(s_3) = 0$, $\beta(s_3) = 2$, $\operatorname{corank} A(G(s_3)) = \operatorname{corank}(0) = 1$; $\alpha(s_4) = 1$, $\beta(s_4) = 1$, $\operatorname{corank} A(G(s_4)) = \operatorname{corank}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0$. We have the Kauffman bracket polynomial $\langle G\rangle(a) = -a^4 - a^{-4}$. There are two orientations on this graph-link. Consider only one orientation given by the sequence (v_1) with the sign $(-)$, see Fig. 17. Then the graph $G\setminus\{v_1\}$ consists of one vertex v_2 with the label $(0,+)$, and the writhe number of this graph equals -1. Thus, the writhe number of the oriented labeled graph G equals -2, and the Jones polynomial is $-a^{10} - a^2$.

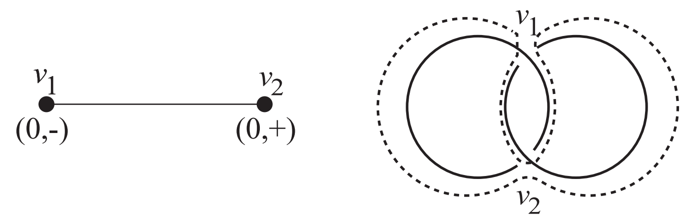

Fig. 16. A graph-link and the Hopf link (the dashed line is a rotating circuit)

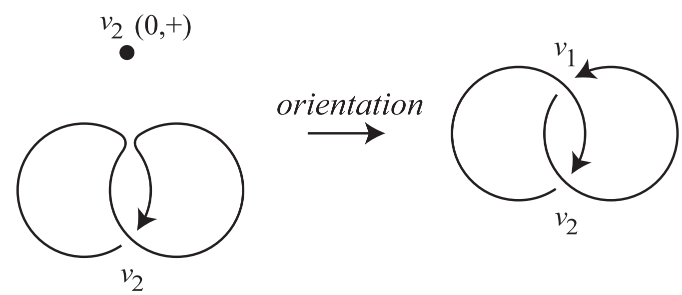

Fig. 17. An orientation on the Hopf link

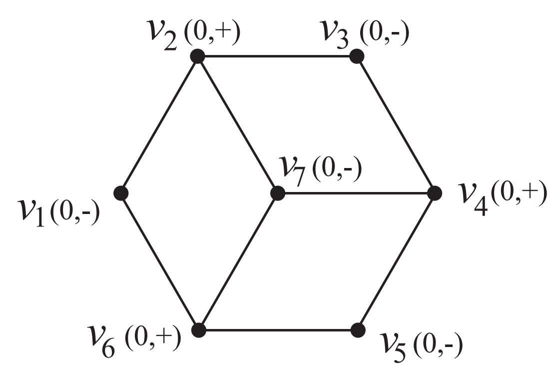

Fig. 18. The labeled second Bouchet graph

3. Let us consider the graph-link generated by the second Bouchet graph BW_3 with labels depicted in Fig. 18. We know, see Example 3.1, that this graph-link contains four components and is nonrealizable.

The Kauffman bracket polynomial is $a^{-15} - 3a^{-11} + 6a^{-7} - 3a^{-3} + 7a - 3a^5 + 4a^9 - a^{13}$. Define an orientation on the graph with the sequence (v_7, v_2, v_3) with the signs $(-, +, -)$.

Note that the oriented labeled graph $BW_3 \setminus \{v_7\}$ can be realized by the Borromean rings; see Fig. 19. Then the writhe number of the labeled graph BW_3 is equal to -1. As a result, we get the Jones polynomial $-a^{-12} + 3a^{-8} - 6a^{-4} + 3 - 7a^4 + 3a^8 - 4a^{12} + a^{16}$.

4. Let us consider the graph-link generated by the first Bouchet graph W_5 with labels depicted in Fig. 20 (the looped graph corresponding to this labeled graph (see [8]) was revealed to the authors by L. Zulli). It is easy to check that this graph-link contains only one component, i.e., it is a graph-knot.

Using parity and the equivalence between the set of graph-knots and the set of homotopy classes of looped graphs [8, 11, 18, 21], we can easily show that the given graph-knot is nonrealizable, and, therefore, is nontrivial. It turns out that the Kauffman bracket polynomial of it is equal to 1, and the writhe number is equal to 0. Thus, *we have an example of a nontrivial graph-knot with the trivial Jones polynomial.*

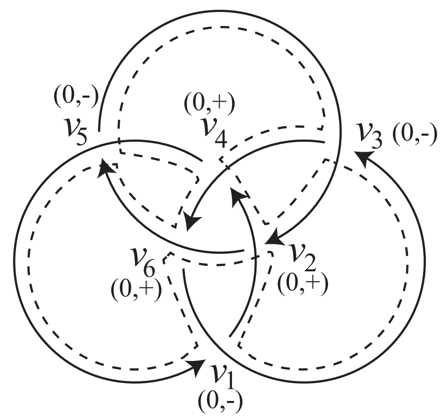

Fig. 19. The Borromean rings (the dashed line is a rotating circuit)

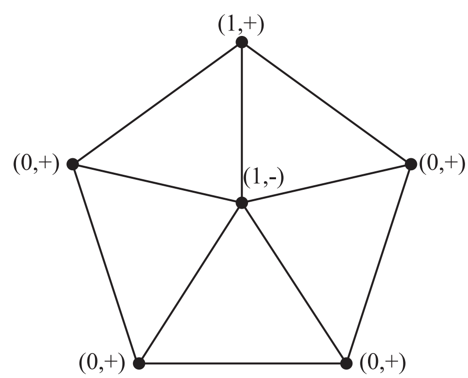

Fig. 20. A graph-link with the trivial Jones polynomial

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