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FINAL DISTRIBUTION OF A DIFFUSION PROCESS WITH FINAL STOP

B. P. Harlamov* UDC 519.2

A one-dimensional diffusion process is considered. The characteristic operator of this process is assumed to be a linear differential operator of the second order with negative coefficient at the term with zero derivative. Such an operator determines the measure of a Markov diffusion process with break (the first interpretation), and also the measure of a semi-Markov diffusion process with final stop (the second interpretation). Under the second interpretation, the existence of the limit of the process at infinity (the final point) is characterized. This limit exists on any interval almost surely with respect to the conditional measure generated by the condition that the process never leaves this interval. The distribution of the final point expressed in terms of two fundamental solutions of the corresponding ordinary differential equation, and also the distribution of the instant final stop are derived. A homogeneous process is considered as an example. Bibliography: 6 *titles.*

1. Diffusion process with final distribution

Characteristic operator. We consider a one-dimensional diffusion process. It is assumed that the characteristic operator of this process, defined on the set of twice differentiable functions, is a trinomial of the form

$$\mathcal{D}f = a(x)f'' + b(x)f' - c(x)f,$$

the coefficients $a(x)$ and $c(x)$ of which are positive [2, p. 726].

In the theory of Markov processes, where the term "characteristic operator" first appeared and actively used (see [2, p. 207]), the presence of the term $-c(x)f$ suggests that the process $X(t)$, $t \geq 0$, we are interested in, has a break at some random time $\zeta < \infty$. This means that the trajectory of the process is determined only until the break and the fact of termination results from the defect of Markov transition probability of the process with this characteristic operator. Namely, the function $u(t,x) \equiv \mathbf{P}_x(X(t) \in \mathbb{R})$ (where $\mathbb{R} = (-\infty, \infty)$ is the range of possible values of the process) satisfies the equation

$$\frac{\partial u}{\partial t} = \mathcal{D}u,$$

which implies that for some $t > 0$ and x,

$$\mathbf{P}\left(X(t) \in \mathbb{R} \,|\, X(0) = x\right) < 1.$$

Another interpretation of the characteristic operator is associated with a class of semi-Markov processes of diffusion type (see below). By definition, the class of strictly Markov processes is a part of the class of semi-Markov processes, see [4]. Therefore speaking of the semi-Markov processes, we will sometimes specify them as semi-Markov, but not Markov. For such semi-Markov processes, Markov transition probability can be defined in the usual way, but it does not determine the measure of the process and, more importantly, does not satisfy the previous parabolic partial differential equation. However, for the semi-Markov process of diffusion type, there are other functionals that satisfy a similar equation of elliptic type. In particular, the functional

$$f_\Delta(x) \equiv \mathbf{P}_x(\sigma_\Delta < \infty),$$

*Institute of Problems of Mechanical Engineering, St.Petersburg, Russia, e-mail: b.p.harlamov@gmail.com.

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where σ_Δ is the first exit of the process trajectory outside the open interval $\Delta \subset \mathbb{R}$, satisfies the equation

$$\mathcal{D} f_\Delta = 0,$$

taking the value 1 in the boundary points of the interval. But this implies that $f_\Delta(x) < 1$ inside the interval. And if we assume that the trajectories of the process are not terminated, then the condition

$$\mathbf{P}_x(\sigma_\Delta = \infty) > 0$$

must be satisfied for some points $x \in \Delta$. And this leads to a completely different interpretation of the characteristic operator. A process with break is replaced by a process with stop, which is an infinite interval of constancy at the end of the trajectory. In the present paper, we prove this statement and find the limit (final) distribution of the process.

A continuous process with limit at infinity. Thus, we assume that the process in question belongs to a broader class than the class of strictly Markov processes. Note that other than that the semi-Markov diffusion process remains similar to Markov diffusion process with break. Only at the time of a break, the process is not terminated, but stops forever at the reached point of the trajectory.

Consider a one-dimensional continuous random process $X(t)(t \geq 0)$. The distribution $\mathbf{P}_x$ of the process with initial state $x \in \mathbb{R}$ is a probability measure on the measurable space $(\mathcal{C}, F)$, where $\mathcal{C}$ is the set of all continuous sample functions $\xi : [0.\infty) \to \mathbb{R}$ and $\mathcal{F}$ is the Borel sigma-algebra of the subsets of this set. In terms of the sample functions, the object $X(t) \equiv X_t$ plays the role of projection, i.e., gives the value of the sample function at the time $t : X_t(\xi) = \xi(t)$.

Let $-\infty \leq a < b \leq \infty$, and let $\sigma_{(a,b)}$ be the first exit of the process outside the interval (a, b), i.e., the function on the set $\mathcal{C}$ of the form

$$\sigma_{(a,b)}(\xi) = \inf\{t \geq 0 : \xi(t) \notin (a, b)\} \quad (\xi \in \mathcal{C}).$$

The problem on final distribution of the process arises from the interpretation of the events of the form $\{\sigma_{(a,b)} = \infty\}$, especially if there exist points $x \in (a, b)$, for which

$$\mathbf{P}_x(\sigma_{(a,b)} = \infty) > 0.$$

In what follows, we use a special notation for the event, the meaning of which is that from a certain moment, the process does not leave the interval Δ. Let

$$\rho(\Delta) \equiv \bigcup_{t \geq 0} \theta_t^{-1}\{\sigma_\Delta = \infty\},$$

where θ_t is a translation operator on the set $\mathcal{C}$:

$$(\forall s \geq 0) \quad (\theta_t(\xi))(s) = \xi(t + s).$$

Let (t_n) be a sequence of random variables, which tends to infinity as $n \to \infty$ almost surely. Then almost surely

$$\rho(\Delta) = \bigcup_{n=1}^{\infty} \theta_{t_n}^{-1}\{\sigma_\Delta = \infty\}. \tag{1}$$

Clearly, $\{\sigma_\Delta = \infty\} \subset \rho(\Delta)$ and

$$\theta_t^{-1}\rho(\Delta) = \rho(\Delta)$$

for every $t > 0$.

One of the reasons that a process starting at a certain time does not leave the interval, is that the sample trajectory ξ of the process has a limit as $t \to \infty$, where $\lim\limits_{t\to\infty} \xi(t) \in (a, b)$. Denote by $\mathcal{C}^{\lim}$ the set of the sample trajectories with limit at infinity.

Lemma 1. *On the set $\mathcal{C}$, the relation*

$$\mathcal{C}^{\lim} \cap \{\sigma_\Delta = \infty\} = \bigcap_{\epsilon>0} \bigcup_{\Delta_1 \in \Pi(\epsilon)} \rho(\Delta_1)$$

holds, where Δ is an open interval and $\Pi(\epsilon)$ is a covering of the set Δ by open intervals of length at most ϵ, $\bigcup_{\Delta_1 \in \Pi(\epsilon)} \Delta_1 = \Delta$.

Proof. Every function $\xi \in \bigcap_{\epsilon>0} \bigcup_{\Delta \in \Pi_\epsilon} \rho(\Delta)$ satisfies the Cauchy criterion at infinity and, hence, converges (converges in itself). The reverse inclusion is trivial. □

The set $\mathcal{C}^{\lim}$ contains as a special subclass the class of functions having the so-called infinite stop, which is an infinite interval of constancy on its domain. The term "infinite stop" do not mean the break of a function in some finite time. The break of a process is usually interpreted as a process getting to some additional state $\vartheta \notin \mathbb{R}$, from which there is no return. Infinite stop is defined to be a stay of the process with continuous sample trajectories in the same state belonging to the set $\mathbb{R}$ for all $t \geq t_0$ with respect to a certain $t_0 \geq 0$. It is easy to determine the beginning of the infinite stop if some information about the entire trajectory as a whole is given. But it is impossible to know that the infinite stop has already begun even if the process prehistory is given. This means that the beginning of the infinite stop (a random moment ζ) is a measurable function with respect to $\mathcal{F}$, but is not a Markov moment with respect to the natural filtration of the process $(\mathcal{F}_t)_0^\infty$, where $\mathcal{F}_t$ is the σ-algebra generated by all the events of the form $\{X_s \leq x\}$ $(s \leq t,\ x \in \mathbb{R})$.

Denote by $\mathcal{C}^\infty$ the set of sample trajectories with infinite stop. Obviously, $\mathcal{C}^\infty \subset \mathcal{C}^{\lim}$. It is easy to prove that both of these sets are measurable with respect to $\mathcal{F}$.

In the final distribution problem solved by a semi-Markov method, the difference between the processes with infinite stop and processes with limits at infinity of a general form is not essential as long as the problem of beginning the infinite stop is not considered. We will return to this problem at the end of the paper.

Continuous semi-Markov process. Let $(\mathbf{P}_x)$ be a consistent measurable family of measures on $(\mathcal{C}, \mathcal{F})$ and indexed by the starting points of the trajectories of the process $x \in \mathbb{R}$. We assume that the consistency of $(\mathbf{P}_x)$ is defined by the condition

$$\mathbf{P}_x(\theta_{\sigma_\Delta}^{-1} B,\ A,\ \sigma_\Delta < \infty) = \mathbf{E}_x(P_{X(\sigma_\Delta)}(B);\ A,\ \sigma_\Delta < \infty) \tag{2}$$

for all $x \in \mathbb{R}$, $B \in \mathcal{F}$, and $A \in \mathcal{F}_{\sigma_\Delta}$, where Δ is an interval contained in the set of the process values and $\mathcal{F}_{\sigma_\Delta}$ is a σ-algebra defined in a standard way with respect to the Markov moment σ_Δ. Thus, the probability that depends on the value of the process at random time σ_Δ, coincides almost surely with conditional probability with respect to the entire prehistory of the process up to this point, i.e., the homogeneous Markov condition holds at the time of the first exit outside open interval included to the range of the process [4]. In general, this process is not Markov. We call this process a continuous semi-Markov one [4]. The adjective "continuous" is used to distinguish our definition from the definition, which is used in the theory of semi-Markov step-processes (although the definition of continuous semi-Markov process with no restrictions on the measures domain area is also suitable for step-processes). We note that according to this definition, every continuous strictly Markov process is a continuous semi-Markov process, but not vice versa. However, to study processes with infinite stop, the Markov model is not suitable, except for some special cases of the stop choice.

Semi-Markov transition functions. For each interval (a,b) and $x \in (a,b)$, we define the semi-Markov transition functions

$$G_{(a,b)}(dt \,|\, x) \equiv \mathbf{P}_x(\sigma_{(a,b)} \in dt,\ X(\sigma_{(a,b)}) = a),$$

$$H_{(a,b)}(dt \,|\, x) \equiv \mathbf{P}_x(\sigma_{(a,b)} \in dt,\ X(\sigma_{(a,b)}) = b),$$

and the corresponding Laplace transform in t $(\lambda \geq 0)$:

$$g_{(a,b)}(\lambda, x) \equiv \int_0^\infty e^{-\lambda t} G_{(a,b)}(dt \,|\, x),$$

$$h_{(a,b)}(\lambda, x) \equiv \int_0^\infty e^{-\lambda t} H_{(a,b)}(dt \,|\, x).$$

Thus,

$$g_{(a,b)}(\lambda, x) = \mathbf{E}_x(\exp(-\lambda\sigma_{(a,b)});\ \sigma_{(a,b)} < \infty,\ X(\sigma_{(a,b)}) = a),$$

$$h_{(a,b)}(\lambda, x) = \mathbf{E}_x(\exp(-\lambda\sigma_{(a,b)});\ \sigma_{(a,b)} < \infty,\ X(\sigma_{(a,b)}) = b).$$

We assume that the limit values satisfy the conditions

$$g_{(a,b)}(\lambda, a+) = h_{(a,b)}(\lambda, b-) = 1, \quad g_{(a,b)}(\lambda, b-) = h_{(a,b)}(\lambda, a+) = 0$$

and at the boundary points of the interval, the values of the above functions are defined by continuity.

Denote the sum "with shift" of any two nonnegative functions τ_1 and τ_2 on $\mathcal{D}$ (which possibly have infinite values) by

$$\tau_1 \dot{+} \tau_2 = \tau_1 + \tau_2 \circ \theta_{\tau_1}$$

if $\tau_1 < \infty$ and by $\tau_1 \dot{+} \tau_2 = \infty$ if $\tau_1 = \infty$. The operation $\dot{+}$ is associative, but not commutative.

It is known that for any interval (c,d) such that $(c,d) \subset (a,b)$, we have

$$\sigma_{(a,b)} = \sigma_{(c,d)} \dot{+} \sigma_{(a,b)}.$$

From this relation, using the found values of the functions $g_{(c,d)}$ and $h_{(c,d)}$ at the ends of the interval $h_{(c,d)}$ and the semi-Markov property of the process, we obtain the equations (see [4])

$$g_{(a,b)}(\lambda, x) = g_{(c,d)}(\lambda, x) g_{(a,b)}(\lambda, c) + h_{(c,d)}(\lambda, x) g_{(a,b)}(\lambda, d), \tag{3}$$

$$h_{(a,b)}(\lambda, x) = g_{(c,d)}(\lambda, x) h_{(a,b)}(\lambda, c) + h_{(c,d)}(\lambda, x) h_{(a,b)}(\lambda, d) \tag{4}$$

for any point x such that $x \in [c,d] \subset [a,b]$ and $\lambda \geq 0$. In particular,

$$\begin{aligned}
g_{(a,b)}(\lambda, x) &= h_{(a,d)}(\lambda, x) g_{(a,b)}(\lambda, d) + g_{(a,d)}(\lambda, x),\\
h_{(a,b)}(\lambda, x) &= h_{(a,d)}(\lambda, x) h_{(a,b)}(\lambda, d),\\
g_{(a,b)}(\lambda, x) &= g_{(c,b)}(\lambda, x) g_{(a,b)}(\lambda, c),\\
h_{(a,b)}(\lambda, x) &= g_{(c,b)}(\lambda, x) h_{(a,b)}(\lambda, c) + h_{(c,b)}(\lambda, x).
\end{aligned}$$

We will deal with processes, for which there exist the ranges (a,b) of values and points $x \in (a,b)$ such that $\mathbf{P}_x(\sigma_{(a,b)} = \infty) > 0$. Note that if for such a point x, this probability is less than one, then it may happen that the process leaves and returns to the interval (a,b) many times prior to being remained in this range forever after one such return. This is the meaning of the event (subset) $\rho(\Delta)$, where $\Delta = (a,b)$.

Semi-Markov diffusion process. A semi-Markov process is said to be diffusion in a neighborhood of the point x if there exist the functions $A(x)$ and $B(\lambda, x)$, such that

$$g_{(x-r,x+r)}(\lambda, x) = \frac{1}{2}(1 - A(x)r - B(\lambda, x)r^2) + o(r^2), \tag{5}$$

$$h_{(x-r,x+r)}(\lambda, x) = \frac{1}{2}(1 + A(x)r - B(\lambda, x)r^2) + o(r^2) \tag{6}$$

as $r \to 0$. It is assumed that $A(x)$ is continuously differentiable in a neighborhood of x and $B(\lambda, x)$ is positive, continuous in the second argument in a neighborhood of x, does not decrease, and has completely monotone partial derivative with respect to the first argument. If the condition of diffusion is satisfied for any point in the open interval (a, b) for certain admissible functions $A(x)$ and $B(\lambda, x)$ $(x \in (a, b), \lambda \geq 0)$, then the functions $g_{(a,b)}(\lambda, x)$, $h_{(a,b)}(\lambda, x)$ satisfy on this interval the differential equation [4]

$$\frac{1}{2}f'' + A(x)f' - B(\lambda, x)f = 0 \tag{7}$$

with boundary values

$$g_{(a,b)}(\lambda, a) = h_{(a,b)}(\lambda, b) = 1, \quad g_{(a,b)}(\lambda, b) = h_{(a,b)}(\lambda, a) = 0,$$

and, hence, for any $\lambda \geq 0$, form the fundamental system of solutions of this equation.

In the present paper, we are especially interested in diffusion semi-Markov processes such that for any $x \in (a, b)$ the condition $B(0, x) > 0$ is satisfied. Exactly for these coefficients and $\lambda = 0$, equation (7) has solutions $g_{(c,d)}$ and $h_{(c,d)}$, for which $g_{(c,d)}(0, x) + h_{(c,d)}(0, x) < 1$ for all $(c, d) \subset [a, b]$ and $x \in (c, d)$. Note that the latter is equivalent to the condition $\mathbf{P}_x(\sigma_{(c,d)} = \infty) > 0$.

In the theory of Markov processes, equation (7) with $B(0, x) > 0$ is used to describe a process with break at some Markov moment $\zeta < \infty$. Namely, as initial object, one takes a measurable with respect to $\mathcal{F}$ nonnegative random variable $\zeta < \infty$. Next, a new space of elementary events, which is the set of trajectories broken at time ζ, is defined. On this set, a growing family of σ-algebras of subsets (filtration) defined in a standard way with the help of cylindrical subsets is considered (see [2, p. 116], [3, p.110], etc.). It is clear that with respect to this filtration, the time ζ is a Markov moment. But in general, it is not a Markov moment on the set of stopped (not broken) trajectories, for which a natural filtration of subsets is given. The final distribution problem is meaningful for a continuous semi-Markov process stopped at a random time ζ, but not for a Markov process broken at this point. An analogue of this problem in the theory of Markov processes is the problem on distribution of process at the point "immediately before the break." It is difficult to give any real interpretation for this step other than a stop at the moment of "break." Semi-Markov approach to solving these problems consists in using semi-Markov transition functions to formulate the results and to derive the relevant formulas.

The Markov diffusion processes form a subclass of the class of semi-Markov diffusion processes. The Markov property of a semi-Markov diffusion process can be defined in terms of the coefficients of equation (7). A semi-Markov diffusion process is locally Markov on the interval (a, b) if there exists a function $B_0(x) > 0$ such that for any $x \in (a, b)$ and $\lambda \geq 0$, the representation

$$B(\lambda, x) = B_0(x) + c\lambda \quad (c > 0)$$

holds (see, e.g., [4, p.115]). In this case, the coefficients of the equation can be expressed via the Feller diffusion coefficients $b(x)$ and $a(x)$ of drift and local variance, respectively. Namely, $A(x) = b(x)/a(x)$, $B_0(x) = 1/a(x)$, and $c = 1$ [4].

Limit at infinity. Denote by

$$G_{(a,b)}(x) \equiv g_{(a,b)}(0,x) = \mathbf{P}_x(\sigma_{(a,b)} < \infty,\, X(\sigma_{(a,b)}) = a),$$

$$H_{(a,b)}(x) \equiv h_{(a,b)}(0,x) = \mathbf{P}_x(\sigma_{(a,b)} < \infty,\, X(\sigma_{(a,b)}) = b)$$

the marginal transition probabilities, where $-\infty < a < x < b < \infty$. On the interval (a,b), these functions satisfy the equation

$$\frac{1}{2} f'' + A(x) f' - B(0,x) f = 0, \tag{8}$$

(in what follows, $B(0,x) \equiv B(x)$) with boundary values

$$G_{(a,b)}(a) = H_{(a,b)}(b) = 1 \quad \text{and} \quad G_{(a,b)}(b) = H_{(a,b)}(a) = 0.$$

The solution of the final distribution problem will be given in terms of these functions.

Theorem 1. *If the continuous function $B(x)$ is positive everywhere on a closed interval $[a,b]$, then $\mathbf{P}_x(\sigma_\Delta = \infty) > 0$ for any point $x \in \Delta \equiv (a,b)$ and*

$$\mathbf{P}_x(\mathcal{C}^{\lim} \,|\, \sigma_\Delta = \infty) = 1.$$

Proof. Under the theorem condition, we consider a process such that $\mathbf{P}_x(\sigma_{(c,d)} = \infty) \equiv 1 - G_{(c,d)}(x) - H_{(c,d)}(x) > 0$ for any $(c,d) \subset [a,b]$ and $x \in (c,d)$. Let $a = a_1 < a_2 < b_1 < b_2 = b$. Set

$$\Delta_i = (a_i, b_i) \quad (i = 1, 2).$$

Further, for brevity, we introduce the following notation: if $\Delta = (a,b)$, then

$$\mu(\Delta) \equiv \{\sigma(\Delta) = \infty\}, \quad \phi(\Delta) \equiv \phi_\Delta = \{\sigma(\Delta) < \infty,\, X(\sigma(\Delta)) = a\},$$

$$\psi(\Delta) \equiv \psi_\Delta = \{\sigma(\Delta) < \infty,\, X(\sigma(\Delta)) = b\}.$$

Similarly, for a continuous process, we will distinguish the points of exit outside any other interval.

In what follows, considering relations between sets, we replace the sign $\cup$ (disjunction) by the sign $+$ if the operands of the union are disjoint, and omit the intersection sign $\cap$ (conjunction) or replace it by a point, where it does not lead to confusion.

Let $x \in \Delta_1$. This determines the order of the points of the first exit. Other location of the starting point inside the interval Δ is considered similarly.

Set $\tau_0 = 0$, $\tau_1 = \sigma_{\Delta_1}$ on $\psi(\Delta_1)$, $\tau_2 = \tau_1 \dot{+} \sigma_{\Delta_2}$ on $\theta_{\tau_1}^{-1}\phi(\Delta_2)$, $\tau_3 = \tau_2 \dot{+} \sigma_{\Delta_1}$ on $\theta_{\tau_2}^{-1}\psi(\Delta_1)$, and so on (each first exit outside the intervals Δ_1 and Δ_2 occurs inside the interval Δ),

$$\tau_{2n+1} = \tau_{2n} \dot{+} \sigma_{\Delta_1} \;\text{ on }\; \theta_{\tau_{2n}}^{-1}\psi(\Delta_1) \quad (n \geq 0),$$

$$\tau_{2n+2} = \tau_{2n+1} \dot{+} \sigma_{\Delta_2} \;\text{ on }\; \theta_{\tau_{2n+1}}^{-1}\phi(\Delta_2), \quad (n \geq 0),$$

$B_0 = \psi(\Delta_1)$, $\quad B_1 = \theta_{\tau_1}^{-1}\phi(\Delta_2)$, $\quad B_2 = \theta_{\tau_2}^{-1}\psi(\Delta_1)$, and so on,

$$B_{2n+1} = \theta_{\tau_{2n+1}}^{-1}\phi(\Delta_2), \quad B_{2n+2} = \theta_{\tau_{2n+2}}^{-1}\psi(\Delta_1) \quad (n \geq 0),$$

$A_0 = \mu(\Delta_1)$, $\quad A_1 = A_0 + B_0\theta_{\tau_1}^{-1}\mu(\Delta_2)$, $\quad A_2 = A_1 + B_0 B_1 \theta_{\tau_2}^{-1}\mu(\Delta_1)$, and so on,

$$A_{2n+1} = A_{2n} + B_0 B_1 \ldots B_{2n} \theta_{\tau_{2n+1}}^{-1}\mu(\Delta_2),$$

$$A_{2n+2} = A_{2n+1} + B_0 B_1 \ldots B_{2n+1} \theta_{\tau_{2n+2}}^{-1}\mu(\Delta_1), \quad (n \geq 0).$$

In these terms, we form equations corresponding to the successive stages of the series expansion for the set $\mu(\Delta)$. From the previous notation, it follows that

$$\mu(\Delta) = A_0 + B_0\theta_{\tau_1}^{-1}\mu(\Delta),$$

Using the representation $\mu(\Delta)$ on the right-hand side of this equation with respect to the starting point belonging to the set Δ_2, distributivity of the inverse translation map θ_t^{-1}, and associativity of the translation operator $\theta_{\tau_1}^{-1}\theta_{\tau_2}^{-1} = \theta_{\tau_1 \dot{+} \tau_2}^{-1}$, we obtain

$$\mu(\Delta) = A_0 + B_0\theta_{\tau_1}^{-1}[\mu(\Delta_2) + \phi(\Delta_2)\theta_{\sigma(\Delta_2)}^{-1}\mu(\Delta)] = A_1 + B_0B_1\theta_{\tau_2}^{-1}\mu(\Delta),$$

and so on. By induction, we get the formula

$$\mu(\Delta) = A_n + B_0B_1 \dots B_n\theta_{\tau_{n+1}}^{-1}\mu(\Delta) \quad (n \geq 0).$$

The sequence of the sets (A_n) is monotonically increasing and sequence of the residues

$$B_0B_1 \dots B_n\theta_{\tau_{n+1}}^{-1}\mu(\Delta) \quad (n \geq 0)$$

decreases monotonically. Set $\mathbf{P}_x(\sigma_\Delta = \infty) \equiv M_\Delta(x)$. Using the semi-Markov property of the process (the Markov regeneration with respect to the first exit outside open set), we obtain

$$\mathbf{P}_x(B_0B_1 \dots B_n\theta_{\tau_{n+1}}^{-1}\mu(\Delta)) = \begin{cases} H_{\Delta_1}(x)[G_{\Delta_2}(b_1)H_{\Delta_1}(a_2)]^{n/2}G_{\Delta_2}(b_1)M_\Delta(a_2) & n \text{ is even}, \\ H_{\Delta_1}(x)[G_{\Delta_2}(b_1)H_{\Delta_1}(a_2)]^{(n+1)/2}M_\Delta(b_1) & n \text{ is odd}. \end{cases}$$

The probability tends to zero as $n \to \infty$. Hence, the sequence (A_n) tends to the limit $\mu(\Delta)$ with $\mathbf{P}_x$-probability 1. The equality

$$\mu(\Delta) = \mu(\Delta_1) + \sum_{n=1}^{\infty} B_0B_1 \dots B_{n-1} \cdot \theta_{\tau_n}^{-1} \left\{ \begin{array}{ll} \mu(\Delta_1) & n \text{ is even} \\ \mu(\Delta_2) & n \text{ is odd} \end{array} \right\}$$

holds almost surely with respect to $\mathbf{P}_x$. We divide the right-hand side of this equation into two parts

$$\mu(\Delta) = \left(\mu(\Delta_1) + \sum_{n=1}^{\infty} B_0B_1 \dots B_{2n-1} \cdot \theta_{\tau_{2n}}^{-1}\mu(\Delta_1)\right) + \sum_{n=0}^{\infty} B_0B_1 \dots B_{2n} \cdot \theta_{\tau_{2n+1}}^{-1}\mu(\Delta_2).$$

Removing the components B_i from the right-hand side can not decrease it. This implies almost surely the equality

$$\mu(\Delta) = \mu(\Delta)\left(\bigcup_{n=0}^{\infty} \theta_{\tau_{2n}}^{-1}\mu(\Delta_1) \cup \bigcup_{n=0}^{\infty} \theta_{\tau_{2n+1}}^{-1}\mu(\Delta_2)\right).$$

Taking into account the configuration of the ends of overlapping intervals, we conclude that the sequence (τ_n) tends to infinity almost surely with respect to $\mathbf{P}_x$. From formula (1), it follows that

$$\mu(\Delta) = \mu(\Delta)[\rho(\Delta_1) \cup \rho(\Delta_2)], \tag{9}$$

which implies the equality

$$\mathbf{P}_x(\rho(\Delta_1) \cup \rho(\Delta_2) \,|\, \mu(\Delta)) = 1.$$

On the other hand,

$$\rho(\Delta) = \bigcup_{t \geq 0} \theta_t^{-1}\mu(\Delta) \subset \bigcup_{t \geq 0} \theta_t^{-1}(\rho(\Delta_1) \cup \rho(\Delta_2)) = \bigcup_{t \geq 0}(\rho(\Delta_1) \cup \rho(\Delta_2)) = \rho(\Delta_1) \cup \rho(\Delta_2).$$

Let us now consider a finite covering $\gamma_n = (\Delta_k)_1^n$ of the interval (a, b) by the intervals Δ_k:

$$\begin{gathered} \bigcup_{k=1}^{n} \Delta_k = (a, b), \quad \Delta_k = (a_k, b_k), \quad 1 \leq k \leq n, \quad n \geq 2, \\ a = a_1 < a_2 < b_1 < a_3 < b_2 < a_4 < b_3 < a_5 < b_4 < \dots < a_n < b_{n-1} < b_n = b. \end{gathered} \tag{10}$$

Applying this formula to subcoverings of this covering, we obtain

$$\mu(\Delta) = \mu(\Delta) \bigcup_{k=1}^{n} \rho(\Delta_k). \tag{11}$$

Clearly, in this way one can get a covering γ_n with arbitrarily small elements. Let ϵ_n be the maximum length of the elements of γ_n, and let $\epsilon_n \to 0$. Then from the above formula, it follows that

$$\mathbf{P}_x\Bigl(\bigcup_{\Delta_k \in \gamma_n}^{n} \rho(\Delta_k) \Bigm| \sigma_\Delta = \infty\Bigr) = 1.$$

Therefore, for any $N \geq 1$ we have

$$\mathbf{P}_x\Bigl(\bigcap_{n=1}^{N} \bigcup_{\Delta_k \in \gamma_n}^{n} \rho(\Delta_k) \Bigm| \sigma_\Delta = \infty\Bigr) = 1,$$

and, hence, by Lemma 1,

$$\mathbf{P}_x\Bigl(\mathcal{C}^{\lim} \,|\, \sigma_\Delta = \infty\Bigr) = 1.$$

Thus, the process has limit at infinity with conditional probability 1. □

2. Final distribution

Any process having limit at infinity is not ergodic, but for any $x \in (a, b)$ there exists a limit one-dimensional distribution (the final distribution) of this process:

$$Q_{(a,b)}(y \,|\, x) \equiv \mathbf{P}_x(\lim_{t\to\infty} X(t) < y \,|\, \sigma_{(a,b)} = \infty),$$

where $a < y \leq b$. This kernel characterizes the conditional distribution of the final point $X(\zeta)$, where the passage to the limit or the stop of the process occurred before the first exit outside the interval (a, b). The distribution $Q_{(a,b)}(y \,|\, x)$ can be found directly in terms of marginal transition probabilities.

Theorem 2. *Suppose that the conditions of Theorem* 1 *hold,* $\Delta_1 = (a, y)$*, and* $\Delta_2 = (y, b)$*.*

(1) *If* $x < y$*, then*

$$Q_\Delta(y \,|\, x) = M_\Delta^{-1}(x)\Bigl(1 - G_\Delta(x) - H_{\Delta_1}(x) + H_\Delta(x)\frac{H'_{\Delta_1}(y-)}{H'_{\Delta_2}(y+)}\Bigr).$$

(2) *If* $x > y$*, then*

$$Q_\Delta(y \,|\, x) = M_\Delta^{-1}(x)\Bigl(-G_\Delta(x) + G_{\Delta_2}(x) - G_\Delta(x)\frac{G'_{\Delta_2}(y+)}{G'_{\Delta_1}(y-)}\Bigr).$$

Proof. (1) Let $a = a_1 < a_2 < b_1 < b_2 = b$, $\Delta = (a, b)$, $\Delta_1 = (a_1, b_1)$, and $\Delta_2 = (a_2, b_2)$. Consider the process starting from the point $x \in \Delta_1$. In the proof of Theorem 1, we derived the formula

$$\mu(\Delta) = \Psi_1 + \Psi_2,$$

where

$$\Psi_1 = \mu(\Delta_1) + \sum_{n=1}^{\infty} B_0 B_1 \dots B_{2n-1} \cdot \theta_{\tau_{2n}}^{-1} \mu(\Delta_1),$$

$$\Psi_2 = \sum_{n=0}^{\infty} B_0 B_1 \dots B_{2n} \cdot \theta_{\tau_{2n+1}}^{-1} \mu(\Delta_2).$$

From the definition of Δ_1 and Δ_2, it follows that

$$\Psi_1 \subset \{\lim_{t\to\infty} X(t) \le b_1\} \quad \text{and} \quad \Psi_2 \subset \{\lim_{t\to\infty} X(t) \ge a_2\}.$$

Hence,

$$\begin{aligned}
\mathbf{P}_x(\Psi_1 \,|\, \sigma_\Delta = \infty) &\le \mathbf{P}_x\big(\lim_{t\to\infty} X(t) \le b_1 \,|\, \sigma_\Delta = \infty\big),\\
\mathbf{P}_x(\Psi_2 \,|\, \sigma_\Delta = \infty) &= 1 - \mathbf{P}_x\big(\Psi_1 \,|\, \sigma_\Delta = \infty\big) \le \mathbf{P}_x\big(\lim_{t\to\infty} X(t) \ge a_2 \,|\, \sigma_\Delta = \infty\big)\\
&= 1 - \mathbf{P}_x\big(\lim_{t\to\infty} X(t) < a_2 \,|\, \sigma_\Delta = \infty\big).
\end{aligned}$$

Consequently,

$$\mathbf{P}_x(\lim_{t\to\infty} X(t) < a_2 \,|\, \sigma_\Delta = \infty) \le \mathbf{P}_x(\Psi_1 \,|\, \sigma_\Delta = \infty) \le \mathbf{P}_x(\lim_{t\to\infty} X(t) \le b_1 \,|\, \sigma_\Delta = \infty)$$

and as $a_2 \to b_1$, the left-hand side of the double inequality tends to the right-hand side at any point of continuity of the distribution function of the random variable $\lim_{t\to\infty} X(t)$. This implies that for almost all b_1 (with respect to the Lebesgue measure),

$$\mathbf{P}_x(\lim_{t\to\infty} X(t) \le b_1 \,|\, \sigma_\Delta = \infty) = \lim_{a_2\to b_1} \mathbf{P}_x(\Psi_1 \,|\, \sigma_\Delta = \infty).$$

From the definition, it follows that

$$\begin{aligned}
\mathbf{P}_x(\Psi_1) &= \mathbf{P}_x\left(\mu(\Delta_1) + \sum_{n=1}^{\infty} B_0 B_1 \ldots B_{2n-1} \cdot \theta_{\tau_{2n}}^{-1} \mu(\Delta_1)\right)\\
&= M_{\Delta_1}(x) + \sum_{n=1}^{\infty} \mathbf{P}_x(B_0 B_1 \ldots B_{2n-1} \cdot \theta_{\tau_{2n}}^{-1} \mu(\Delta_1))\\
&= M_{\Delta_1}(x) + H_{\Delta_1}(x) G_{\Delta_2}(b_1) M_{\Delta_1}(a_2) + \sum_{n=1}^{\infty} H_{\Delta_1}(x)(G_{\Delta_2}(b_1) H_{\Delta_1}(a_2))^n G_{\Delta_2}(b_1) M_{\Delta_1}(a_2)\\
&= M_{\Delta_1}(x) + H_{\Delta_1}(x) G_{\Delta_2}(b_1) \frac{M_{\Delta_1}(a_2)}{1 - G_{\Delta_2}(b_1) H_{\Delta_1}(a_2)}.
\end{aligned}$$

Let us find the limit of this expression as $a_2 \to b_1 \equiv y$. Set $b_1 - a_2 = h$. From the boundary conditions, it follows that

$$\begin{aligned}
G_{\Delta_1}(y-h) &= G_{\Delta_1}(y-h) - G_{\Delta_1}(y) = -G'_{\Delta_1}(y-)h + o(h),\\
H_{\Delta_1}(y-h) &= H_{\Delta_1}(y-h) - H_{\Delta_1}(y) + 1 = 1 - H'_{\Delta_1}(y-)h + o(h).
\end{aligned}$$

The interval $\Delta_2 \equiv (y-h, b)$ also depends on h. We have

$$G_\Delta(y) = G_{\Delta_2}(y) G_\Delta(y-h),$$

whence

$$G_{\Delta_2}(y) = \frac{G_\Delta(y)}{G_\Delta(y-h)} \to 1 \quad (h \to 0),$$

$$\frac{1}{h}\,(G_{\Delta_2}(y) - 1) = \frac{1}{h}\,(G_\Delta(y)/G_\Delta(y-h) - 1) \to \frac{G'_\Delta(y-)}{G_\Delta(y)}.$$

From the semi-Markov property, it follows that $G_\Delta(y+h) = G_{(y,b)}(y+h) G_\Delta(y)$ and, hence,

$$\frac{1}{h}\,(G_{(y,b)}(y+h) - 1) = \frac{1}{h}\,(G_\Delta(y+h)/G_\Delta(y) - 1) \to \frac{G'_\Delta(y+)}{G_\Delta(y)}.$$

If the derivative G'_Δ is continuous at the point y, then

$$G_{\Delta_2}(y) = 1 + G'_{(y,b)}(y+)h + o(h).$$

Consequently,

$$\mathbf{P}_x(\Psi_1) \to M_{\Delta_1}(x) + H_{\Delta_1}(x)\frac{H'_{\Delta_1}(y-) + G'_{\Delta_1}(y-)}{H'_{\Delta_1}(y-) - G'_{(y,b)}(y+)}$$

for almost all y.

To simplify this expression, we use formulas proved in [6] (see Appendix). For almost all $y \in (a,b) \equiv \Delta$, we have

$$H_\Delta(y) = \frac{H'_{\Delta_2}(y+)}{H'_{\Delta_1}(y-) - G'_{\Delta_2}(y+)}, \quad G_\Delta(y) = \frac{-G'_{\Delta_1}(y-)}{H'_{\Delta_1}(y-) - G'_{\Delta_2}(y+)}, \tag{12}$$

where $\Delta_1 = (a,y)$, $\Delta_2 = (y,b)$. It follows that

$$\lim_{h\to 0} \mathbf{P}_x(\Psi_1) = 1 - G_\Delta(x) - H_{\Delta_1}(x) + H_\Delta(x)\,\frac{H'_{\Delta_1}(y-)}{H'_{\Delta_2}(y+)}$$

and since $\Psi_1 \subset \{\sigma_\Delta = \infty\}$, the proof of the first statement is complete.

(2) Let us consider the process starting at the point $x \in \Delta_2$, where $\Delta = (a,b)$, $\Delta_1 = (a_1,b_1)$, and $\Delta_2 = (a_2,b_2)$.

Let

$$\tau_1 = \sigma_{\Delta_2},\ \tau_2 = \tau_1 \dot{+} \sigma_{\Delta_1},\ \tau_3 = \tau_2 \dot{+} \sigma_{\Delta_2},\ \tau_4 = \tau_3 \dot{+} \sigma_{\Delta_1},\ \text{and so on,}$$

$$B_0 = \phi(\Delta_2),\ B_1 = \theta_{\tau_1}^{-1}\psi(\Delta_1),\ B_2 = \theta_{\tau_2}^{-1}\phi(\Delta_2),\ B_3 = \theta_{\tau_3}^{-1}\psi(\Delta_1),\ \text{and so on,}$$

$$\Psi_3 = \mu(\Delta_2) + \sum_{n=1}^{\infty} B_0 B_1 \ldots B_{2n-1} \cdot \theta_{\tau_{2n}}^{-1}\mu(\Delta_2),$$

$$\Psi_4 = \sum_{n=0}^{\infty} B_0 B_1 \ldots B_{2n} \cdot \theta_{\tau_{2n+1}}^{-1}\mu(\Delta_1).$$

Here, $\mathbf{P}_x(\Psi_3 + \Psi_4 \,|\, \sigma_\Delta = \infty) = 1$, and also

$$\Psi_3 \subset \{\lim_{t\to\infty} X(t) \ge a_2\}, \quad \Psi_4 \subset \{\lim_{t\to\infty} X(t) \le b_1\}.$$

It follows that

$$\begin{aligned}
&\mathbf{P}_x(\Psi_3 | \, \sigma_\Delta = \infty) \le \mathbf{P}_x(\lim_{t\to\infty} X(t) \ge a_2 \,|\, \sigma_\Delta = \infty),\\
&\qquad \mathbf{P}_x(\Psi_4 | \, \sigma_\Delta = \infty) = 1 - \mathbf{P}_x(\Psi_3 | \, \sigma_\Delta = \infty) \le \mathbf{P}_x(\lim_{t\to\infty} X(t) \le b_1 \,|\, \sigma_\Delta = \infty)\\
&\qquad\qquad = 1 - \mathbf{P}_x(\lim_{t\to\infty} X(t) > b_1 \,|\, \sigma_\Delta = \infty).
\end{aligned}$$

Consequently,

$$\mathbf{P}_x(\lim_{t\to\infty} X(t) > b_1 \,|\, \sigma_\Delta = \infty) \le \mathbf{P}_x(\Psi_3 | \, \sigma_\Delta = \infty) \le \mathbf{P}_x\big(\lim_{t\to\infty} X(t) \ge a_2 \,|\, \sigma_\Delta = \infty\big).$$

As $b_1 \to a_2$, the left term of this double inequality tends to the right one at any point of continuity of the required distribution function. Therefore,

$$\mathbf{P}_x\big(\lim_{t\to\infty} X(t) \ge a_2 \,|\, \sigma_\Delta = \infty\big) = \lim_{b_1 \to a_2} \mathbf{P}_x(\Psi_3 | \, \sigma_\Delta = \infty).$$

Let $y = a_2$ and $h = b_1 - a_2$. From the definition, it follows that

$$\begin{aligned}\mathbf{P}_x(\Psi_3) &= M_{\Delta_2}(x) + \sum_{n=0}^{\infty} G_{\Delta_2}(x) H_{\Delta_1}(y) (G_{\Delta_2}(y+h) H_{\Delta_1}(y))^n M_{\Delta_2}(y+h) \\ &= M_{\Delta_2}(x) + G_{\Delta_2}(x) H_{\Delta_1}(y) \frac{M_{\Delta_2}(y+h)}{1 - G_{\Delta_2}(y+h) H_{\Delta_1}(y)}.\end{aligned}$$

Here,

$$\begin{aligned}G_{\Delta_2}(y+h) &= G_{\Delta_2}(y+h) - G_{\Delta_2}(y) + 1 = 1 + G'_{\Delta_2}(y+)h + o(h), \\ H_{\Delta_2}(y+h) &= H_{\Delta_2}(y+h) - H_{\Delta_2}(y) = H'_{\Delta_2}(y+)h + o(h), \\ H_{\Delta_1}(y) &= \frac{H_\Delta(y)}{H_\Delta(y+h)} = 1 - \frac{H'_\Delta(y+)h}{H_\Delta(y)} + o(h).\end{aligned}$$

From the semi-Markov property, it follows that $H_\Delta(y-h) = H_{(a,y)}(y-h) H_\Delta(y)$ and hence,

$$\frac{1}{h}\left(H_{(a,y)}(y-h) - 1\right) \to \frac{-H'_\Delta(y-)}{H_\Delta(y)}.$$

And if the derivative H'_Δ is continuous at the point y, then

$$H_{\Delta_1}(y) = 1 - H'_{\Delta_1}(y-)h + o(h).$$

Consequently,

$$\mathbf{P}_x(\Psi_3) \to M_{\Delta_2}(x) + G_{\Delta_2}(x) \frac{-H'_{\Delta_2}(y+) - G'_{\Delta_2}(y+)}{H'_{\Delta_1}(y-) - G'_{\Delta_2}(y+)}$$

for almost all y. It follows that

$$\mathbf{P}_x(\lim_{t\to\infty} X(t) \ge y \,|\, \sigma_\Delta = \infty) = M_\Delta^{-1}(x) \left(M_{\Delta_2}(x) + G_{\Delta_2}(x) \frac{-H'_{\Delta_2}(y+) - G'_{\Delta_2}(y+)}{H'_{(a,y)}(y) - G'_{\Delta_2}(y+)} \right)$$

or

$$\begin{aligned}\mathbf{P}_x(\lim_{t\to\infty} X(t) < y \,|\, \sigma_\Delta = \infty) &= 1 - \mathbf{P}_x(\lim_{t\to\infty} X(t) \ge y \,|\, \sigma_\Delta = \infty) \\ &= M_\Delta^{-1}(x) \left(M_\Delta(x) - M_{\Delta_2}(x) + G_{\Delta_2}(x) \frac{G'_{\Delta_2}(y+) + H'_{\Delta_2}(y+)}{-G'_{\Delta_2}(y+) + H'_{(a,y)}(y-)} \right).\end{aligned}$$

After using formulas (12), we get

$$\lim_{h\to 0} \mathbf{P}_x(\Psi_3) = -G_\Delta(x) + G_{\Delta_2}(x) - G_\Delta(x) \frac{G'_{\Delta_2}(y+)}{G'_{\Delta_1}(y-)},$$

where $\Delta_1 = (a, y)$ and $\Delta_2 = (y, b)$. □

Let

$$f(x,y) \equiv \mathbf{P}_x(\lim_{t\to 0} X(t) < y,\, \sigma_\Delta = \infty) \equiv Q_\Delta(y \,|\, x) M_\Delta(x),$$

where $x, y \in \Delta \equiv (a,b)$. According to Theorem 2, this function has different analytic representations on the areas $\{x < y\}$ and $\{x > y\}$. Denote the first and the second representations by $f_1(x,y)$ and $f_2(x,y)$, respectively. We show that this function is continuous on the line

$\{x = y\}$, i.e., that the two limits $\lim_{x\to y} f_1(x, y) \equiv f_1(y, y)$ and $\lim_{x\to y} f_2(x, y) \equiv f_2(y, y)$ are equal. Indeed, if $x \to y$, then

$$f_1(x, y) = 1 - G_\Delta(x) - H_{\Delta_1}(x) + H_\Delta(x)\,\frac{H'_{\Delta_1}(y-)}{H'_{\Delta_2}(y+)} \to -G_\Delta(y) + H_\Delta(y)\,\frac{H'_{\Delta_1}(y-)}{H'_{\Delta_2}(y+)} \equiv f_1(y, y);$$

$$f_2(x, y) = -G_\Delta(x) + G_{\Delta_2}(x) - G_\Delta(x)\frac{G'_{\Delta_2}(y+)}{G'_{\Delta_1}(y-)} \to -G_\Delta(y) + 1 - G_\Delta(y)\frac{G'_{\Delta_2}(y+)}{G'_{\Delta_1}(y-)} \equiv f_2(y, y).$$

Here, using formulas (12) we conclude that for almost all y,

$$f_1(y, y) - f_2(y, y) = \frac{H'_{\Delta_1}(y-)}{H'_{\Delta_1}(y-) - G'_{\Delta_2}(y+)} - 1 + \frac{-G'_{\Delta_2}(y+)}{H'_{\Delta_1}(y-) - G'_{\Delta_2}(y+)} = 0.$$

Corollary 1. *If the conditions of Theorem 2 are satisfied, then*

$$f_1(x, y) = 1 - G_{\Delta_1}(x) - H_{\Delta_1}(x) + H_{\Delta_1}(x) f(y, y), \tag{13}$$

$$f_2(x, y) = G_{\Delta_2}(x) f(y, y), \tag{14}$$

where in each formula, one can use any of the two presentations of $f(y, y)$:

$$f(y, y) \equiv f_1(y, y) = -G_\Delta(y) + H_\Delta(y)\,\frac{H'_{\Delta_1}(y-)}{H'_{\Delta_2}(y+)},$$

$$f(y, y) \equiv f_2(y, y) = 1 - G_\Delta(y) - G_\Delta(y)\,\frac{G'_{\Delta_2}(y+)}{G'_{\Delta_1}(y-)}.$$

Proof. Set

$$r_G = \frac{G'_{\Delta_2}(y+)}{G'_{\Delta_1}(y-)}, \quad r_H = \frac{H'_{\Delta_1}(y-)}{H'_{\Delta_2}(y+)}.$$

From the two presentations of $f(y, y)$, we obtain the following formulas for r_H and r_G:

$$r_G = \frac{f(y, y) + G_\Delta(y) - 1}{-G_\Delta(y)}, \quad r_H = \frac{f(y, y) + G_\Delta(y)}{H_\Delta(y)}.$$

Substituting these expressions into the corresponding formulas, we complete the proof of the corollary. □

Note that $G'_{\Delta_1}(y-) \to -\infty$ and $G'_{\Delta_2}(y+) \to G'_\Delta(a+) > -\infty$ if $y \to a$, and also $H'_{\Delta_2}(y+) \to \infty$ and $H'_{\Delta_1}(y-) \to H'_\Delta(b-) < \infty$ if $y \to b$. It follows that

$$Q_\Delta(a \,|\, x) = 0, \quad Q_\Delta(b \,|\, x) = 1 \quad (\forall x \in \Delta).$$

Example. Let us consider equation (8) with constant coefficients $A(x) \equiv A$, $B(0, x) \equiv B$ on the interval $\Delta = (a, b)$. Solving the homogeneous differential equation with given boundary conditions, we obtain

$$G_\Delta(x) = e^{-A(x-a)}\frac{\operatorname{sh}(b - x)r}{\operatorname{sh}(b - a)r}, \quad H_\Delta(x) = e^{A(b-x)}\frac{\operatorname{sh}(x - a)r}{\operatorname{sh}(b - a)r},$$

where $r = \sqrt{A^2 + 2B}$. It follows that

$$G_\Delta(x) \sim e^{-A(x-a)}\frac{2\operatorname{sh}(b - x)r}{e^{(b-a)r}} = e^{-(x-a)(r+A)}\frac{2\operatorname{sh}(b - x)r}{e^{(b-x)r}} \to 0 \quad (a \to -\infty),$$

$$H_\Delta(x) \sim e^{A(b-x)}\frac{2\operatorname{sh}(x - a)r}{e^{(b-a)r}} = e^{-(b-x)(r-A)}\frac{2\operatorname{sh}(x - a)r}{e^{(x-a)r}} \to 0 \quad (b \to \infty).$$

Let us find the limit of the final distribution as $a \to -\infty$ and $b \to \infty$. Obviously, $\mathbf{P}_x(\mu(\Delta)) \to 1$, and if $x < y$, then

$$Q_\Delta(y\,|\,x) \sim 1 - H_{\Delta_1}(x) + H_\Delta(x)\frac{H'_{\Delta_1}(y-)}{H'_{\Delta_2}(y+)}.$$

Next,

$$H_{\Delta_1}(x) = e^{(y-x)A}\frac{\operatorname{sh}(x-a)r}{\operatorname{sh}(y-a)r} \sim e^{(y-x)A}\frac{e^{(x-a)r}}{e^{(y-a)r}} = e^{-(y-x)(r-A)} \qquad (a \to -\infty),$$
$$H'_{\Delta_1}(y-) = -A + r\,\frac{\operatorname{ch}(y-a)r}{\operatorname{sh}(y-a)r}, \quad H'_{\Delta_2}(y+) = r\,e^{(b-y)A}/\operatorname{sh}(b-y)r,$$

$$H_\Delta(x)\frac{H'_{\Delta_1}(y-)}{H'_{\Delta_2}(y+)} \sim e^{-(b-x)(r-A)}\left(\frac{2\operatorname{sh}(x-a)r}{e^{(x-a)r}}\right)\frac{-A + r\operatorname{ch}(y-a)r/\operatorname{sh}(y-a)r}{re^{(b-y)A}/\operatorname{sh}(b-y)r}$$
$$\sim \frac{1}{2}\left(1 - \frac{A}{r}\right)e^{-(y-x)(r-A)},$$

whence

$$Q_\Delta(y\,|\,x) \sim 1 - \frac{1}{2}\left(1 + \frac{A}{r}\right)e^{-(y-x)(r-A)}.$$

For the case $x > y$, we have

$$Q_\Delta(y\,|\,x) \sim G_{\Delta_2}(x) - G_\Delta(x)\frac{G'_{\Delta_2}(y+)}{G'_{\Delta_1}(y-)},$$
$$G'_{\Delta_2}(y+) = -A - r\frac{\operatorname{ch}(b-y)r}{\operatorname{sh}(b-y)r}, \quad G'_{\Delta_1}(y-) = -r\,\frac{e^{-(y-a)A}}{\operatorname{sh}(y-a)r},$$

$$G_\Delta(x)\frac{G'_{\Delta_2}(y+)}{G'_{\Delta_1}(y-)} \sim e^{-(x-a)(r+A)}\left(\frac{2\operatorname{sh}(b-x)r}{e^{(b-x)r}}\right)\frac{-A - r\operatorname{ch}(b-y)r/\operatorname{sh}(b-y)r}{-r\,e^{-(y-a)A}/\operatorname{sh}(y-a)r}$$
$$\sim \frac{1}{2}\left(1 + \frac{A}{r}\right)e^{-(x-y)(r+A)}.$$

Therefore,

$$Q_\Delta(y\,|\,x) \sim \frac{1}{2}\left(1 - \frac{A}{r}\right)e^{-(x-y)(r+A)}.$$

The density of the final measure for the one-dimensional diffusion process with characteristic operator

$$\mathcal{A}f = \frac{1}{2}\,f'' + A\,f' - B\,f$$

and the domain $(-\infty, \infty)$ is of the form

$$q(y\,|\,x) = \begin{cases} \frac{B}{r}\,e^{-(x-y)(r+A)}, & x > y, \\ \frac{B}{r}\,e^{-(y-x)(r-A)}, & x < y, \end{cases} \tag{15}$$

where $B > 0$ and $r = \sqrt{A^2 + 2B}$.

3. Distribution of time before stop

Having the formula $\mathbf{P}_x(C^{\lim} \mid \sigma_{(a,b)} = \infty) = 1$ and distribution $Q_{(a,b)}(y \mid x)$, it is natural to ask whether it is possible to replace $C^{\lim}$ in the first expression by its proper subset C^{∞} (of continuous trajectories with infinite interval of constancy, i.e., with stop), and if so, then how to allocate the time to the beginning of the stop; we denote this time by ζ. The lack of the stop can be identified with infinite time before stopping. Thus, we need to distinguish between the finite moment ζ and the infinite one, in which for every t_0 there exists $t_1 > t_0$ such that $X(t_1) \neq \lim_{t\to\infty} X(t)$.

To determine the distribution of the variable ζ, we consider the sequence of the semi-Markov step-processes generated by the process and sequence of the so-called correct coverings of the interval Δ by intervals of finite length. Let the covering $\gamma = (\Delta_k)_1^m$ be constructed by a finite number of boundary points inside the interval $\Delta = (a, b)$,

$$a \equiv c_0 < c_1 < c_2 < \cdots < c_m < c_{m+1} \equiv b \quad (m \geq 1),$$

so that

$$\Delta_1 = (c_0, c_2),\ \Delta_2 = (c_1, c_3),\ \Delta_3 = (c_2, c_4), \ldots, \Delta_m = (c_{m-1}, c_{m+1}).$$

Such a covering is said to be correct. The rank of the covering is defined to be the greatest length of the intervals belonging to the covering.

Fix a correct covering and the smallest index $N(x)$ of an element of the covering that contains the starting point $x \in \Delta$ of the process (there are at most two such indices). Let $N(x) = k$ $(1 \leq k \leq m)$. We define a mapping $L_\gamma : \mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is the Skorokhod space (the set of functions continuous from the right that have limit from the left at any point $t > 0$). Let $\xi(0) = x$. The function $L_\gamma\xi$ is a piecewise constant function taking a finite number of values. In addition to the starting point x, it takes values from the set of boundary points of the intervals of the covering γ and makes a finite number of jumps on every finite time interval. Each possible value of the step-function corresponds to a unique interval of values of the initial function ξ. In particular, for a and b, the intervals are $(-\infty, c_1)$ and (c_m, ∞), respectively. The function $L_\gamma\xi$ (if it is not equal to the constant x) makes a jump at the points (τ_n) $(n \geq 1)$ of the first exit of the function ξ or shifted functions $\theta_{\tau_{n-1}}(\xi)$ outside the corresponding intervals (if such an exit for ξ does not exist, then the function stops at the last reached point). Thus, if $\tau_n(\xi)$ is the next jump from the state c_p to state c_q, then the process $L_\gamma\xi$ keeps the constant value c_q on the interval $[\tau_n(\xi),\, (\tau_n \dot{+} \sigma_{\Delta_q}))$ (here, under Δ_q we mean infinite intervals if $q = 0$ or $q = m + 1$).

The measurable mapping L_γ induces a measure of the semi-Markov step-process $(X \circ L_\gamma)(t)$, which we call the embedded semi-Markov chain. The evolution of the semi-Markov chain until the moment $\sigma_\Delta \circ L_\gamma$ is given by the following objects: (1) the initial state x and the index $N(x)$ of the initial interval of the covering γ, to which it belongs, (2) the set of regular possible states $\{c_1, c_2, \ldots, c_m\}$ giving a sequence of the intervals $\Delta_k = (c_{k-1}, c_{k+1})$, where $c_0 = a$ and $c_{m+1} = b$, (3) the time of being in the initial state or the infinite time with probability $M_{\Delta_{N(x)}}(x)$, or the finite time defined by the functions $g_{\Delta_{N(x)}}(\lambda, x)$ and $h_{\Delta_{N(x)}}(\lambda, x)$, (4) the matrix $S(\lambda)$ of the Laplace images of the time of regular transitions from the boundary point of an interval belonging to the covering to the border of the next interval belonging to the covering, (5) the vector of stop probabilities at the points of the regular set of the states

$$M = (M_{\Delta_1}(c_1), M_{\Delta_2}(c_2), \ldots, M_{\Delta_m}(c_m))^T$$

(here, T denotes the transposition).

The element $S_{ij}(\lambda)$ of the matrix $S(\lambda)$ determines the expectation of the variable $\exp(-\lambda\sigma_{\Delta_i})$ on the set $\{\sigma_{\Delta_i} < \infty,\ X(\sigma_{\Delta_i}) = c_j\}$ provided that the process starts from the point c_i. The value $\sum_{j=1}^{m} S_{ij}(0)M_{\Delta_j}(c_j)$ is the stop probability of the semi-Markov chain exactly after the first jump with the same initial condition.

From the semi-Markov property of the original process, it follows that the nth power of the matrix S characterizes the state of the process at the time of the nth jump of the embedded semi-Markov chain, which is denoted by τ_n, on the set $\{\tau_n < \sigma_\Delta \circ L_\gamma\}$. By definition, we put

$$\tau_n = \tau_{n-1} \dot{+} \sigma_{\Delta_k} \quad (n \geq 1)$$

on the set $\{\tau_{n-1} < \infty,\, X(\tau_{n-1}) = c_k\}$.

Let us prove that

$$S_{ij}^n(\lambda) = \mathbf{E}_{c_i}(e^{-\lambda\tau_n};\, X(\tau_n) = c_j,\, \tau_n < \sigma_\Delta) \quad (i,\, j = 1, \dots m). \tag{16}$$

Indeed, we can rewrite (16) in the form

$$S_{ij}^n(\lambda) = \mathbf{E}_{c_i}(e^{-\lambda\tau_n};\, \tau_n < \infty,\, X(\tau_1), \dots, X(\tau_{n-1}) \in \Delta,\, X(\tau_n) = c_j).$$

For $n = 1$, the formula is true if we set $\tau_1 = \sigma_{\Delta_i}$. Further, from assumption (16) and the semi-Markov property, it follows that

$$\begin{aligned} S_{kp}^{n+1}(\lambda) &= \sum_{i=1}^{m} \mathbf{E}_{c_k}\Big(e^{-\lambda\tau_n};\, \tau_n < \infty,\, X(\tau_1), \dots, X(\tau_{n-1}) \in \Delta,\, X(\tau_n) = c_i\Big) \\ &\qquad \times \mathbf{E}_{c_i}\Big(e^{-\lambda\sigma_{\Delta_i}};\, \sigma_{\Delta_i} < \infty, X(\sigma_{\Delta_i}) = c_p\Big) \\ &= \sum_{i=1}^{m} \mathbf{E}_{c_k}\Big(e^{-\lambda\tau_n} e^{-\lambda\sigma_{\Delta_i}\circ\theta_{\tau_n}};\, \sigma_{\Delta_i} \circ \theta_{\tau_n} < \infty, X(\sigma_{\Delta_i}) \circ \theta_{\tau_n} = c_p, \\ &\qquad \tau_n < \infty, X(\tau_1), \dots, X(\tau_{n-1}) \in \Delta,\, X(\tau_n) = c_i\Big) \\ &= \sum_{i=1}^{m} \mathbf{E}_{c_k}\Big(e^{-\lambda\tau_{n+1}};\, \tau_{n+1} < \infty,\, X(\tau_{n+1}) = c_p, X(\tau_1), \dots, X(\tau_{n-1}) \in \Delta,\, X(\tau_n) = c_i\Big) \\ &= \mathbf{E}_{c_k}\Big(e^{-\lambda\tau_{n+1}};\, \tau_{n+1} < \infty,\, X(\tau_1), \dots, X(\tau_n) \in \Delta,\, X(\tau_{n+1}) = c_p\Big). \end{aligned}$$

Thus, formula (16) is true for all n.

For $\lambda = 0$, from formula (16) it follows that

$$S_{ij}^n(0) = \mathbf{P}_{c_i}\big(\tau_n < \infty,\, X(\tau_1), \dots, X(\tau_{n-1}) \in \Delta,\, X(\tau_n) = c_j\big).$$

Thus, $S^n(0)M$ is the vector of stop probabilities of the semi-Markov chain exactly after the nth jump and $S^n(\lambda)M$ is the vector of the expectations of the variable $e^{-\lambda\tau_n}$ for different stop points in the set $\{\tau_n < \infty,\, \tau_{n+1} = \infty\}$.

Denote by ζ_γ the stopping time of the embedded semi-Markov chain generated by the covering γ. By definition, $\zeta_\gamma = \tau_n$ on the set $\{\tau_n < \infty, \tau_{n+1} = \infty\}$. For the process controlled by equation (7), assuming $B(0, x) > 0$ we have

$$\mathbf{P}_x(\sigma_\Delta = \infty) > 0 \quad \text{and} \quad \mathbf{P}_x(\zeta_\gamma < \infty \,|\, \sigma_\Delta = \infty) = 1.$$

This probability can be expressed in terms of the functions G_{Δ_k}, H_{Δ_k}, and M_{Δ_k}. We derive this probability as a consequence of a formula that represents the expectation

$$\mathbf{E}_x(e^{-\lambda\zeta_\gamma};\, \zeta_\gamma < \infty \,|\, \sigma_\Delta = \infty)$$

expressed in terms of $g_{\Delta_k}(\lambda, \cdot)$, $h_{\Delta_k}(\lambda, \cdot)$, and M_{Δ_k}. To group terms with respect to x, we use the terminal property of the time ζ_γ, which is formulated for the Markov moments τ, namely, if $\tau < \zeta_\gamma$, then $\zeta_\gamma = \tau \dot{+} \zeta_\gamma$. For $x \in \Delta$, this implies that

$$\begin{aligned}
&\mathbf{E}_x\big(e^{-\lambda\zeta_\gamma};\, \zeta_\gamma < \infty,\, \sigma_\Delta = \infty\big) = \sum_{k=1}^{m} \mathbf{E}_x\Big(e^{-\lambda\zeta_\gamma};\, \zeta_\gamma < \infty,\, X(\zeta_\gamma) = c_k,\, \sigma_\Delta = \infty\Big) \\
&= \sum_{c_k<x} \mathbf{E}_x\Big(e^{-\lambda\sigma_{(c_k,b)}\dot{+}\zeta_\gamma};\, \sigma_{(c_k,b)}\dot{+}\zeta_\gamma < \infty,\, X(\sigma_{(c_k,b)}\dot{+}\zeta_\gamma) = c_k, \sigma_{(c_k,b)}\dot{+}\sigma_\Delta = \infty\Big) \\
&\quad + \sum_{c_k>x} \mathbf{E}_x\Big(e^{-\lambda\sigma_{(a,c_k)}\dot{+}\zeta_\gamma};\, \sigma_{(a,c_k)}\dot{+}\zeta_\gamma < \infty,\, X(\sigma_{(a,c_k)}\dot{+}\zeta_\gamma) = c_k, \sigma_{(a,c_k)}\dot{+}\sigma_\Delta = \infty\Big) \\
&= \sum_{c_k<x} \mathbf{E}_x\Big(e^{-\lambda\sigma_{(c_k,b)}};\, \sigma_{(c_k,b)} < \infty,\, X(\sigma_{(c_k,b)}) = c_k\Big)\mathbf{E}_{c_k}\Big(e^{-\lambda\zeta_\gamma};\, \zeta_\gamma < \infty,\, X(\zeta_\gamma) = c_k,\, \sigma_\Delta = \infty\Big) \\
&\quad + \sum_{c_k>x} \mathbf{E}_x\Big(e^{-\lambda\sigma_{(a,c_k)}};\, \sigma_{(a,c_k)} < \infty,\, X(\sigma_{(a,c_k)}) = c_k\Big)\mathbf{E}_{c_k}\Big(e^{-\lambda\zeta_\gamma};\, \zeta_\gamma < \infty,\, X(\zeta_\gamma) = c_k,\, \sigma_\Delta = \infty\Big) \\
&= \sum_{c_k<x} g_{(c_k,b)}(\lambda, x)\mathbf{E}_{c_k}\Big(e^{-\lambda\zeta_\gamma};\, \zeta_\gamma < \infty,\, X(\zeta_\gamma) = c_k,\, \sigma_\Delta = \infty\Big) \\
&\quad + \sum_{c_k>x} h_{(a,c_k)}(\lambda, x)\mathbf{E}_{c_k}\Big(e^{-\lambda\zeta_\gamma};\, \zeta_\gamma < \infty,\, X(\zeta_\gamma) = c_k,\, \sigma_\Delta = \infty\Big).
\end{aligned}$$

We note that the problem is reduced to finding the required expectation in the regular initial points of the process. The problem is particularly simple if $m = 3$, where both multiples not depending on x are determined by the process emerging from the point c_2.

Let us show that for any $m \ge 3$ and $1 < k < m$, finding the function $\mathbf{E}_x(e^{-\lambda\zeta_\gamma};\, \zeta_\gamma < \infty,\, X(\zeta_\gamma) = c_k,\, \sigma_\Delta = \infty)$ can be reduced to the case $m = 3$ (and to the case $m = 2$ for the "boundary" intervals of the covering, where $k = 1$ or $k = m$).

If $1 < k < m$, we pass to the case $m = 3$ by removing all the internal boundary points with numbers less than $k-1$ and greater than $k+1$. The new three-point correct covering contains three intervals $\widetilde{\Delta}_1 = (a, c_k)$, $\widetilde{\Delta}_2 = \Delta_k \equiv (c_{k-1}, c_{k+1})$, and $\widetilde{\Delta}_3 = (c_k, b)$. Obviously, the removal of the boundary points of the covering does not affect the initial process, by means of which the semi-Markov chains are constructed. On the other hand, removal of the boundary points of the covering, for example, from the right group of points, leads to the use of the two functions $g_{(c_k,b)}(\lambda, c_{(k+1)})$ and $M_{(c_k,b)}(c_{(k+1)})$, which can be expressed by the semi-Markov process law via the composition of the functions

$$g_{\Delta_{k+i}}(\lambda, c_{k+i}), \quad h_{\Delta_{k+i}}(\lambda, c_{k+i}), \quad M_{\Delta_{k+i}}(c_{k+i}),$$

$(i = 1, \ldots, m-k)$, which describe the probabilities of all trajectories of the original semi-Markov chain from the moment of the first contact with the point c_{k+1} to the first exit outside the interval (c_k, c_{m+1}) through the left border or to the stop there. For example, if $k = m-2$ and only one boundary c_m is removed, then

$$\begin{aligned}
g_{(c_k,b)}(\lambda, c_{k+1}) &= \frac{g_{\Delta_{k+1}}(\lambda, c_{k+1})}{1 - h_{\Delta_{k+1}}(\lambda, c_{k+1})g_{\Delta_{k+2}}(\lambda, c_{k+2})}, \\
M_{(c_k,b)}(c_{k+1}) &= \frac{M_{\Delta_{k+1}}(c_{k+1}) + h_{\Delta_{k+1}}(\lambda, c_{k+1})M_{\Delta_{k+2}}(c_{k+2})}{1 - h_{\Delta_{k+1}}(\lambda, c_{k+1})g_{\Delta_{k+2}}(\lambda, c_{k+2})}.
\end{aligned}$$

The formulas become more cumbersome if we remove two or more boundary points. In the present paper, we do not need these formulas.

For $m = 3$, we get the matrix

$$S(\lambda) = \begin{pmatrix} 0 & h_{\Delta_1}(\lambda, c_1) & 0 \\ g_{\Delta_2}(\lambda, c_2) & 0 & h_{\Delta_2}(\lambda, c_2) \\ 0 & g_{\Delta_3}(\lambda, c_3) & 0 \end{pmatrix}$$

of regular transition functions. To find its powers, we consider the matrix

$$A = \begin{pmatrix} 0 & a & 0 \\ b & 0 & c \\ 0 & d & 0 \end{pmatrix} \quad (a,\, b,\, c,\, d > 0).$$

The odd powers of this matrix have the same order of placement of zeros and positive elements as A has, and zeros of the even powers are located in the places of the positive values of the original matrix. Hence, we obtain the values of the elements of the second column for all powers of the matrix A,

$$A_{12}^{2(n+1)+1} = A_{12}^{2n+1}(ab + cd), \quad A_{12}^{2n+1} = a(ab + cd)^n \quad (n \geq 0),$$

and, similarly,

$$A_{22}^{2n} = (ab + cd)^n \quad (n \geq 1), \quad A_{32}^{2n+1} = d(ab + cd)^n \quad (n \geq 0).$$

It follows that

$$S_{22}^{2n}(\lambda) = (h_{\Delta_1}(\lambda,\, c_1)g_{\Delta_2}(\lambda,\, c_2) + h_{\Delta_2}(\lambda,\, c_2)g_{\Delta_3}(\lambda,\, c_3))^n.$$

Consequently,

$$\begin{aligned}
&\mathbf{E}_{c_2}\big(e^{-\lambda\zeta_\gamma};\, \zeta_\gamma < \infty,\, X(\zeta_\gamma) = c_2,\, \sigma_\Delta = \infty\big) \\
&= M_{\Delta_2}(c_2) + \sum_{n=1}^{\infty} \mathbf{E}_{c_2}\big(e^{-\lambda\tau_n};\, \tau_n < \infty,\, \tau_{n+1} = \infty,\, X(\tau_n) = c_2,\, X(\tau_1), \ldots X(\tau_{n-1}) \in \Delta\big) \\
&= M_{\Delta_2}(c_2) + \sum_{n=1}^{\infty} M_{\Delta_2}(c_2)\mathbf{E}_{c_2}\big(e^{-\lambda\tau_n};\, \tau_n < \infty,\, X(\tau_n) = c_2,\, X(\tau_1), \ldots X(\tau_{n-1}) \in \Delta\big) \\
&= M_{\Delta_2}(c_2) + \sum_{n=1}^{\infty} M_{\Delta_2}(c_2)S_{22}^{n}(\lambda) \\
&= M_{\Delta_2}(c_2) + \sum_{n=1}^{\infty} M_{\Delta_2}(c_2)(h_{\Delta_1}(\lambda, c_1)g_{\Delta_2}(\lambda, c_2)) + h_{\Delta_2}(\lambda,\, c_2)g_{\Delta_3}(\lambda, c_3))^n \\
&= \frac{M_{\Delta_2}(c_2)}{1 - h_{\Delta_1}(\lambda,\, c_1)g_{\Delta_2}(\lambda,\, c_2)) - h_{\Delta_2}(\lambda,\, c_2)g_{\Delta_3}(\lambda,\, c_3)}.
\end{aligned}$$

For $\lambda = 0$, we obtain

$$P_{c_2}\big(\zeta_\gamma < \infty,\, X(\zeta_\gamma) = c_2,\, \sigma_\Delta = \infty\big) = \frac{M_{\Delta_2}(c_2)}{1 - H_{\Delta_1}(c_1)G_{\Delta_2}(c_2)) - H_{\Delta_2}(c_2)G_{\Delta_3}(c_3)}.$$

For $m = 2$, the matrix of regular transition functions has the form

$$S(\lambda) = \begin{pmatrix} 0 & h_{\Delta_1}(\lambda, c_1) \\ g_{\Delta_2}(\lambda, c_2) & 0 \end{pmatrix}.$$

To study its powers, we consider the matrix

$$B = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \quad (a,\, b > 0).$$

The odd powers of this matrix have the same order of placement of zeros and positive elements as B has, and zeros of the even powers are located in the places of the positive values of the original matrix. Hence, we obtain the values of the elements of the powers of the matrix B,

$$B_{11}^{2n} = (ab)^n \quad (n \geq 1), \quad B_{12}^{2n+1} = a(ab)^n \quad (n \geq 0),$$
$$B_{21}^{2n+1} = b(ab)^n \quad (n \geq 0), \quad B_{22}^{2n} = (ab)^n \quad (n \geq 1)$$

and, hence,

$$S_{11}^{2n}(\lambda) = S_{22}^{2n}(\lambda) = (h_{\Delta_1}(\lambda, c_1) g_{\Delta_2}(\lambda, c_2))^n.$$

Next,

$$\begin{aligned}
&\mathbf{E}_{c_1}(e^{-\lambda\zeta_\gamma};\, \zeta_\gamma < \infty,\, X(\zeta_\gamma) = c_1,\, \sigma_\Delta = \infty) \\
&= M_{\Delta_1}(c_1) + \sum_{n=1}^{\infty} \mathbf{E}_{c_1}(e^{-\lambda\tau_n};\, \tau_n < \infty,\, \tau_{n+1} = \infty, X(\tau_n) = c_1,\, X(\tau_1), \ldots, X(\tau_{n-1}) \in \Delta) \\
&= M_{\Delta_1}(c_1) + \sum_{n=1}^{\infty} M_{\Delta_1}(c_1) \mathbf{E}_{c_1}(e^{-\lambda\tau_n};\, \tau_n < \infty, X(\tau_n) = c_1,\, X(\tau_1), \ldots, X(\tau_{n-1}) \in \Delta) \\
&= M_{\Delta_1}(c_1) + \sum_{n=1}^{\infty} M_{\Delta_1}(c_1) S_{11}^n(\lambda) \\
&= M_{\Delta_1}(c_1) + \sum_{n=1}^{\infty} M_{\Delta_1}(c_1)(h_{\Delta_1}(\lambda, c_1) g_{\Delta_2}(\lambda, c_2))^n = \frac{M_{\Delta_1}(c_1)}{1 - h_{\Delta_1}(\lambda, c_1) g_{\Delta_2}(\lambda, c_2)}
\end{aligned}$$

and also

$$\mathbf{E}_{c_2}\big(e^{-\lambda\zeta_\gamma};\, \zeta_\gamma < \infty,\, X(\zeta_\gamma) = c_2,\, \sigma_\Delta = \infty\big) = \frac{M_{\Delta_2}(c_2)}{1 - h_{\Delta_1}(\lambda, c_1) g_{\Delta_2}(\lambda, c_2)}.$$

Thus, for $x \in \Delta$,

$$\begin{aligned}
\mathbf{E}_x(e^{-\lambda\zeta_\gamma};\, \zeta_\gamma < \infty,\, \sigma_\Delta = \infty) &= \sum_{k=1}^{m} \mathbf{E}_x(e^{-\lambda\zeta_\gamma};\, \zeta_\gamma < \infty,\, X(\zeta_\gamma) = c_k,\, \sigma_\Delta = \infty) \\
&= \sum_{a<c_k<x} \mathbf{E}_x(e^{-\lambda\zeta_\gamma};\, \zeta_\gamma < \infty,\, X(\zeta_\gamma) = c_k,\, \sigma_\Delta = \infty) \\
&\quad + \sum_{x<c_k<b} \mathbf{E}_x(e^{-\lambda\zeta_\gamma};\, \zeta_\gamma < \infty,\, X(\zeta_\gamma) = c_k,\, \sigma_\Delta = \infty) \\
&= \sum_{a<c_k<x} g_{(c_k,b)}(\lambda, x) \mathbf{E}_{c_k}(e^{-\lambda\zeta_\gamma};\, \zeta_\gamma < \infty,\, X(\zeta_\gamma) = c_k,\, \sigma_\Delta = \infty) \\
&\quad + \sum_{x<c_k<b} h_{(a,c_k)}(\lambda, x) \mathbf{E}_{c_k}(e^{-\lambda\zeta_\gamma};\, \zeta_\gamma < \infty,\, X(\zeta_\gamma) = c_k,\, \sigma_\Delta = \infty) \\
&= \sum_{a<c_k<x} \frac{g_{(c_k,b)}(\lambda, x)\, M_{\Delta_k}(c_k)}{1 - h_{(a,c_k)}(\lambda, c_{k-1}) g_{\Delta_k}(\lambda, c_k) - g_{(c_k,b)}(\lambda, c_{k+1}) h_{\Delta_k}(\lambda, c_k)} \\
&\quad + \sum_{x<c_k<b} \frac{h_{(a,c_k)}(\lambda, x)\, M_{\Delta_k}(c_k)}{1 - h_{(a,c_k)}(\lambda, c_{k-1}) g_{\Delta_k}(\lambda, c_k) - g_{(c_k,b)}(\lambda, c_{k+1}) h_{\Delta_k}(\lambda, c_k)},
\end{aligned}$$

where $h_{(a,c_1)}(\lambda, c_0) = g_{(c_m,b)}(\lambda, c_{m+1}) = 0$.

Our next task is to prove the convergence $\zeta_\gamma \to \zeta$ as $\epsilon \to 0$, where ϵ is the rank of the covering γ. For this purpose, it is convenient to use some regular method for constructing a sequence of correct covering with decreasing ranks.

Given a covering $\gamma(\epsilon)$, a correct covering of rank $\epsilon/2$ is constructed by dividing each odd interval in half, followed by adding to thus formed new sequence of odd intervals the even elements of the cover, which are the intervals between the midpoints of adjacent odd intervals. We call this operation a correct subdivision. Suppose that the covering γ_2 is obtained by the correct subdivision from a right covering γ_1. Clearly, in this case, $\zeta_{\gamma_1} \leq \zeta_{\gamma_2} \leq \zeta$. Let $(\gamma_n)_1^\infty$ be a sequence of correct coverings, each of which is obtained from the previous by the correct subdivision. The stop moments (ζ_n) corresponding to these coverings form a nondecreasing sequence of measurable functions defined on the same probability space. This sequence converges almost surely.

Lemma 2. *The sequence (ζ_n) converges to ζ on the set $\{\sigma_\Delta = \infty\}$ almost surely with respect to $\mathbf{P}_x$.*

Proof. From the definition of ζ_n, it follows that for every function $\xi \in \mathcal{C}^{\lim}$,

$$\sup_{t \geq \zeta_n} |\xi(t) - \xi(\zeta_n)| \leq \epsilon_n.$$

The fact that $\zeta_n \leq \zeta_{n+1}$ for every $n \geq 1$, implies the existence of the limit $\lim_{n\to\infty} \zeta_n \equiv \zeta_\infty$ almost surely with respect to $\mathbf{P}_x$, where $\zeta_\infty \geq \zeta_n$. Obviously, $\zeta \geq \zeta_n$. This implies that for any $t > 0$, we have $|\xi(\zeta_\infty + t) - \xi(\zeta)| \leq 2\epsilon_n$. Since n and ξ are arbitrary, it follows that these two values are equal almost surely with respect to $\mathbf{P}_x$. Therefore, $\zeta_\infty = \zeta$ almost surely with respect to $\mathbf{P}_x$. □

It follows that

$$\mathbf{E}_x(e^{-\lambda\zeta}; \sigma_\Delta = \infty) = \mathbf{E}_x(e^{-\lambda\zeta} - e^{-\lambda\zeta_n}; \sigma_\Delta = \infty) + \sum_{k=1}^{m_n} \mathbf{E}_x(e^{-\lambda\zeta_n}; \zeta_n < \infty,\, X(\zeta_n) = c_{n,k},\, \sigma_\Delta = \infty),$$

where m_n and $c_{n,k}$ are the number of intervals and the internal boundary point of the covering γ_n, respectively. Consequently,

$$\begin{aligned}\mathbf{E}_x(e^{-\lambda\zeta_\gamma}; \zeta_\gamma < \infty,\, \sigma_\Delta = \infty) &= \mathbf{E}_x(e^{-\lambda\zeta} - e^{-\lambda\zeta_n}; \sigma_\Delta = \infty)\\ &+ \sum_{k:a<c_{n,k}<x} \frac{g_{(c_{n,k},b)}(\lambda, x) M_{\Delta_{n,k}}(c_{n,k})}{1 - h_{(a,c_{n,k})}(\lambda, c_{n,k-1}) g_{\Delta_{n,k}}(\lambda, c_{n,k}) - g_{(c_{n,k},b)}(\lambda, c_{n,k+1}) h_{\Delta_{n,k}}(\lambda, c_{n,k})}\\ &+ \sum_{k:x<c_{n,k}<b} \frac{h_{(a,c_{n,k})}(\lambda, x) M_{\Delta_{n,k}}(c_{n,k})}{1 - h_{(a,c_{n,k})}(\lambda, c_{n,k-1}) g_{\Delta_{n,k}}(\lambda, c_{n,k}) - g_{(c_{n,k},b)}(\lambda, c_{n,k+1}) h_{\Delta_{n,k}}(\lambda, c_{n,k})}.\end{aligned}$$

Denote by $\epsilon_{n,k}$ the length of the interval $\Delta_{n,k}$. From the definition of semi-Markov process of diffusion type, it follows that as $n \to \infty$, we have

$$\begin{aligned}M_{\Delta_{n,k}}(c_{n,k}) &= B(0, c_{n,k})\epsilon_{n,k}^2 + o(\epsilon_{n,k}^2),\\ g_{\Delta_{n,k}}(\lambda, c_{n,k}) &= \frac{1}{2}(1 - A(c_{n,k})\epsilon_{n,k}) + o(\epsilon_{n,k}),\\ h_{\Delta_{n,k}}(\lambda, c_{n,k}) &= \frac{1}{2}(1 + A(c_{n,k})\epsilon_{n,k}) + o(\epsilon_{n,k}).\end{aligned}$$

The existence of the derivatives of the functions $g_{(c_{n,k},b)}(\lambda, x)$ and $h_{(a,c_{n,k})}(\lambda, x)$ on the interval boundaries implies that

$$\begin{aligned}g_{(c_{n,k},b)}(\lambda, c_{n,k+1}) &= g_{(c_{n,k},b)}(\lambda, c_{n,k+1}) - g_{(c_{n,k},b)}(\lambda, c_{n,k}) + 1 = 1 + g'_{(c_{n,k},b)}(\lambda, c_{n,k}+)\epsilon_{n,k} + o(\epsilon_{n,k}),\\ h_{(a,c_{n,k})}(\lambda, c_{n,k-1}) &= h_{(a,c_{n,k})}(\lambda, c_{n,k-1}) - h_{(a,c_{n,k})}(\lambda, c_{n,k}) + 1 = 1 - h'_{(c_{n,k},b)}(\lambda, c_{n,k}+)\epsilon_{n,k} + o(\epsilon_{n,k}).\end{aligned}$$

It follows that

$$\mathbf{E}_x(e^{-\lambda\zeta_\gamma}; \zeta_\gamma < \infty, \sigma_\Delta = \infty) = \mathbf{E}_x(e^{-\lambda\zeta} - e^{-\lambda\zeta_n}; \sigma_\Delta = \infty)$$
$$+ \sum_{k:\, a<c_{n,k}<x} \frac{g_{(c_{n,k},b)}(\lambda, x) B(0, c_{n,k})\epsilon_{n,k}^2 + o(\epsilon_{n,k}^2)}{(1/2)(h'_{(a,c_{n,k})}(\lambda, c_{n,k}-) - g'_{(c_{n,k},b)}(\lambda, c_{n,k}+))\epsilon_{n,k} + o(\epsilon_{n,k})},$$
$$+ \sum_{k:\, x<c_{n,k}<b} \frac{h_{(a,c_{n,k})}(\lambda, x) B(0, c_{n,k})\epsilon_{n,k}^2 + o(\epsilon_{n,k}^2)}{(1/2)(h'_{(a,c_{n,k})}(\lambda, c_{n,k}-) - g'_{(c_{n,k},b)}(\lambda, c_{n,k}+))\epsilon_{n,k} + o(\epsilon_{n,k})}.$$

As $n \to \infty$, we obtain the integral representation

$$\begin{aligned} \mathbf{E}_x(e^{-\lambda\zeta}; \zeta < \infty, \sigma_\Delta = \infty) &= \int_a^x g_{(y,b)}(\lambda, x) \frac{2B(0,y)\,dy}{h'_{(a,y)}(\lambda, y-) - g'_{(y,b)}(\lambda, y+)} \\ &+ \int_x^b h_{(a,y)}(\lambda, x) \frac{2B(0,y)\,dy}{h'_{(a,y)}(\lambda, y-) - g'_{(y,b)}(\lambda, y+)}. \end{aligned} \tag{17}$$

Applying formulas (12) and semi-Markov formulas (3, 4), we get

$$\mathbf{E}_x(e^{-\lambda\zeta}; \zeta < \infty, \sigma_\Delta = \infty) = g_\Delta(\lambda, x) \int_a^x \frac{2B(0,y)\,dy}{-g'_{\Delta_1}(\lambda, y-)} + h_\Delta(\lambda, x) \int_x^b \frac{2B(0,y)\,dy}{h'_{\Delta_2}(\lambda, y+)}. \tag{18}$$

Example. Consider this formula for A and $B(\lambda, \cdot)$ to be constant. We have

$$\begin{aligned} g_\Delta(\lambda, x) &= e^{-A(x-a)} \frac{\operatorname{sh}(b-x)r}{\operatorname{sh}(b-a)r}, \\ h_\Delta(\lambda, x) &= e^{A(b-x)} \frac{\operatorname{sh}(x-a)r}{\operatorname{sh}(b-a)r}, \end{aligned} \tag{19}$$

where $r = \sqrt{A^2 + 2B(\lambda)}$, and also

$$\begin{aligned} g'_{\Delta_1}(\lambda, y-) &= e^{-A(y-a)} \frac{-r}{\operatorname{sh}(y-a)r}, \\ h'_{\Delta_2}(\lambda, y+) &= e^{A(b-y)} \frac{r}{\operatorname{sh}(b-y)r}. \end{aligned}$$

It follows that

$$\begin{aligned} &\mathbf{E}_x(e^{-\lambda\zeta}; \zeta < \infty, \sigma_\Delta = \infty) \\ &= g_\Delta(\lambda, x) \int_a^x \frac{2B(0) \operatorname{sh}(y-a)r\,dy}{re^{-A(y-a)}} + h_\Delta(\lambda, x) \int_x^b \frac{2B(0) \operatorname{sh}(b-y)r\,dy}{r\,e^{A(b-y)}} \\ &= \frac{B(0)}{B(\lambda)} g_\Delta(\lambda, x) \Big(-1 + e^{A(x-a)} \operatorname{ch}(x-a)r - \frac{A}{r} e^{A(x-a)} \operatorname{sh}(x-a)r \Big) \\ &+ \frac{B(0)}{B(\lambda)} h_\Delta(\lambda, x) \Big(-1 + e^{-A(b-x)} \operatorname{ch}(b-x)r + \frac{A}{r} e^{-A(b-x)} \operatorname{sh}(b-x)r \Big) \\ &= \frac{B(0)}{B(\lambda)} \Big(-g_\Delta(\lambda, x) + \frac{\operatorname{sh}(b-x)r}{\operatorname{sh}(b-a)r} \operatorname{ch}(x-a)r - \frac{A}{r} \frac{\operatorname{sh}(b-x)r}{\operatorname{sh}(b-a)r} \operatorname{sh}(x-a)r \end{aligned}$$

$$
- h_\Delta(\lambda, x) + \frac{\operatorname{sh}(x-a)r}{\operatorname{sh}(b-a)r}\operatorname{ch}(b-x)r + \frac{A}{r}\frac{\operatorname{sh}(x-a)r}{\operatorname{sh}(b-a)r}\operatorname{sh}(b-x)r\Bigg)
$$

$$
= \frac{B(0)}{B(\lambda)}(1 - g_\Delta(\lambda, x) - h_\Delta(\lambda, x). \quad (20)
$$

In some cases, one can find the inverse image of the Laplace transform of this expression. If the original process is developed as a diffusion Markov process until the stop, then the coefficient $B(\lambda)$ depends linearly on λ, i.e., $B(\lambda) = B + c\lambda$, where $B > 0, c > 0$. The inverse image of the relation $\frac{B}{B+c\lambda}$ is the density of exponential distribution. At the same time, the inverse images of the functions $g_\Delta(\lambda, x)$ and $h_\Delta(\lambda, x)$ are known (see [1, p. 303]):

$$
\mathcal{L}_\lambda^{-1} g_\Delta(\lambda, x) = e^{-A(x-a)-(A^2/2+B)t} \sum_{k=-\infty}^{\infty} \frac{x-a+2k(b-a)}{\sqrt{2\pi t^3}} e^{-(x-a+2k(b-a))^2/(2t)},
$$

$$
\mathcal{L}_\lambda^{-1} h_\Delta(\lambda, x) = e^{A(b-x)-(A^2/2+B)t} \sum_{k=-\infty}^{\infty} \frac{b-x+2k(b-a)}{\sqrt{2\pi t^3}} e^{-(b-x+2k(b-a))^2/(2t)}.
$$

To obtain the full inverse image, we use a known result, according to which the product of images corresponds to the convolution of the inverse images.

An unexpected result is obtained from this formula as $a \to -\infty$ and $b \to \infty$. In this case,

$$
\mathbf{E}_x(e^{-\lambda\zeta}; \zeta < \infty, \sigma_\Delta = \infty) \to \frac{B(0)}{B(\lambda)}
$$

and the limit does not depend on x. It follows that in "Markov's" case, for any starting point of the process, the time before the final stop of the process is exponentially distributed with the same parameter B/c.

Appendix. Let us prove formulas (12). Let $a < y-h < y < y+h < b$, $\Delta = (a, b)$, $\Delta_1 = (a, y)$, and $\Delta_2 = (y, b)$. Then

$$
\begin{aligned}
H_\Delta(y-h) &= H_{\Delta_1}(y-h)H_\Delta(y); \\
H_\Delta(y-h) - H_\Delta(y) &= H_\Delta(y)(H_{\Delta_1}(y-h) - H_{\Delta_1}(y)); \\
H'_\Delta(y-) &= H'_{\Delta_1}(y-)H_\Delta(y), \qquad (21) \\
H_\Delta(y+h) &= H_{\Delta_2}(y+h) + G_{\Delta_2}(y+h)H_\Delta(y); \\
H_\Delta(y+h) - H_\Delta(y) &= H_{\Delta_2}(y+h) - H_{\Delta_2}(y) + (G_{\Delta_2}(y+h) - G_{\Delta_2}(y))H_\Delta(y); \\
H'_\Delta(y+) &= H'_{\Delta_2}(y+) + G'_{\Delta_2}(y)H_\Delta(y). \qquad (22) \\
G_\Delta(y-h) &= G_{\Delta_1}(y-h) + H_{\Delta_1}(y-h)G_\Delta(y); \\
G_\Delta(y-h) - G_\Delta(y) &= G_{\Delta_1}(y-h) - G_{\Delta_1}(y) + G_\Delta(y)(H_{\Delta_1}(y-h) - H_{\Delta_1}(y)); \\
G'_\Delta(y-) &= G'_{\Delta_1}(y-) + H'_{\Delta_1}(y-)G_\Delta(y), \qquad (23) \\
G_\Delta(y+h) &= G_{\Delta_2}(y-h)G_\Delta(y); \\
G_\Delta(y+h) - G_\Delta(y) &= G_\Delta(y)(G_{\Delta_2}(y+h) - G_{\Delta_2}(y)), \\
G'_\Delta(y+) &= G'_{\Delta_2}(y+)G_\Delta(y). \qquad (24)
\end{aligned}
$$

Comparing formulas (21) and (22) at the points y, where the derivative from the right equals the derivative from the left, we obtain

$$
H_\Delta(y) = \frac{H'_{\Delta_2}(y+)}{H'_{\Delta_1}(y-) - G'_{\Delta_2}(y+)}.
$$

Comparing formulas (23) and (24) at the points y, where the derivative from the right equals the derivative from the left, we obtain

$$G_{\Delta}(y) = \frac{-G'_{\Delta_1}(y-)}{H'_{\Delta_1}(y-) - G'_{\Delta_2}(y+)}.$$

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