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PROBABILITIES OF SMALL DEVIATIONS OF THE WEIGHTED SUM OF INDEPENDENT RANDOM VARIABLES WITH COMMON DISTRIBUTION THAT DECREASES AT ZERO NOT FASTER THAN A POWER

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The paper presents estimates of small deviation probabilities of the sum $\sum_{j\ge1} \lambda_j X_j$*, where* $\{\lambda_j\}$ *are positive numbers and* $\{X_j\}$ *are i.i.d. positive random variables satisfying weak restrictions at zero and infinity. Bibliography:* 16 *titles.*

1. Introduction and results. Let $S = \sum_{j\ge1} \lambda(j)\, X_j$, where $\{X_i\}$ are independent copies of a positive random variable X with distribution function $V(x)$, and let $\lambda(\,\cdot\,)$ be a bounded positive *nonincreasing* function defined on the interval $[1,\infty]$. We assume that the series S converges almost surely, which is equivalent to the condition

$$\sum_{j\ge1} \mathbf{E}\min\left(1, \lambda(j)\, X\right) < \infty. \tag{1.1}$$

It is known that a convergence rate of the probability $\mathbf{P}(S < r)$ to zero as $r \searrow 0$ depends on the behavior of the distribution V at zero and at infinity, and the latter, in a sense, is determined by condition (1.1). So, if $\lambda(n) \asymp n^{-A}$, $A > 1$, or $\lambda(n) \asymp q^n$, $0 < q < 1$, then (1.1) is equivalent to the conditions $\mathbf{E}X^{1/A} < \infty$ and $\mathbf{E}\log(1 + X) < \infty$, respectively.

Henceforth, $u(y) \asymp v(y) \iff \log\{u(y)/v(y)\} = O(1)$, $y \to \infty$.

If the behavior of V at zero is not too restricted, then it is possible to get an optimal *logarithmic* asymptotics for $\mathbf{P}(S < r)$ (see, for instance, [1–6]).

In order to obtain more exact estimates, it is usually assumed that V decreases at zero as a power, for example, it is a regularly varying function (see [7, 8, 16]), or satisfies essentially milder condition proposed in [9] (see also [10]):

L. *There exist constants* $b \in (0,1)$, $c_1, c_2 > 1$, *and* $\varepsilon > 0$ *such that for each* $r \le \varepsilon$,

$$c_1\, V(b\,r) \le V(r) \le c_2\, V(b\,r). \tag{1.2}$$

Later on, results from [9] were improved in [11] and [12]. In the last paper, the condition **L** was replaced by a more general assumption **R** (see below), which, in particular, allows $V(r)$, in addition, to decrease at zero slower than *any* power of r (for instance, to be a slowly varying function).

For $y > 0$, set

$$\nu(y) = \frac{1}{y}\int_0^y u\, dV(u)$$

and introduce the following condition:

R. *There exist constants* $b \in (0,1)$, $c_1 > b$, $c_2 > 1$, *and* $\varepsilon > 0$ *such that for every* $r \le \varepsilon$,

$$c_1\, \nu(b\,r) \le \nu(r) \le c_2\, \nu(b\,r). \tag{1.3}$$

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It was shown in [12] that $\mathbf{L} \Longleftrightarrow \mathbf{R}\Big|_{c_1>1}$. At the same time, if

$$\nu(y) \asymp l(y), \ y \to +0, \tag{1.4}$$

where $l(y)$ is a function slowly varying at zero, then

$$V(y) \asymp \widetilde{l}(y) = \int\limits_0^y l(u)/u \, du, \ y \to +0 \tag{1.5}$$

and $\widetilde{l}(y)$ is also slowly varying at zero. From (1.4) and (1.5), it follows that (1.3) is satisfied for b small enough, whereas the left-hand side of condition (1.2) is violated. Thus, $\mathbf{R}$ is weaker than $\mathbf{L}$.

Set

$$f(u) = \mathbf{E}e^{-uX}, \quad L(u) = \sum_{n\geq 1} \log f(u\lambda(n)), \ u \geq 0. \tag{1.6}$$

We assume (see [14, (3.2)]) that the distribution V satisfies at infinity the condition

$\mathbf{F}$. $\limsup\limits_{s\to\infty} s^2\, \mathbf{P}(X \geq s)/\mathbf{E}X^2\, \mathbf{1}[X < s] < \infty$.

The condition $\mathbf{F}$ holds if X belongs to the attraction domain of a stable law and, in particular, has a finite variance. It also implies that $\mathbf{E}X^\delta < \infty$ for some positive δ (see [11]).

Theorem 1. ([12]). *Let conditions* (1.1), $\mathbf{R}$, *and* $\mathbf{F}$ *be satisfy and, in addition,*

$$\tau^2(u) = u^2\, L''(u) \to \infty, \quad u \to \infty. \tag{1.7}$$

Then

$$\mathbf{P}(S < r) \sim \frac{\exp\left(L(u) + ur\right)}{\tau(u)\sqrt{2\pi}}, \quad r \to 0, \tag{1.8}$$

where the function $u = u(r)$ is a unique solution of the equation

$$L'(u) + r = 0. \tag{1.9}$$

Remark 1. Condition (1.7) follows from $\mathbf{L}$. Also, it is valid (without any assumption concerning the behavior of V at zero) if the conditions $\mathbf{F}$ and $\lambda(j) \asymp e^{-g(j)}$ hold, where the function g is nondecreasing and $g(u)/u$ monotonically tends to zero at infinity (see [12]). On the other hand, if, for example, $\lambda(j) \asymp \exp(-j^2)$, $\mathbf{E}X^2 < \infty$, and (see (1.4)) $l(y) \asymp \log^{-2} y$, $y \to +0$, then $\liminf\limits_{u\to\infty} \tau(u) = 0$.

For rapidly decreasing weights $\lambda(j)$, the necessary condition (1.1) is significantly weaker than the condition $\mathbf{F}$ playing a key role in the proof of Theorem 1.

In [13], the asymptotics of the probability $\mathbf{P}(S < r)$ was studied in the case where the condition $\mathbf{L}$ is satisfied and the condition $\mathbf{F}$ is not. In particular, the following result was obtained there.

Theorem 2. *Let conditions* (1.1), $\mathbf{L}$, *and*

$$\sup_{m\geq 1} \lambda(m\,l)/\lambda(m) \leq A\,\lambda(l), \quad l \geq 1, \tag{1.10}$$

hold, where A is a positive constant. Then

$$\mathbf{P}(S < r) \asymp \frac{\exp\left(L(u) + ur\right)}{\tau(u)}, \quad r \to 0, \tag{1.11}$$

where the function $u = u(r)$ satisfies equation (1.9).

Note that (1.10) holds if $\lambda(n) \asymp e^{-g(\log n)}$, where the function $g(y)/y$ does not decrease for all y large enough. For example, as $\lambda(n)$, one can take $n^{-\delta}$ or $e^{-n^{\delta}}$ with $\delta > 0$.

In the present paper, the behavior of the probability $\mathbf{P}(S < r)$ is studied provided that $\mathbf{L}$ is replaced by $\mathbf{R}$ and the condition $\mathbf{F}$ is relaxed.

Let us formulate the results.

Theorem 3. *Let conditions* (1.1), $\mathbf{R}$, *and* (*see* (1.6))

$$\limsup_{s\to 0} s\,|f'''(s)|/f''(s) < \infty \tag{1.12}$$

hold. Then

$$\mathbf{P}(S < r) \asymp \frac{\exp\left(L(u) + ur\right)}{1 + \tau(u)}, \quad r \to 0, \tag{1.13}$$

where the function $u = u(r)$ *satisfies the equation* (1.9).

Remark 2. Condition (1.12) holds if the function

$$\beta(x) = \mathbf{E}\, X^2\, \mathbf{I}[X < x]$$

satisfies the condition

$$\sup_{x\ge u} x^{-k}\,\beta(x) = O\left(u^{-k}\,\beta(u)\right), \quad u \to \infty, \tag{1.14}$$

for some $k > 0$, or, more generally, if

$$\sup_{t\ge 1} \beta(tu)/\beta(t) = O\left(g(u)\right), \quad u \to \infty,$$

where $\int_1^\infty g(u)\,u\,e^{-u}\,du < \infty$.

Note that (1.14) is weaker than $\mathbf{F}$. Indeed, $\mathbf{F}$ implies

$$x^{-2}\,\beta(x) \le x^{-2}\,\mathbf{E}\min\left(x^2, X^2\right) \le u^{-2}\,\mathbf{E}\min\left(u^2, X^2\right) = O\left(u^{-2}\,\beta(u)\right), \quad u \to \infty$$

for any $x \ge u > 0$.

Remark 3. Theorem 3 is still valid if the function u in (1.13) satisfies the conditions

$$\frac{L'(u) + r}{\sqrt{L''(u)}} = O\,(1), \quad r \to 0, \tag{1.15}$$

and

$$L'(\varepsilon u) + r \le 0 \le L'(u/\varepsilon) + r, \tag{1.16}$$

where $\varepsilon \in (0, 1)$ is a constant and $r > 0$ is small enough.

In other words, the exact solution of equation (1.9) can be replaced by an approximate one.

Remark 4. Set $I_0(u) = \int_1^\infty \log f(u\lambda(t))\,dt$. If the function $\lambda(\,\cdot\,)$ satisfies the condition

$$\int_1^\infty |\log''\lambda(t)|\,dt < \infty \tag{1.17}$$

and, moreover,

$$\int_1^\infty |(s\,\log' f(s))'|\,ds < \infty, \tag{1.18}$$

then (see (1.13))

$$L(u) = I_0(u) + 0.5 \log f(u) + O(1), \quad u \to \infty. \tag{1.19}$$

We remark that (1.18) holds if the function $s \log' f(s)$ is monotone at infinity. Some other conditions can be found in [10] and [15].

The following consequence of Theorem 3 is similar to [13, Theorem 4].

Theorem 4. *Let conditions* (1.1), **R** , (1.12), (1.17), (1.18), *and* $s^2 I_0''(s) \to \infty$, $s \to \infty$, *hold. Then*

$$\mathbf{P}(S < r) \asymp \sqrt{\frac{f(u)}{u^2 I_0''(u)}}\, e^{I_0(u)+ur}, \quad r \to 0,$$

where the function $u = u(r)$ *satisfies the conditions*

$$I_0'(\varepsilon u) + r \le 0 \le I_0'(u/\varepsilon) + r \quad \text{and} \quad \Big|\frac{I_0'(u) + r}{\sqrt{I_0''(u)}}\Big| < 1/\varepsilon$$

with constant $\varepsilon \in (0,1)$ *and all positive* r *small enough.*

2. Proofs. Below, we essentially use results from [12] and [13].

Let a random variable $X(u)$, $u \ge 0$, have distribution $e^{-u\,r}\, V(dr)/f(u)$.

From [13, (3.4), (3.5)], it follows that

$$\mathbf{P}(S < r) \le \frac{e^{L(u)+ur}}{\tau(u)} \left(\frac{1}{\sqrt{2\pi}} + 6\,\tau(u)\,\mu(u)\right), \tag{2.1}$$

where

$$\mu(u) = \frac{1}{\sigma^3(u)} \sum_{j\ge1} \lambda_j^3\, \mathbf{E}|X(u\,\lambda(j)) - \mathbf{E}X(u\,\lambda(j)|^3 \tag{2.2}$$

and the function u satisfies condition (1.9). Moreover,

$$\mathbf{P}(S < r) \le e^{L(u)+ur}. \tag{2.3}$$

We have,

$$\mu(u) \le 8 \sum_{j\ge1} \lambda^3(j)\, \mathbf{E}X^3(u\lambda(j)) \le 8A/\tau(u), \tag{2.4}$$

where owing to [12, (2.5) for $h \ge 1$] and (1.12) (see also [14, Lemma 2.2]),

$$A = \sup_{\gamma>0} \gamma\, \mathbf{E}X^3(\gamma)/\mathbf{Var}X(\gamma) < \infty. \tag{2.5}$$

The upper bound in (1.13) follows from (2.1) and (2.3)–(2.5).

Next, from [13, (3.6), (3.7), and (3.10)] with regard to (2.2), (2.4), and (2.5), it follows that for any $K > 0$,

$$\mathbf{P}(S < r) \ge e^{L(u)+ur-2K} \max\left(1 - \frac{\tau^2(\bar u)}{K^2}, \frac{1}{\tau(\bar u)}\left(K\,\bar\Phi(K/\tau(\bar u)) - 48A\right)\right), \tag{2.6}$$

where u and $\bar u$ satisfy conditions (1.9) and $\bar u(\,r + L'(\bar u)) = K$, respectively, $\bar\Phi(t) = \frac{1}{t}\int_0^t d\Phi(X)$, and $\Phi(\,\cdot\,)$ is the standard normal distribution. Moreover,

$$|\tau^2(\bar u) - \tau^2(u)| \le (2 + 8\,A)\,K. \tag{2.7}$$

The lower bound in (1.13) follows from (2.6) and (2.7), provided that K is large enough. Theorem 3 is proved.

Let us consider Remark 1. The first statement follows from [9]. To check the second one, we observe that if $u\,\lambda(n+1) < 1 \le u\,\lambda(n)$, then

$$\tau^2(u) \le \biggl(\sum_{1\le j\le n} + \sum_{j>n}\biggr)(u\,\lambda(j))^2\,\mathbf{E}X^2(u\,\lambda(j)) = J_1 + J_2.$$

In our example,

$$J_1 = O\biggl(u^2\sum_{j>n}\lambda^2(j)\biggr), \quad J_2 = O\biggl(u^2\sum_{1\le j\le n}\frac{1}{|\log u\lambda(j)|}\biggr), \quad u\to\infty.$$

Let $u = u_N = e^{(N+1/2)^2}$, where $N = 1, 2, \ldots$. Then $n = N$ and, accordingly to the previous estimates, for $z_N = (N+1/2)^2$ and $N\to\infty$, we have

$$J_1 = O\biggl(\sum_{j\ge N} e^{-2(j^2 - z_N)}\biggr) = O(e^{-cN}), \quad J_2 = O\biggl(\sum_{1\le j\le N}\frac{1}{z_N - j^2}\biggr) = O(\log N/N).$$

Thus, $\lim\limits_{N\to\infty}\tau(u) = 0$. Remark 1 is completely checked.

Remark 2 is a consequence of the obvious estimates $f''(s) \ge \beta(1/s)/e$ and $s\,|f''(s)| \le \beta(1/s) + \int\limits_1^\infty \beta(x/s)\,e^{-x}(x-1)\,dx$.

Remark 3 is a consequence of the following relations (see [13, (3.10) and below]), in which the function $h = h(r)$ satisfies the equation $L'(h) + r = 0$:

$$\Bigl|\log\frac{\tau^2(u)}{\tau^2(h)}\Bigr| = \Bigl|\int\limits_h^u \frac{d\tau^2(u)}{\tau^2(u)}\Bigr| \le (2+8\,A)\Bigl|\int\limits_h^u\frac{du}{u}\Bigr| = (2+8\,A)\Bigl|\log\frac{u}{h}\Bigr|,$$

$$0 \le L(u) + u\,r - (L(h) + h\,r) = \int\limits_h^u (L'(t)+r)\,dt \le |(u-h)|\,|L'(u)+r| = (u-h)^2\,L''(\widetilde{u}),\ \widetilde{u}\in(u,h).$$

Remark 4 and Theorem 4 with regard to (2.5) are checked in the same way as similar statements from [10].

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