DOI 10.1007/s10958-016-2790-5
Journal of Mathematical Sciences, Vol. 214, No. 4, April, 2016

ON THE LITTLEWOOD–OFFORD PROBLEM

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UDC 519.2

The paper deals with studying a connection between the Littlewood–Offord problem and estimating the concentration functions of some symmetric infinitely divisible distributions. Some multivariate generalizations of Arak's results (1980) *are given. They establish a relationship of the concentration function of the sum and arithmetic structure of supports of the distributions of independent random vectors for arbitrary distributions of summands. Bibliography: 21 titles.*

Let $X, X_1, \dots, X_n$ be independent identically distributed random variables. Let $a = (a_1, \dots, a_n)$, where $a_k = (a_{k1}, \dots, a_{kd}) \in \mathbf{R}^d$, $k = 1, \dots, n$. The concentration function of a real d-dimensional random vector Y with distribution $F = \mathcal{L}(Y)$ is defined by the equality

$$Q(F, \lambda) = \sup_{x \in \mathbf{R}^d} \mathbf{P}(Y \in x + \lambda B), \quad \lambda \geq 0,$$

where $B = \{x \in \mathbf{R}^n : \|x\| \leq 1/2\}$. In the present paper, we study the behavior of the concentration function of weighted sum $S_a = \sum\limits_{k=1}^{n} X_k a_k$ depending on properties of the vectors a_k. Recently, interest in this subject has increased considerably in connection with the study of eigenvalues of random matrices (see, for instance, [9, 13, 16–20]). For a detailed history of the problem, we refer to a recent survey of Nguen and Vu in [14]. The authors of the above articles (see also [10]) called this question the Littlewood–Offord problem, since for the first time, this problem was considered in 1943 by Littlewood and Offord [12] in connection with the study of random polynomials. They considered a special case, where the coefficients $a_k \in \mathbf{R}$ are one-dimensional and X takes values ± 1 with probabilities $1/2$.

Let us introduce some notation. In the sequel, let F_a denote the distribution of the sum S_a, let E_y be the probability measure concentrated at a point y, and let G be the distribution of the random variable $\widetilde{X}$, where $\widetilde{X} = X_1 - X_2$ is a symmetrized random variable.

The symbol c will be used for absolute positive constants, which may be different even in the same formula. Below, $A \ll B$ means that $|A| \leq cB$. Also, we write $A \asymp B$ if $A \ll B$ and $B \ll A$. We write $A \ll_d B$ if $|A| \leq c(d)B$, where $c(d) > 0$ depends only on d. Similarly, $A \asymp_d B$ if $A \ll_d B$ and $B \ll_d A$. The scalar product in $\mathbf{R}^d$ is denoted by $\langle \cdot\,, \cdot \rangle$. In what follows, $\lfloor x \rfloor$ is the largest integer k such that $k < x$. For $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, we use the norms $\|x\|^2 = x_1^2 + \dots + x_n^2$ and $|x| = \max_j |x_j|$. We denote by $\widehat{F}(t)$, $t \in \mathbf{R}^d$, the characteristic function of a d-dimensional distribution F.

Products and powers of measures are understood in the convolution sense. For an infinitely divisible distribution F, the infinitely divisible distribution with characteristic function $\widehat{F}^{\lambda}(t)$, $\lambda \geq 0$, is denoted by F^{λ}.

The elementary properties of concentration functions are well studied (see, for instance, [3, 11, 15]). It is known that

$$Q(F, \mu) \ll_d (1 + \lfloor \mu/\lambda \rfloor)^d \, Q(F, \lambda) \tag{1}$$

for any $\mu, \lambda > 0$. Hence,

$$Q(F, c\lambda) \asymp_d Q(F, \lambda). \tag{2}$$

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Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 431, 2014, pp. 72–81. Original article submitted November 5, 2014.

Let us formulate a generalization of the classical Esséen inequality [7] to the multivariate case ([8], see also [11]).

Lemma 1. *Let $\tau > 0$, and let F be a d-dimensional probability distribution. Then*

$$Q(F,\tau) \ll_d \tau^d \int\limits_{|t|\le 1/\tau} |\widehat{F}(t)|\,dt. \tag{3}$$

In the general case, the concentration function $Q(F,\lambda)$ cannot be estimated from below by the right-hand side of inequality (3). However, if we assume additionally that the distribution F is symmetric and its characteristic function is nonnegative for all $t \in \mathbf{R}$, then

$$Q(F,\tau) \gg_d \tau^d \int\limits_{|t|\le 1/\tau} |\widehat{F}(t)|\,dt \tag{4}$$

and, therefore,

$$Q(F,\tau) \asymp_d \tau^d \int\limits_{|t|\le 1/\tau} |\widehat{F}(t)|\,dt \tag{5}$$

(see Lemma 1.5 for $d = 1$ in [3, Chap. II] or [1]). In the multivariate case, relations (4) and (5) were obtained by Zaitsev in [21], see also [4]. The use of relation (5) allows us to simplify the arguments in [9, 17, 20], which were applied to the Littlewood–Offord problem (see [4–6]).

The main result of the present paper is a general inequality, which reduces the estimation of concentration functions in the Littlewood–Offord problem to the estimation of the concentration functions of some infinitely divisible distributions. This result is formulated in Theorem 1 below.

For $z \in \mathbf{R}$, we introduce the distribution H_z with characteristic function

$$\widehat{H}_z(t) = \exp\Big(-\frac{1}{2}\sum_{k=1}^{n}\big(1-\cos(\langle\, t, a_k\rangle\, z)\big)\Big). \tag{6}$$

It depends on the vector a. It is clear that H_z is a symmetric infinitely divisible distribution. Therefore, its characteristic function is positive for all $t \in \mathbf{R}^d$.

Theorem 1. *Let V be an arbitrary d-dimensional Borel measure such that $\lambda = V\{\mathbf{R}\} > 0$ and $V \le G$, i.e., $V\{B\} \le G\{B\}$ for any Borel set B. Then, for any $\varepsilon > 0$ and $\tau > 0$, we have*

$$Q(F_a,\tau) \ll_d Q(H_1^{\lambda},\varepsilon)\,\exp\Bigg(d\int\limits_{z\in\mathbf{R}} \log\big(1+\lfloor\tau(\varepsilon|z|)^{-1}\rfloor\big)\,F\{dz\}\Bigg), \tag{7}$$

where $F = \lambda^{-1}V$.

Note that $\log\big(1+\lfloor\tau(\varepsilon|z|)^{-1}\rfloor\big) = 0$ for $|z| \ge \tau/\varepsilon$. Therefore, the integration in (7) is taken, in fact, over the set $\big\{z : |z| < \tau/\varepsilon\big\}$ only.

Corollary 1. *Let $\delta > 0$ and*

$$p(\delta) = G\big\{\{z : |z| \ge \delta\}\big\} > 0. \tag{8}$$

Then, for any $\varepsilon,\tau > 0$, we have

$$Q(F_a,\tau) \ll_d e^{\Delta}\, Q(H_1^{p(\delta)},\varepsilon), \tag{9}$$

where

$$\Delta = \Delta(\tau,\varepsilon,\delta) = \frac{d}{p(\delta)}\int\limits_{|z|\ge\delta} \log\big(1+\lfloor\tau(\varepsilon|z|)^{-1}\rfloor\big)\,G\{dz\}. \tag{10}$$

In particular, choosing $\delta = \tau/\varepsilon$, we get the following statement.

Corollary 2. *For any $\varepsilon, \tau > 0$, we have*

$$Q(F_a, \tau) \ll_d Q(H_1^{p(\tau/\varepsilon)}, \varepsilon). \tag{11}$$

Corollary 2 (usually for $\tau = \varepsilon$) is actually a starting point for almost all recent studies on the Littlewood–Offord problem (see, for instance, [9, 10, 13, 16, 17, 20]). More precisely, with the help of Lemma 1 or its analogs, the authors of the above-mentioned papers have obtained estimates of the form

$$Q(F_a, \tau) \ll_d \sup_{z \geq \tau/\varepsilon} \tau^d \int\limits_{|t| \leq 1/\tau} \widehat{H}_z^{p(\tau/\varepsilon)}(t)\, dt. \tag{12}$$

The fact that (1) and (5) imply

$$\begin{aligned} \sup_{z \geq \tau/\varepsilon} \tau^d \int\limits_{|t| \leq 1/\tau} \widehat{H}_z^{p(\tau/\varepsilon)}(t)\, dt &\asymp_d \sup_{z \geq \tau/\varepsilon} Q\big(H_z^{p(\tau/\varepsilon)}, \tau\big) \\ &= \sup_{z \geq \tau/\varepsilon} Q\big(H_1^{p(\tau/\varepsilon)}, \tau/z\big) = Q\big(H_1^{p(\tau/\varepsilon)}, \varepsilon\big), \end{aligned} \tag{13}$$

went apparently unnoticed by the authors of these papers, which significantly complicated further evaluation of the right-hand side of inequality (12).

Choosing V so that

$$V\{dz\} = \big(\max\big\{1, \log\big(1 + \lfloor \tau(\varepsilon|z|)^{-1} \rfloor\big)\big\}\big)^{-1} G\{dz\}, \tag{14}$$

we obtain one more corollary.

Corollary 3. *For any $\varepsilon, \tau > 0$, we have*

$$Q(F_a, \tau) \ll_d Q(H_1^{\lambda}, \varepsilon) \exp\big(d\lambda^{-1} G\big\{\{z : |z| < \tau/\varepsilon\}\big\}\big), \tag{15}$$

where

$$\lambda = \lambda(G, \tau/\varepsilon) = V\{\mathbf{R}\} = \int\limits_{z \in \mathbf{R}} \big(\max\big\{1, \log\big(1 + \lfloor \tau(\varepsilon|z|)^{-1} \rfloor\big)\big\}\big)^{-1} G\{dz\}. \tag{16}$$

In Corollaries 1–3, we choose the measure V in the form $V\{dz\} = f(z)\, G\{dz\}$ with $0 \leq f(z) \leq 1$. Choosing the optimal function f, minimizing the right-hand sides of inequalities (9), (11), and (15), is a difficult problem. It is clear that its solution depends on a and G. Certainly, it is sufficient to consider nondecreasing functions f only.

For a fixed ε, the increase of λ implies the decrease of $Q(H_1^{\lambda}, \varepsilon)$. Theorem 1 can be applied to $V = G$. Then $\lambda = 1$. This is the maximal possible value of λ. However, in this case, the integral on the right-hand side of (7) can be infinite. In particular, it diverges if the distribution G has a nonzero atom at zero. This atom must be excluded in constructing the measure V in any case if we expect to get a meaningful bound for $Q(F_a, \tau)$. For a fixed measure V, the decrease of ε implies the decrease of $Q(H_1^{\lambda}, \varepsilon)$, but the integral on the right-hand side of inequality (7) increases.

In Corollary 3, we used the measure V defined in (14) so that the integral on the right-hand side of inequality (7) would converge always no matter what the measure G is.

The proof of Theorem 1 is based on elementary properties of concentration functions, which will be given below. Note that H_1^{λ} is an infinitely divisible distribution with Lévy spectral measure $M_\lambda = \frac{\lambda}{4}\, M^*$, where $M^* = \sum\limits_{k=1}^{n} \big(E_{a_k} + E_{-a_k}\big)$. It is clear that the assertions of Theorem 1 and Corollaries 1–3 reduce the Littlewood–Offord problem to the study of the measure M^* uniquely corresponding to the vector a. In fact, almost all the results obtained in solving this problem are formulated in terms of the coefficients a_j or, equivalently, in terms of the properties of the measure M^*. Sometimes, this leads to the loss of information on the distribution of the random variable X, which may help in obtaining more precise estimates. In particular, if $\mathcal{L}(X)$ is the standard normal distribution, then F_a is a Gaussian distribution with zero mean and covariance operator, which can easily be computed. Thus, there are situations, in which it is possible to obtain estimates for $Q(F_a,\tau)$ that do not follow from the results formulated in terms of the measure M^*.

Note that using the results of Arak [1,2] (see also [3]), one could derive from Theorem 1 estimates similar to estimates of the concentration functions in the Littlewood–Offord problem that were obtained in a recent paper [13] (see also [14]). A detailed discussion of this fact is presented in a joint paper of the authors and Friedrich Götze, which is preparing for the publication. In the same paper, the proof of multidimensional analogs of some results of Arak in [1] is given. In Theorems 2 and 3 below, we cite these results without proofs. They show a relationship between the order of smallness of the concentration function of the sum and arithmetic structure of the supports of distributions of independent random vectors for *arbitrary* distributions of summands. This is different from the results in [9, 13, 16–20], where a similar relationship was found in a special case of summands with distributions arising in the Littlewood–Offord problem.

We need some notation. Let $\mathbf{Z}_+$ be the set of nonnegative integers. For any $r \in \mathbf{Z}_+$ and $u = (u_1, \dots, u_r) \in (\mathbf{R}^d)^r$ with $u_j \in \mathbf{R}^d$ for $j = 1, \dots, r$, we introduce the set

$$K_1(u) = \Big\{ \sum_{j=1}^{r} n_j u_j : n_j \in \{-1, 0, 1\} \text{ for } j = 1, \dots, r \Big\}. \tag{17}$$

Denote by $[B]_\tau$ the closed τ-neighborhood of a set B in the sense of the norm $|\cdot|$.

Theorem 2. *Let $\tau \ge 0$, and let F_j be a d-dimensional probability distribution, $j = 1, \dots, n$. Set $\rho = Q\Big(\prod_{j=1}^{n} F_j, \tau\Big)$. Then there exist an integer $r \in \mathbf{Z}_+$ and vectors $u_1, \dots, u_r,\ x_1, \dots, x_r \in \mathbf{R}^d$ such that*

$$r \ll_d |\log \rho| + 1 \tag{18}$$

and

$$\sum_{j=1}^{n} F_j\{\mathbf{R}^d \setminus [K_1(u)]_\tau + x_j\} \ll_d \big(|\log \rho| + 1\big)^3, \tag{19}$$

where $u = (u_1, \dots, u_r) \in (\mathbf{R}^d)^r$ and the set $K_1(u)$ is defined in (17).

Theorem 3. *Let D be a d-dimensional infinitely divisible distribution with characteristic function of the form $\exp\big\{\alpha(\widehat{M}(t) - 1)\big\}$, $t \in \mathbf{R}^d$, where $\alpha > 0$ and M is a probability distribution. Let $\tau \ge 0$ and $\gamma = Q(D, \tau)$. Then there exist an integer $r \in \mathbf{Z}_+$ and vectors $u_1, \dots, u_r \in \mathbf{R}^d$ such that*

$$r \ll_d |\log \gamma| + 1 \tag{20}$$

and

$$\alpha\, M\{\mathbf{R}^d \setminus [K_1(u)]_\tau\} \ll_d \big(|\log \gamma| + 1\big)^3, \tag{21}$$

where $u = (u_1, \dots, u_r) \in (\mathbf{R}^d)^r$.

Proof of Theorem 1. Let us show that for arbitrary probability distribution F and $\lambda, T > 0$,

$$\log \int\limits_{|t|\le T} \exp\Big(-\frac{1}{2}\sum_{k=1}^{n}\int\limits_{z\in\mathbf{R}} \big(1-\cos(\langle t, a_k\rangle z)\big)\,\lambda\, F\{dz\}\Big)\,dt$$

$$\le \int\limits_{z\in\mathbf{R}} \Big(\log \int\limits_{|t|\le T} \exp\Big(-\frac{\lambda}{2}\sum_{k=1}^{n}\big(1-\cos(\langle t, a_k\rangle z)\big)\Big)\,dt\Big)\,F\{dz\} \tag{22}$$

$$= \int\limits_{z\in\mathbf{R}} \Big(\log \int\limits_{|t|\le T} \widehat{H}_z^{\lambda}(t)\,dt\Big)\,F\{dz\}.$$

It suffices to prove (22) for the discrete distribution $F = \sum\limits_{j=1}^{\infty} p_j E_{z_j}$, where $0 \le p_j \le 1$, $z_j \in \mathbf{R}$, and $\sum\limits_{j=1}^{\infty} p_j = 1$. Applying in this case the Hölder inequality, we obtain

$$\int\limits_{|t|\le T} \exp\Big(-\frac{1}{2}\sum_{k=1}^{n}\int\limits_{z\in\mathbf{R}} \big(1-\cos(\langle t, a_k\rangle\, z)\big)\,\lambda\, F\{dz\}\Big)\,dt$$

$$= \int\limits_{|t|\le T} \exp\Big(-\frac{\lambda}{2}\sum_{j=1}^{\infty} p_j \sum_{k=1}^{n}\big(1-\cos(\langle t, a_k\rangle\, z_j)\big)\Big)\,dt \tag{23}$$

$$\le \prod_{j=1}^{\infty}\Big(\int\limits_{|t|\le T} \exp\Big(-\frac{\lambda}{2}\sum_{k=1}^{n}\big(1-\cos(\langle t, a_k\rangle\, z_j)\big)\Big)\,dt\Big)^{p_j}.$$

Taking the logarithms of the left- and right-hand sides of (23), we get (22). In the general case, we can approximate the distribution F by discrete distributions in the sense of weak convergence and to pass to the limit. We use the fact that the weak convergence of probability distributions is equivalent to the convergence of characteristic functions, which is uniform on the bounded sets. Moreover, the weak convergence of symmetric infinitely divisible distributions is equivalent to the weak convergence of the corresponding spectral measures. Note also that the integrals $\int\limits_{|t|\le T}$ can be replaced in (22) by the integrals $\int\limits_{t\in B}$ over an arbitrary Borel set B.

For the characteristic function $\widehat{W}(t)$ of a random vector Y, we have

$$|\widehat{W}(t)|^2 = \mathbf{E}\exp(i\langle t, \widetilde{Y}\rangle) = \mathbf{E}\cos(\langle t, \widetilde{Y}\rangle),$$

where $\widetilde{Y}$ is the corresponding symmetrized random vector. Therefore,

$$|\widehat{W}(t)| \le \exp\Big(-\frac{1}{2}\big(1-|\widehat{W}(t)|^2\big)\Big) = \exp\Big(-\frac{1}{2}\mathbf{E}\,\big(1-\cos(\langle t, \widetilde{Y}\rangle)\big)\Big). \tag{24}$$

According to Theorem 1 and formulas $V = \lambda F \le G$, (22), and (24), we have

$$
\begin{aligned}
Q(F_a,\tau) &\ll_d \tau^d \int\limits_{\tau|t|\le 1} |\widehat{F}_a(t)|\,dt \\
&\ll_d \tau^d \int\limits_{\tau|t|\le 1} \exp\Big(-\frac12\sum_{k=1}^n \mathbf{E}\,\big(1-\cos(\langle\, t, a_k\rangle\,\widetilde{X})\big)\Big)\,dt \\
&= \tau^d \int\limits_{\tau|t|\le 1} \exp\Big(-\frac12\sum_{k=1}^n \int\limits_{z\in\mathbf{R}} \big(1-\cos(\langle\, t, a_k\rangle\, z)\big)\,G\{dz\}\Big)\,dt \qquad (25)\\
&\le \tau^d \int\limits_{\tau|t|\le 1} \exp\Big(-\frac12\sum_{k=1}^n \int\limits_{z\in\mathbf{R}} \big(1-\cos(\langle\, t, a_k\rangle\, z)\big)\,\lambda\, F\{dz\}\Big)\,dt \\
&\le \exp\Big(\int\limits_{z\in\mathbf{R}} \log\Big(\tau^d \int\limits_{\tau|t|\le 1} \widehat{H}^{\lambda}_z(t)\,dt\Big)\,F\{dz\}\Big).
\end{aligned}
$$

Using (1) and (5), we get

$$
\tau^d \int\limits_{\tau|t|\le 1} \widehat{H}^{\lambda}_z(t)\,dt \asymp_d Q(H^{\lambda}_z,\tau) = Q\big(H^{\lambda}_1,\tau|z|^{-1}\big) \le \big(1+\lfloor \tau(\varepsilon|z|)^{-1}\rfloor\big)^d\, Q(H^{\lambda}_1,\varepsilon). \qquad (26)
$$

Substituting this estimate into (25), we obtain (7). □

Acknowledgments. The research of the first and the second authors are supported by the RFBR grants No. 13-01-00256 and NSh-2504.2014.1. The research of the first author was supported by the Chebyshev Laboratory of the St.Petersburg State University under the grant of the Russian Federation Government 11.G34.31.0026 and by the grant of the St.Petersburg State University No. 6.38.672.2013. The research of the second author was supported by the Program of Fundamental Researches of the Russian Academy of Sciences "Modern Problems of Fundamental Mathematics."

Translated by A. Yu. Zaitsev.

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