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MERGING OF ASYMPTOTICS IN THE ILLUMINATED PART OF THE FOCK DOMAIN

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Investigation of the shortwave diffraction by elongated bodies of revolution requires a detailed consideration of matching of local asymptotics in the illuminated part of the Fock domain. In the paper, that task is solved by means of a straightforward construction of the reflected wave with the help of the ray method. The main problem on the way, which was judged by V. A. Fock as a rather complicated one, is the calculation of the eikonal and the geometric spreading in curvilinear coordinates used in the boundary layer method in the vicinity of the light-shadow zone. Bibliography: 9 titles.

The application of the method of local asymptotic expansions requires a matching procedure for the asymptotics constructed in different domains of the original problems. For diffraction and wave propagation problems, this matching procedure has been considered in various situations in papers and monographs [1–4].

V. A. Fock's paper concerning the wave field close to the surface of a conductive body (see [1, 2]) is among the first to propose and develop the parabolic equation method, known today in the diffraction theory as the Leontovich–Fock method. However, that article contains several assumptions of a physical nature, which restrict the applicability of its results to small neighborhoods of incidence of a limiting ray, i.e., a ray of the incident wave that touches the surface of the scatterer at a point on the light-shadow boundary. Thus the 3D problem was regarded initially in a neighborhood of the point of incidence $x = y = z = 0$ of a certain limiting ray, where the equation of the scatterer surface is described by the first quadratic form

$$z = -1/2(ax^2 + 2bxy + cy^2). \tag{0.1}$$

Moreover, z is directed along a normal to the surface, x varies along the limiting ray into the shaded zone, and the axis y is orthogonal to the plane of incidence x, z. The parabolic equation for the required function V of two variables ξ, ζ is derived on the plane of incidence x, z, and ξ and ζ are internal (prolate) variables of the corresponding boundary layer; their direction coincides with x and z, respectively. Further, it is assumed that the dependence of the desired solution of the original 3D problem on x, y, z is described by the formulas

$$\xi = m(ax + by),$$

$$\zeta = 2am[z + 1/2(ax^2 + 2bxy + cy^2)], \tag{0.2}$$

where

$$m = \left(\frac{k}{2a}\right)^{1/3} = \left(\frac{kR_0}{2}\right)^{1/3}$$

is a large dimensionless Fock parameter, and $R_0 = 1/a$ is the radius of surface curvature in a normal section along the limiting ray of the incident wave.

It seems plausible from the physical point of view that the constructed solution describes the wave field approximately in a certain neighborhood of the plane of incidence x, z as well, provided that the two main radii of curvature of the scatterer surface differ slightly. However

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it remains unclear how that can be confirmed mathematically. Moreover, in [1, 2] the main oscillating multiplier is taken in the form $\exp(ikx)$, i.e., along a tangent to the body surface. But the concept of short-wave approximation determines instead that the wave field "slides" along the geodesics of the body surface. This is especially essential in the shadowed part of the Fock domain, where creeping waves appear.

For these reasons the parabolic equation method was modernized in the problems of short-wave diffraction by prolate 3D bodies (for this, see [5, 6] and the bibliography therein). Considering the diffraction by strongly elongated bodies, the authors had to investigate in detail the problem of merging of local asymptotics in the illuminated part of the Fock domain by means of a straightforward construction of the reflected wave with the use of the ray method. The main difficulty of this approach, which V. A. Fock judged as a "rather complicated" one, lies in the calculation of the eikonal and the geometric spreading (of the amplitude) of the reflected wave in the curvilinear coordinates employed in the Fock domain.

Paper [7] has dealt with the problem of matching of ray asymptotics in the illuminated part of the Fock domain for the problem of diffraction on a smooth convex contour on the basis of properties of homogeneous functions. In paper [8], a similar problem was solved in a much more general than in [7] situation of impedance boundary conditions on a contour also on the basis of homogeneous functions.

Our approach rests solely upon ray considerations and differs from the method employed in papers [7, 8]. Concerning this approach, see [9] where a new boundary layer in the vicinity of the light-shadow zone for a strongly elongated scatterer was proposed. This layer is defined by the conditions $k^{1/7}s = O(1), k^{4/7}n = O(1)$ and appears more extended along the coordinate s compared to the Fock domain. In the distant part of the illuminated zone of this layer, the Fock's parabolic equation arises and therefore a problem of merging of local asymptotics, similar to the one under discussion, appears.

The aim of the present article is to give the solution of the indicated problem in detail. Let us emphasize that the formulas that we derived coincide with the results of V. A. Fock [1, 2], which were obtained by the asymptotic investigation of the integral formulas for the solution of the parabolic equation.

1. The ray method for calculating the reflected wave in the illuminated part of the light-shadow boundary of the Fock domain

We consider the short-wave diffraction of the plane wave on a body of revolution Σ, generated by the rotation of a plane convex curve $x = f(z)$ around the axis z of the Cartesian coordinates x, y, z:

$$r = f(z), \quad r = \sqrt{x^2 + y^2}, \quad x = r\cos\varphi, \quad y = r\sin\varphi.$$

The section of the surface $\partial\Sigma$ by the plane $z = 0$ represents the equator, and it coincides with the light-shadow boundary of the incident plane wave $U^{\text{inc}} = e^{ikz}, k \gg 1$, where k is the wave number.

Along with the Cartesian coordinates, we introduce curvilinear coordinates (s, φ, n), where s is the arc length along the meridians (geodesic lines) on the body surface $\partial\Sigma$, φ represents the azimuthal angle, the length along the external normal of the body of revolution is denoted by n. The arc length s is connected with the coordinate z by the relation

$$s = \int\limits_0^z \sqrt{1 + (f'(z))^2}\, dz, \tag{1.1}$$

i.e., it is counted from the equator and in the illuminated part of the body of revolution $s \leq 0$. The inverse function $z = z(s)$ can be obtained by the inversion of relation (1.1).

In order to obtain the connection between the Cartesian coordinates and the coordinates (s, φ, n), it is convenient to use the following vector equation. We denote the radius vector of a point M in the Cartesian coordinates by $\mathbf{R}_M$, the radius vector of a point on the surface $\partial\Sigma$ by $\mathbf{R}_\Sigma(s, \varphi)$, and let $\mathbf{n} = \mathbf{n}(s, \varphi)$ be the unit vector of the external normal to $\partial\Sigma$. Then, the following vector relation holds:

$$\mathbf{R}_M = \mathbf{R}_\Sigma(s, \varphi) + n\,\mathbf{n}(s, \varphi), \tag{1.2}$$

where the vectors involved have the form

$$\begin{aligned} \mathbf{R}_M &= x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z, \\ \mathbf{R}_\Sigma &= f(z(s))\big(\cos\varphi\,\mathbf{e}_x + \sin\varphi\,\mathbf{e}_y\big) + z(s)\,\mathbf{e}_z, \\ \mathbf{n} &= \frac{\cos\varphi\,\mathbf{e}_x + \sin\varphi\,\mathbf{e}_y - f'(z(s))\,\mathbf{e}_z}{\sqrt{1 + (f'(z(s)))^2}}. \end{aligned} \tag{1.3}$$

Here $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ are the unit vectors of the Cartesian coordinates.

Differentiating relation (1.2), we get the following formula for the square of the length element:

$$dS^2 = (d\mathbf{R}_M, d\mathbf{R}_M) = h_s^2 ds^2 + dn^2 + h_\varphi^2 d\varphi^2,$$

where the Lamé coefficients h_s and h_φ have the form

$$\begin{aligned} h_s &= 1 - n\frac{f''(z(s))}{[1 + (f'(z(s)))^2]^{3/2}} = 1 + n\,K(s), \\ h_\varphi &= f(z(s)) + \frac{n}{\sqrt{1 + (f'(z(s)))^2}} = f(z(s)) + n\frac{dz}{ds}. \end{aligned} \tag{1.4}$$

In formulas (1.1)-(1.4), f' and f'' mean the first- and second-order derivatives of the function $f(z)$ with respect to the argument z, which in turn is a function of the arc length s and is obtained by the inversion of the function (1.1), and $K(s)$ is the curvature of the geodesic line (meridian) at the point s.

Further, explicit formulas for the coordinates (x, y, z) in terms of the coordinates (s, φ, n) are obtained from relation (1.2) by the scalar multiplication of both parts by $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$. By the symmetry of the problem under consideration relative to the rotation through the angle φ, it is sufficient to get connection formulas only in one section by the plane $\varphi = \text{const}$. For definiteness, we assume that $\varphi = 0$, then relations (1.2)–(1.4) imply that

$$x = f(z(s)) + n\,\frac{dz(s)}{ds}; \quad z = z(s) - n\,f'(z(s))\frac{dz(s)}{ds}. \tag{1.5}$$

The construction of the reflected wave by means of the ray method presents no difficulties. Indeed, let us consider a set of rays of the incident wave that form a circular cylinder with radius r_* and axis $x = y = 0$. For $r_* < f(0)$, this cylinder intersects the body of revolution in a parallel, say, $s = s_*$, on the illuminated part of the surface $\partial\Sigma$, where $s_* < 0$. Each ray reflects from the surface according to the Snell's law at a point of this parallel, and, as a result, a circular cone of reflected rays arises. For $s_* = f(0)$, the corresponding cylinder of incident rays obviously touches the body surface at equator points, forming a light-shadow boundary on $\partial\Sigma$ and then a cylinder of limiting rays. The set of these rays for every fixed r_* and s_*, respectively, admits a simple description in Cartesian coordinates, especially as, by the symmetry, it is sufficient to examine this process only in the half-plane $\varphi = 0$. Denote the Cartesian coordinates of the point of incidence (reflection) of the ingoing plane wave by $z_* = z(s_*)$ and $x_* = f(z_*)$. Further, let $\mathbf{p} = p_x\,\mathbf{e}_x + p_z\,\mathbf{e}_z$ be the unit vector directed along the reflected ray. Then the equation for the reflected ray can be written in the form

$$\frac{x - x_*}{p_x(x_*, z_*)} = \frac{z - z_*}{p_z(x_*, z_*)} = t, \tag{1.6}$$

where the parameter t has obviously the meaning of the length along the reflected ray from the reflection point s_* to the observation point, because $p_x^2 + p_z^2 = 1$ for every z_*. The components p_x and p_z of the unit vector $\mathbf{p}$ can be expressed via the angle of inclination γ of the tangent to the curve $x = f(z)$ at the point of reflection.

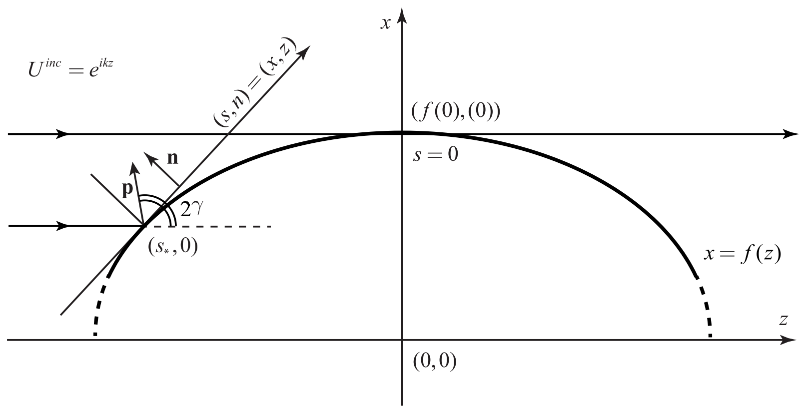

Fig. 1. Connection between the coordinates and construction of reflected rays.

It is easy to see from Fig. 1 that the angle between the vector $\mathbf{p}$ and the axis z equals 2γ. Thus, for p_x and p_z we get the expressions

$$p_z = \cos 2\gamma, \quad p_x = \sin 2\gamma, \tag{1.7}$$

where the right-hand sides of the equations are calculated at the ray reflection point $z_* = z(s_*)$. Since $\tan\gamma = f'(z)$, the components p_x and p_z are expressed in terms of the derivative $f'(z)$.

Formulas (1.6) describe the set of reflected rays in ray coordinates, which are $z_* = z(s_*)$ and t. Indeed, by setting a value of z_*, we fix a ray, and the value of t determines a point on it. For $z_* = 0$, we get the limiting ray that touches $\partial\Sigma$ at the point $z = 0$, $x = f(0)$.

For a convex curve $x = f(z)$, this ray set generates a regular ray field, because the rays fill some part of the half-plane $x > 0$ outside the body of revolution without intersection (i.e., through every point inside this domain one and only one reflected ray passes). This means that inside the given domain, a one-to-one correspondence exists between the ray coordinates s_*, t, the Cartesian coordinates x, z, and the curvilinear coordinates s, n. In turn, this enables one to construct, at least in principle, a ray formula for the reflected wave, namely, the eikonal τ and the geometric spreading J as functions of the Cartesian coordinates x, z or the curvilinear coordinates s, n. This can be carried out according to the following plan.

For J we have the formula

$$J = \left|\frac{D(x,z)}{D(t,s_*)}\right| = \left|\frac{D(x,z)}{D(s,n)}\right|\left|\frac{D(s,n)}{D(t,s_*)}\right|. \tag{1.8}$$

With the help of the ray equation (1.6), the functional determinants on the right-hand side of (1.8) are calculated elementarily in the ray coordinates s_*, t. After that the ray coordinates should be expressed through the Cartesian x, z or curvilinear s, n coordinates. This last-mentioned procedure is most difficult.

To construct the eikonal τ, we consider the differential form $p_x\,dx + p_z\,dz$, where p_x and p_z are the components of the unit vector $\mathbf{p}$ directed along the reflected ray. Since in the problem under consideration the rays are straight lines, it follows that p_x and p_z are constant on the reflected ray and thus depend only on one ray parameter s_*. At the first step, this parameter s_* should be expressed as a function of the Cartesian x, z or the curvilinear s, n coordinates in the domain of regularity of the field of reflected rays. (Note that this domain is called the Lagrange manifold.) Thus, at the first step we obtain an I-form $\omega^I = p_x(x,z)dx + p_z(x,z)dz$ on this Lagrange manifold, and this form ω^I is closed, i.e., a curvilinear integral of it does not depend on the path of integration. Moreover, it is exact, because the fundamental group of this Lagrange manifold is trivial. In other words, that means the following.

The differential form ω^I is the total differential of a function τ, defined on part of the plane where the field of reflected rays is regular, because it contains no closed paths that cannot be deformed into one point by means of continuous deformation. Thus, $d\tau = p_x\,dx + p_z\,dz$ and to reconstruct τ from this equality no quantization conditions are required. The function τ thereby is restored by calculation of the curvilinear integral

$$\tau = \int p_x\,dx + p_z\,dz, \tag{1.9}$$

which, as has already been mentioned, is independent of the path of integration.

We mention in conclusion that τ automatically satisfies the eikonal equation $(\nabla\tau)^2 = 1$, because from (1.9) we derive that $\frac{d\tau}{dx} = p_x$, $\frac{d\tau}{dz} = p_z$, and $p_x^2 + p_z^2 = 1$ according to the definition of p_x, p_z.

We note once again that the passage from the ray coordinates s_*, t to the Cartesian x, z or curvilinear s, n ones presents the main difficulties on the way of realization of the plan described above.

To merge the ray asymptotics of the reflected wave with a solution of the parabolic equation, we need to consider only the illuminated part of the Fock domain in a neighborhood of the light-shadow boundary. This neighborhood is determined by conditions $s \simeq O(k^{-1/3})$, $n \simeq O(k^{-2/3})$, where the wave number k is regarded as a large parameter, $k \gg 1$. For this reason the variables s, n prove to be small and all the functions $f(z)$, $z(s)$, $K(s)$, and so on, occurring in formulas (1.1)–(1.8), can be expanded in Taylor series in powers of s and n. At that, it is necessary to preserve first several terms of the series, because it is sufficient to construct the ray formula for the reflected wave only approximately.

Introduce several auxiliary formulas needed in the sequel:

$$\begin{gathered}
s = z + \frac{1}{6}K^2(0)z^3 + O(z^4),\\
z = s - \frac{1}{6}K^2(0)s^3 + O(s^4),\\
f'(z(s)) = -K(0)s - K'(0)\frac{s^2}{2!} + O(s^3),
\end{gathered} \tag{1.10}$$

where $K(0)$ is the curvature of the geodesic line at the point of tangency $s = 0$ of the limiting ray, and $K'(0)$ is its derivative at the same point. The first of the formulas (1.10) readily follows from relation (1.1), and the second is the result of the inversion of the series for $s(z)$.

Henceforth we shall use only curvilinear coordinates s, n; for this reason, at the first step it is necessary to find explicit formulas for their connection with the Cartesian coordinates.

From relations (1.2)-(1.4) and (1.10) we get

$$\begin{aligned}
x &= f(0) - \frac{1}{2}K(0)s^2 + n + \dots,\\
z &= s - \frac{1}{3!}K^2(0)s^3 + K(0)n\,s + \dots.
\end{aligned} \tag{1.11}$$

We need to consider the reflected rays that get into the illuminated part of the Fock domain. This means that the reflection points s_* should be of order $O(k^{-1/3})$. Expanding the right-hand sides of relations (1.7) in powers of s and setting $s = s_*$, we get

$$\begin{aligned} p_x &= -2K(0)s_* - K'(0)(s_*)^2 + \dots, \\ p_z &= 1 - 2K^2(0)(s_*)^2 + \dots. \end{aligned} \tag{1.12}$$

Now we turn to the equations of reflected rays (1.6) in the part of Fock domain considered. Using relations (1.11) and (1.12) in Eqs. (1.6), we arrive at the relations

$$n - \frac{1}{2}K(0)(s^2 - (s_*)^2) = ts_*[-2K(0) + \dots],$$

$$s - s_* - \frac{1}{6}K^2(0)(s^3 - (s_*)^3) + K(0)ns = t[1 - 2K^2(0)(s_*)^2 + \dots], \tag{1.13}$$

which enable us to express the ray coordinates s_*, t via the curvilinear coordinates s, n in the Fock domain with the required precision.

In the first approximation, from the second relation (1.13) we obtain $t \simeq s - s_*$. The substitution of this value of t in the first relation (1.13) yields a quadratic equation for s_* as a function of s and n:

$$(s_*)^2 - s_*\frac{4}{3}s + \frac{1}{3}s^2 - \frac{2n}{3K(0)} = 0, \tag{1.14}$$

which has two real roots $(s_*)_{1,2} = \frac{2}{3}s \pm \frac{1}{3}\sqrt{s^2 + 6nK^{-1}(0)}$ of the equation. The required root is selected from the following condition. By the definition of coordinates s, n in the illuminated part of the Fock domain, s and s_* are negative and the point s_* is situated to the left of the point s (see Fig. 1), i.e., $|s_*| > |s|$. But if the observation point with coordinates s, n lies on the scatterer surface $\partial\Sigma$ ($n = 0$), then only the reflected ray having $s = s_*$ corresponds to it. If by the square root of a positive number we mean its arithmetic value, which is natural, then for $n = 0$ we get $\sqrt{s^2} = |s|$ and then $-\sqrt{s^2} = -|s| = s$. Therefore for s_* the second root should be taken, i.e.,

$$s_* = \frac{2}{3}s - \frac{1}{3}\sqrt{s^2 + 6\frac{n}{K(0)}}. \tag{1.15}$$

The correction to the second ray parameter t can be found from the relation

$$t = \sqrt{(x - x_*)^2 + (z - z_*)^2},$$

using the connecting formulas (1.11) between x, z and s, n, as well as relation (1.15) for s_*.

The results produced above enable us to present the differential form $p_x dx + p_z dz$ in curvilinear coordinates s, n. The differentials dx and dz are recalculated with the use of formulas (1.11):

$$\begin{aligned} dx &= -K(0)s\,ds + dn, \\ dz &= ds\left(1 - \frac{1}{2}K^2(0)s^2 + K(0)n\right) + dnK(0)s, \end{aligned} \tag{1.16}$$

and formula (1.15) should be substituted for s_* in the expressions (1.12) for p_x, p_z.

As a result, we obtain the following formula:

$$d\tau = \left\{1 + K^2(0)\left(-\frac{5}{18}s^2 + \frac{2}{9}s\sqrt{s^2 + \frac{6n}{K(0)}}\right) - \frac{1}{3}K(0)n\right\} ds$$

$$+K(0)\Big(-\frac{s}{3}+\frac{2}{3}\sqrt{s^2+\frac{6n}{K(0)}}\Big)\,dn, \tag{1.17}$$

where all the terms of order up to $O(k^{-1})$ are preserved and all terms of lesser order are removed for $k \gg 1$.

To obtain the eikonal $\tau_{\rm ref}$ of the reflected wave we integrate the differential form (1.17) along the following path: integration with respect to s is carried out for $n = 0$ and an antiderivative is taken such that it coincides with the value of the eikonal $\tau_{\rm inc}$ of the incident wave on $\partial\Sigma$; then we calculate the integral over n from 0 to n for a fixed s. As a result, we get

$$\tau_{\rm ref} = s - \frac{5}{54}K^2(0)s^3 - \frac{1}{3}K(0)ns + \frac{2}{27}K^2(0)\big(s^2+\frac{6n}{K(0)}\big)^{3/2}. \tag{1.18}$$

Next we turn to calculation of the amplitude of the reflected wave in the illuminated part of the Fock domain, and first we derive a formula for the geometric spreading J (see relation (1.8)). The functional determinant

$$\left|\frac{D(x,z)}{D(t,s_*)}\right|$$

is easily obtained in ray coordinates (the principal terms), using relations (1.6), (1.7), and (1.13):

$$\frac{D(x,z)}{D(t,s_*)} = -K(0)s_* + 2tK(0) + O((s_*)^3) + tO((s_*)^2). \tag{1.19}$$

Substituting the value of s_* from (1.15) in (1.19) and setting $t \simeq s - s_*$, we get the following expression for the principal term of the geometric spreading in the Fock domain:

$$J = K(0)\sqrt{s^2+\frac{6n}{K(0)}}. \tag{1.20}$$

We recall that in relation (1.20) we mean the arithmetic value of the root; therefore, the sign of the modulus can be omitted. On the body surface, $n = 0$, $s = s_*$, and both s and s_* are negative; therefore, the relation $J|_{n=0} = K(0)|s_*|$ follows from (1.20). Denote the amplitude of the reflected wave by $A_{\rm ref} = {\rm const}\ J^{-1/2}$; then $A_{\rm ref}$ is equal to -1 for the Dirichlet condition on $\partial\Sigma$. This allows us to find the arbitrary constant. Finally we get

$$A_{\rm ref} = -\sqrt{\frac{|s_*|}{\sqrt{s^2+\frac{6n}{K(0)}}}}. \tag{1.21}$$

In conclusion, we present formulas for the reflected wave in the stretched coordinates σ, ν, used inside the Fock domain. The connection between the coordinates s, n and σ, ν is given by the relations

$$s = \frac{\sigma}{M_0K(0)}, \quad n = \frac{\nu}{2M_0^2K(0)}, \quad M_0 = \left(\frac{k}{2K(0)}\right)^{1/3}, \tag{1.22}$$

where M_0 is a large Fock parameter. Using (1.22) in formulas (1.18), (1.21), we derive the following expression for the reflected wave $U^{\rm ref}$ in the Fock domain in the principal term of the asymptotics as $M_0 \to \infty$:

$$U^{\rm ref} = -e^{iks}\sqrt{\frac{|\sigma_*|}{\sqrt{\sigma^2+3\nu}}}\exp\Big\{i\Big(-\frac{5}{27}\sigma^3 - \frac{1}{3}\nu\sigma + \frac{4}{27}(\sigma^2+3\nu)^{3/2}\Big)\Big\}, \tag{1.23}$$

where the ray parameters s_*, (σ_*) should be excluded from the amplitude with the help of the formula $\sigma_* = \frac{2}{3}\sigma - \frac{1}{3}\sqrt{\sigma^2+3\nu}$.

2. Calculation of the reflected wave in the illuminated part of the Fock domain by the saddle point method

For reader's convenience we present the main steps of calculation of the asymptotics of the reflected field in the Fock domain. Consider the expression for the reflected field

$$U_0^{\mathrm{ref}} = -\frac{e^{iks_i}}{2\sqrt{\pi}} \int\limits_L e^{i\sigma\zeta} \frac{w_2(\zeta)}{w_1(\zeta)} w_1(\zeta - \nu) d\zeta \tag{2.1}$$

for $\sigma = -|\sigma|$, $|\sigma| \to \infty$. The contour L encloses the roots of the function $w_1(\zeta)$ situated on the ray $\arg \zeta = \pi/3$.

1^0. *The determination of the phase of the reflected wave* U_0^{ref} *as* $|\sigma| \to \infty$, $\sigma < 0$.

To separate out the fast varying functions, we replace the integration parameter ζ by a new parameter z according to the formula $\zeta = \sigma^2 z$ for $\sigma^2 \to \infty$:

$$U_0^{\mathrm{ref}} = -\sigma^2 \frac{e^{iks_i}}{2\sqrt{\pi}} \int\limits_L e^{-i|\sigma|^3 z} \frac{w_2(\sigma^2 z)}{w_1(\sigma^2 z)} w_1(\sigma^2 z - \nu)\, dz.$$

In order that the search of a saddle point be convenient, we make one more change $z = -t$. Then we get

$$U_0^{\mathrm{ref}} = \sigma^2 \frac{e^{iks_i}}{2\sqrt{\pi}} \int\limits_{L_1} e^{i|\sigma|^3 t} \frac{w_2(-\sigma^2 t)}{w_1(-\sigma^2 t)} w_1(-\sigma^2 t - \nu)\, dt; \tag{2.2}$$

here the direction of the contour L_1 is opposite to that of the contour L.

We assume that the saddle point lies on the semiaxis $t > 0$. Then for $\sigma^2 \to \infty$ we replace the Airy functions by their asymptotics as $t \to \infty$:

$$w_1(-\sigma^2 t) \simeq (\sigma^2 t)^{-1/4} \exp\left\{ i\left[\frac{2}{3}(\sigma^2 t)^{3/2} + \frac{\pi}{4} \right] \right\},$$
$$w_2(-\sigma^2 t) = \overline{w_1(-\sigma^2 t)}.$$

Now, we turn back to the contour L. Then, finally, for U_0^{ref} we have

$$U_0^{\mathrm{ref}} = -\sigma^2 e^{-i\pi/4} \frac{e^{iks_i}}{2\sqrt{\pi}} \int\limits_L dt \frac{\exp\{i\psi\}}{(\sigma^2 t + \nu)^{1/4}},$$

$$\psi = |\sigma|^3 t - \frac{4}{3}(\sigma^2 t)^{3/2} + \frac{2}{3}(\sigma^2 t + \nu)^{3/2}; \tag{2.3}$$

here by the radical we mean the arithmetic value of the root. Therefore the phase ψ can be written as

$$\psi = |\sigma|^3 \left[t - \frac{4}{3} t^{3/2} + \frac{2}{3}\left(t + \frac{\nu}{\sigma^2} \right)^{3/2} \right]. \tag{2.4}$$

Using the equation for the critical point

$$1 - 2t^{1/2} + \left(t + \frac{\nu}{\sigma^2} \right)^{1/2} = 0, \tag{2.5}$$

we get a quadratic equation with respect to $t^{1/2}$. Its solution is as follows:

$$t_{1,2}^{1/2} = \frac{2}{3} \pm \frac{1}{3}\sqrt{1 + \frac{3\nu}{\sigma^2}}. \tag{2.6}$$

Multiplying both parts of (2.6) by $|\sigma|$, we arrive at

$$(\sqrt{\sigma^2 t})_{1,2} = \frac{2}{3}|\sigma| \pm \frac{1}{3}\sqrt{\sigma^2 + 3\nu}. \tag{2.7}$$

We draw attention to the similarity of Eq. (2.5) and the equation for s_*, which allows us to formulate the following assertion. A critical or a saddle point for the integral (2.3) corresponds to a point on $\partial\Sigma$ such that the ray of a reflected wave that issues out of it comes to the observation point with the coordinates σ and ν in the Fock domain. For this reason, we take a root in (2.6) that corresponds to the condition $s_* = s$ for $\nu = 0$. Here s_* and s are negative, because they lie on the illuminated part of $\partial\Sigma$ in the Fock domain:

$$\left(\sqrt{\sigma^2 t_*}\right) = \frac{2}{3}|\sigma| + \frac{1}{3}\sqrt{\sigma^2 + 3\nu}. \tag{2.8}$$

We calculate the phase ψ at the critical point (2.8). To this end we represent it in the form

$$\psi = |\sigma|\sigma^2 t - \frac{4}{3}\left(\sqrt{\sigma^2 t}\right)^3 + \frac{2}{3}\left(\sqrt{\sigma^2 t + \nu}\right)^3 \tag{2.9}$$

and make use of the following relation, which is a consequence of Eq. (2.5):

$$\sqrt{\sigma^2 t + \nu} = 2\sqrt{\sigma^2 t} - |\sigma|.$$

As a result, we obtain

$$\psi_* = \frac{5}{27}|\sigma|^3 + \frac{1}{3}|\sigma|\nu + \frac{4}{27}(\sigma^2 + 3\nu)^{3/2}.$$

Finally, in view of $|\sigma| = -\sigma$, we get the phase of the reflected wave

$$\psi_* = -\frac{5}{27}\sigma^3 - \frac{1}{3}\sigma\nu + \frac{4}{27}(\sigma^2 + 3\nu)^{3/2}. \tag{2.10}$$

2^0. *Computation of the asymptotics of the reflected wave* u_0^{ref} *as* $(\sigma) \to \infty$, $\sigma < 0$.

For U_0^{ref} we take the first formula in (2.3). Let t_* be a critical, stationary point and ψ_* be the value of the phase at it. Then in the vicinity of t_*, by the techniques of the saddle point method, we obtain an expression for the reflected wave in the form:

$$U_0^{\mathrm{ref}} = -\sigma^2 e^{-i\pi/4}\frac{e^{iks}i}{2\sqrt{\pi}}\frac{\exp\{i\psi_*\}}{(\sigma^2 t_* + \nu)^{1/4}} \times \left[\frac{\pi}{2\left(-\left.\frac{\partial^2\psi}{\partial t^2}\right|_{t_*}\right)}\right]^{1/2} e^{-i\pi/4}. \tag{2.11}$$

For the second derivative of the phase, we have (see (2.9))

$$\left.\frac{\partial^2\psi}{\partial t^2}\right|_{t_*} = -\frac{\sigma^4\sqrt{\sigma^2 + 3\nu}}{2\sqrt{\sigma^2 t_*}\sqrt{\sigma^2 t_* + \nu}}. \tag{2.12}$$

Substituting the expression (2.12) in (2.11), we arrive at the following result for the reflected wave:

$$U_0^{\mathrm{ref}} \cong -\sqrt[4]{\frac{\sigma^2 t_*}{\sigma^2 + 3\nu}}\exp\left\{i\left(ks - \frac{5}{27}\sigma^3 - \frac{1}{3}\nu\sigma + \frac{4}{27}(\sigma^2 + 3\nu)^{3/2}\right)\right\}. \tag{2.13}$$

Consider the expression $\sqrt{\sigma^2 t_*}$ in (2.13) and in the amplitude of the reflected wave (1.21) in more detail. We recall that the ray formula (2.13) for the reflected wave s_* contains the coordinate of the reflection point of the ray that comes to the observation point s, n in the illuminated part of the Fock domain. Moreover, s_* and s are negative, i.e., $s_* = -|s_*|$ and $s = -|s|$. Therefore, formula (1.15) can be represented in the form $|s_*| = \frac{2}{3}|s| + \frac{1}{3}\sqrt{s^2 + \frac{6n}{K(0)}}$, and, passing to the stretched coordinates σ, ν (see relations (1.22)), we get exactly formula (2.8) for $\sqrt{\sigma^2 t_*}$.

In conclusion, we would emphasize the following fact. If we fix the coordinates of the observation point s, n or σ, ν in formulas (1.15) and (2.8), then their left-hand sides give the coordinate of the reflection point on $\partial\Sigma$ of the ray that comes to this observation point. Moreover, obviously, s_* with the corresponding t_* are both constant along this ray and therefore occur in the amplitude as constants of integration of the transport equation.

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