Stationary statistical experiments and the optimal estimator for a predictable component

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Abstract. A stationary autoregression process given by a difference stochastic equation is characterized by a two-dimensional covariance matrix under stationarity conditions. The optimal estimator function represented by a square variation of the martingale is used to obtain consistent estimators for the parameter of a predictable component.

Keywords. Stationary autoregression process, optimal estimator function, consistent statistics, evaluation parameters.

1. Introduction

The statistical analysis of real measurements is particularly important, when the observation of physical data (biological, chemical, etc.) has often no justification in the form of a physical theory.

In numerous complex systems like biological processes with equilibrium, the dynamics of the concentration or frequencies of the presence of some predefined attribute can be described by a mathematical model of binary statistical experiments. The model is based on the validation of elementary hypotheses about the presence or absence of the predefined attribute in the set of elements that make up a complex system [1].

It is assumed that:

1. all the elements that make up the system can gain or lose the said attribute in the course of time; i.e, the relative frequency of an attribute is a dynamic variable;

2. the basic objects of our study are statistical experiments (SEs) characterized by relative frequencies of the presence of an attribute in a sample of fixed volume at every time instant;

3. the average result of a next experiment depends on the average result of the present experiment. Such a relationship is called the persistent regression and used as a fundamental condition for the subsequent analysis of the model.

The method of construction and exploration of the proposed mathematical model is based on the analysis of the following basic properties [2] of statistical experiments:

1. Persistent regression.
2. Equilibrium value and fluctuations, as well as their asymptotic behavior.
3. Approximation by the normal process of autoregression.

The present work introduces a new computational method and the corresponding statistical estimators of parameters characterizing the transitional mobility of the dynamical frequency (concentration).

The model is based on the stationarity conditions for statistical experiments given by a difference stochastic equation with predictable linear component and with additional stochastic component determined by a stationary, in wide meaning, random sequence.

The martingale approach using the martingale-differences in the analysis of autoregressive models is described in books [3, 4]. In particular, the conditional least square method based on martingale-differences is considered, for example, in works [5–7]. In [8, Ch. 14], Liptser and Shiryaev considered the statistical problems for stationary random sequences with fractionally rational spectrum, by using the filtering equation.

The aim of this article is to apply the martingale approach proposed by Heyde in [3] to the construction of an optimal estimator of the predictable component parameter by involving the square variation of the martingale component.

2. Statement of the problem

Definition 1. *A statistical experiment with persistent regression [1, 2] is given by the solution of the difference stochastic equation*

$$\Delta\alpha_t = -V_0\,\alpha_t + \Delta\mu_t, \quad \Delta\alpha_t := \alpha_{t+1} - \alpha_t, \quad t \in \mathcal{N}_0 = 0,\, 1,\, 2,\, 3,\, \ldots \tag{1}$$

The stochastic component $\Delta\mu_t$, $t \in \mathcal{N}_0$, is given by the martingale-differences

$$\Delta\mu_t := \mu_{t+1} - \mu_t, \quad t \in \mathcal{N}_0 = 0, \qquad \mu_0 = 0,$$

generated by a stationary square integrable martingale μ_t, $t \in \mathcal{N}_0$, which is characterized by two first moments:

$$E(\Delta\mu_t) = 0, \quad E(\Delta\mu_t)^2 = \sigma^2, \quad \forall\, t \in \mathcal{N}_0. \tag{2}$$

The initial value α_0 in (1)–(2) is a random variable independent of $\Delta\mu_t$, $t \geq 1$. The predictable parameter V_0 satisfies the following limitation:

$$0 < V_0 < 2. \tag{3}$$

3. Stationary statistical experiments

A stationary, in a wide meaning, process with discrete time is determined by the covariance function [9, Ch. VII].

In our model with (1)–(3), the stationarity in a wide meaning is ensured by the condition

$$cov\,(\alpha_t, \alpha_t) = E\,\alpha_t^2 = \sigma_0^2, \qquad t \in \mathcal{N}_0. \tag{4}$$

In this connection, it is necessary to establish conditions for the initial distribution which assure the stationarity of Eqs. (1)–(3) in a wide meaning.

Theorem 1. *The stochastic equation (1)–(3) presents a stationary random sequence in a wide meaning iff the following relations take place:*

$$E\,\alpha_0 = 0, \quad E\,\alpha_0^2 = \sigma_0^2 = \sigma^2/V_0(2 - V_0). \tag{5}$$

Proof. The statement of Theorem 1 is carried out by induction.

Let $B_t := E\,\alpha_t^2 = \sigma_0^2$, $t \leq T$. The formula (1) implies that

$$E\,[\Delta\alpha_t]^2 = E\,[-V_0\,\alpha_t + \Delta\mu_t]^2 = V_0^2\sigma_0^2 + \sigma^2. \tag{6}$$

By the relation $\Delta \alpha_t := \alpha_{t+1} - \alpha_t$, dispersion (6) can be represented as follows:

$$E\,[\Delta \alpha_t]^2 = B_{t+1} + (2V_0 - 1)\,\sigma_0^2. \tag{7}$$

In other words, taking together (6) and (7), we have

$$B_{t+1} - B_t = (V_0^2 - 2\,V_0)\,\sigma_0^2 + \sigma^2 = 0.$$

Hence, for all $t \in \mathcal{N}_0$, $B_t = \sigma_0^2$.

The necessity of conditions (5) is obvious. $\square$

Remark 1. The limitation condition (3) provides the correctness of the statistical stationarity condition (5).

4. Statistics for stationary statistical experiments

The covariation analysis can be used for developing the statistics for the stationary statistical experiments (1)–(5).

Theorem 2. *The stationary two-component vectors* $(\alpha_t, \Delta\,\alpha_t)$, $t \in \mathcal{N}_0$ *(1)–(5) can be described by the following covariation functions:*

$$R(s) := E[\alpha_t\,\alpha_{t+s}] = \sigma_0^2\,q^s, \ \ R^0(s) := E[\alpha_t\,\Delta\,\alpha_{t+s}] = -V_0\sigma_0^2\,q^s, \ s \geq 0,$$

$$R^\Delta(s) := E[\Delta\,\alpha_t\,\Delta\,\alpha_{t+s}] = -V_0\sigma_0^2\,q^{s-1}, \ \ s \geq 1, \ \ q := 1 - V_0. \tag{8}$$

The covariations are calculated as follows:

$$R := R(0) = \sigma_0^2, \ \ R^0 := R^0(0) = -V_0\,\sigma_0^2, \ \ R^\Delta = R^\Delta(0) := 2\,V_0\sigma_0^2. \tag{9}$$

Proof. The first and third covariations in (9) are obvious.

The second covariation in (9) is calculated, by using Eq. (1) and relation (6).

The covariation functions (8) for the two-component vectors $(\alpha_t, \Delta\alpha_t)$, $t \in \mathcal{N}_0$ are proved by induction. $\square$

Remark 2. Covariations (9) generate, as estimators, the following statistics:

$$R_T := \frac{1}{T}\sum_{t=0}^{T-1} \alpha_t^2, \ \ R_T^\Delta := \frac{1}{T}\sum_{t=0}^{T-1}(\Delta\alpha_t)^2, \ \ R_T^0 := \frac{1}{T}\sum_{t=0}^{T-1} \alpha_t\Delta\alpha_t. \tag{10}$$

In what follows, we use the following proposition.

Presupposition 1. Estimators (10) are consistent in the mean-square meaning:

$$E[R_T - R]^2 \to 0, \ \ E[R_T^\Delta - R^\Delta]^2 \to 0, \ \ E[R_T^0 - R^0]^2 \to 0, \ \ T \to \infty. \tag{11}$$

Remark 3. The consistency of estimators (11) for Gaussian SE follows from the condition [9, Ch. VII. Annex. and exerc., ex. 23]:

$$\frac{1}{T}\sum_{s=0}^{T-1} R^2(s) \overset{L^2}{\to} 0, \quad T \to \infty. \tag{12}$$

Remark 4. The consistency of estimators (11) also has place for the bounded SE.

Proposition 1. *Statistics (10) generate the following two estimators of the predictable component parameter* V_0:

$$V_T^0 = -R_T^0/R_T, \qquad V_T^\Delta = R_T^\Delta/2\,R_T, \tag{13}$$

which, as will be shown below, are both consistent in mean square and asymptotically unbiased.

5. Martingale approach

The problem of optimal estimators for the predictable parameter V_0 can be solved by the quasi-likelihood approach represented in monograph [3, Ch. 2]. The initial statistics for the stationary SE (1)—(5) are given by the trajectory

$$\dot{G}_T(V) := (\alpha_t, \ 0 \le t \le T),$$
(14)

and the martingale-differences

$$G_T(V) := (\Delta \mu_t, \ 0 \le t \le T),$$
(15)

or, in the equivalent form,

$$G_T(V) := (\Delta \alpha_t + V \alpha_t, \ 0 \le t \le T).$$
(16)

In (15)–(16), the predictable parameter V is unknown and satisfies the condition $0 < V < 2$. Following the idea of C. Heyde, we introduce *quasi-score functional* (QSF).

Definition 2. *The normalized QSF is defined by the covariation*[1]

$$\mathcal{E}(V) := E\left(G_T(V), G_T(V)\right) / E\left(\dot{G}_T(V), \dot{G}_T(V)\right).$$
(17)

Proposition 2. *The covariation function in (17) gives the following result:*

$$\mathcal{E}(V) = V^2 - 2 V V_0 + 2 V_0.$$
(18)

The martingale statistics (15)–(16) and trajectories (14) generate the following covariation functions in (17):

$$E\left(G_T(V), G_T(V)\right) = E \sum_{t=1}^{T} \Delta \mu_t^2 = E \sum_{t=1}^{T} (\Delta \alpha_t + V \alpha_t)^2 = [V^2 - 2 V V_0 + 2 V_0] \cdot T \sigma_0^2;$$
(19)

and

$$E\left(\dot{G}_T(V), \dot{G}_T(V)\right) = E \sum_{t=1}^{T} \alpha_t^2 = T \sigma_0^2,$$
(20)

where V_0 is a true value of the parameter V.

So, the normalized quasi-score function does not depend on the time parameter T.

Corollary 1. *Function (18) is positive definite and admits the following representation:*

$$\mathcal{E}(V) = (V - V_0)^2 + \mathcal{E}^0, \qquad \mathcal{E}^0 := \sigma^2/\sigma_0^2 = V_0(2 - V_0).$$
(21)

The parameter $\mathcal{E}^0$ is called the *stationarity coefficient* and plays an important role in the optimal estimation.

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[1] $(\cdot, \cdot)$ is a scalar product of two vectors

6.   Optimal estimator function (OEF)

Following C. Heyde [3], we introduce the *normalized estimator function.*

Definition 3. *The optimal estimator function is given by the relation*

$$\widehat{\mathcal{E}}_T(V) := \mathcal{E}_T(V)/R_T, \tag{22}$$

where, by definition,

$$\mathcal{E}_T(V) := \frac{1}{T} \sum_{t=1}^{T} [\Delta\alpha_t + V\,\alpha_t]^2, \tag{23}$$

which is a square variation of the martingale

$$\mu_t := \sum_{k=0}^{t-1} \Delta\mu_k, \qquad t \geq 1, \qquad \mu_0 = 0. \tag{24}$$

The normalizing statistics R_T is defined in (10).

Theorem 3. *The optimal estimator statistics V_T^0 for the predictable parameter V_0 is derived from the optimality relation:*

$$\min_{0<V<2} \mathcal{E}_T(V) = \mathcal{E}_T(V_T^0) = \mathcal{E}_T^0, \qquad \mathcal{E}_T^0 := V_T^\Delta - (V_T^0)^2, \tag{25}$$

and has the following explicit form:

$$V_T^0 = -\sum_{t=1}^{T} \alpha_t\,\Delta\alpha_t / \sum_{t=1}^{T} \alpha_t^2 = -R_T^0/R_T, \tag{26}$$

where the statistics V_T^0 and V_T^Δ are defined in (13).

Proof. The calculation of the OEF by formula (23) gives the following result:

$$\mathcal{E}_T(V) = V^2 R_T + 2V\,R_T^0 + R_T^\Delta. \tag{27}$$

Now, the optimal estimator function (22) can be obtained by the division of (27) by the normalized statistics defined in (10):

$$\widehat{\mathcal{E}}_T(V) := V^2 - 2\,V\,V_T^0 + V_T^\Delta = (V - V_T^0)^2 + V_T^\Delta - (V_T^0)^2, \tag{28}$$

where the statistics V_T^0 and V_T^Δ are defined in (13).
   The last relation provides the statement of Theorem 3.   □

Remark 5. It is easy to verify, in Gaussian case, the following consistency of estimates:

$$V_T^\Delta \xrightarrow{P} 2\,V_0, \qquad V_T^0 \xrightarrow{P} -V_0, \qquad T \to \infty. \tag{29}$$

So, the residual term of the optimal estimator function in (25) represents the estimator for the stationary coefficient $\mathcal{E}^0 := \sigma^2/\sigma_0^2$:

$$\mathcal{E}_T^0 \xrightarrow{P} V_0(2 - V_0) = \sigma^2/\sigma_0^2, \qquad T \to \infty. \tag{30}$$

7. Gaussian stationary statistical experiment

In a particular case, the Gaussian stationary statistical experiment is given by a solution of the following difference stochastic equation:

$$\Delta \alpha_t = -V_0 \alpha_t + \Delta W_t, \qquad t \in \mathcal{N}_0. \tag{31}$$

The stochastic component ΔW_t is given by a martingale-difference of the normal sequence W_t, $t \in \mathcal{N}_0$ [10, Ch. 2, p. 6]:

$$\Delta W_t := W_{t+1} - W_t, \qquad t \in \mathcal{N}_0, \qquad W_0 = 0,$$

which is characterized by two first moments:

$$E(\Delta W_t) = 0, \qquad E(\Delta W_t)^2 = \sigma^2, \qquad t \in \mathcal{N}_0. \tag{32}$$

The initial value of SE (1) – (2) is a normally distributed random variable independent of ΔW_t, $t \geq 1$. The predictable parameter V_0 satisfies limitation (3).

The problem of optimal estimator for the predictable parameter V_0 can be solved now, by using the likelihood function for a stationary Gaussian sequence W_t, $t \in \mathcal{N}_0$, which admits the following representation:

$$\ln L(V, W_T) = -\frac{1}{2\sigma^2} \sum_{t=0}^{T-1} (\Delta W_t)^2 + P_{0T}, \tag{33}$$

where

$$W_T := \frac{1}{T} \sum_{t=0}^{T-1} (\Delta W_t)^2, \tag{34}$$

and P_{0T} is a constant which does not depend on V.

Now, the normalized likelihood estimating function for the predictable parameter V_0 can be represented, by using the relation (see (33)–(34) and (22)–(27))

$$\ln L(V, W_T) = -\frac{R_T}{2\sigma^2} \widehat{\mathcal{E}}_T(V) + P_0, \tag{35}$$

in the form

$$\widehat{\mathcal{E}}_T(V) := \frac{1}{T} \sum_{t=0}^{T-1} (\Delta W_t)^2 / R_T = V^2 - 2V V_T^0 + V_T^\Delta = (V - V_T^0)^2 + V_T^\Delta - (V_T^0)^2. \tag{36}$$

By Theorem 3, the optimal estimator statistics V_T^0 for the predictable parameter V_0 is given by formula (26).

Corollary 2. *The optimal estimator statistics V_T^0, reported in Theorem 3 for the normalized estimator function (36) coincides with the maximum likelihood estimation expressed by the logarithm of the likelihood function (33).*

8. Multivariate stationary statistical experiments

The martingal approach to the optimal estimation of the predictable parameter V_0 realized in Sections 5–6 for the stationary statistical experiments (1)–(5) in the one-dimensional case can be extended to multivariate statistical experiments.

Now, the multivariate model of stationary statistical experiments is given by a solution of the difference stochastic equation

$$\Delta \alpha_t = -\mathbb{V}_0 \, \alpha_t + \Delta \mu_t, \qquad t \geq 0. \tag{37}$$

The predictable component is determined by the regression matrix

$$\mathbb{V} := [V_{km}; \, 1 \leq k, \, m \leq M],$$

which satisfies the additional stationarity condition

$$\text{the matrix } \mathbb{Q} := \mathbb{I} - \mathbb{V} \text{ has the eigenvalues strictly inside a unit circle.} \tag{S}$$

The stochastic component is defined by the square integrable stationary martingale $\mu_t = (\mu_t^k, \, 1 \leq k \leq M)$, $t \in \mathcal{N}_0$, which is characterized by two first moments

$$E \, \Delta \mu_t = 0, \qquad E[\Delta \, \mu_t \, \Delta \, \mu_t^{'}] = \mathbb{B}, \qquad t \geq 0. \tag{38}$$

Here, the martingale-differences are defined as follows:

$$\Delta \, \mu_t := \mu_{t+1} - \mu_t, \qquad t \in \mathbb{N}_0, \qquad \mu_0 := 0.$$

Under condition (S) by analogy with Theorem 1, the statistical stationarity for the statistical experiments (37)–(38) supposes that the initial value α_0 is a random variable independent of $\Delta \, \mu_t$, $t \geq 1$, and satisfies the following condition (cf. (5)):

$$E \, \alpha_0 = 0, \qquad E[\alpha_0 \, \alpha_0^{'}] = \mathbb{R}.$$

The positive definite matrix $\mathbb{R}$ is determined by a solution of the following matrix stationarity equation:

$$\mathbb{B} = \mathbb{V}_0 \, \mathbb{R} + \mathbb{R} \mathbb{V}_0^{'} - \mathbb{V}_0 \mathbb{R} \mathbb{V}_0^{'}. \tag{39}$$

In other words, the stationary SE (37)–(38) is a stationary process in a wide meaning:

$$E \, \alpha_t = 0, \; E[\alpha_t \, \alpha_t^{'}] = R, \; E[\alpha_{t+s} \, \alpha_t^{'}] = R(s) = (\mathbb{I} - \mathbb{V})^s R \; \forall \, t \in \mathcal{N}_0. \tag{40}$$

By analogy with Theorem 2 below, the stationary SE defined by (37)–(40) is completely determined by the following covariation matrices:

$$E[\alpha_t \, \alpha_t^{'}] = R, \qquad E[\Delta \, \alpha_t \, \Delta \, \alpha_t^{'}] = R^{\Delta},$$

$$E[\Delta \, \alpha_t \, \alpha_t^{'}] = R^0, \qquad E[\alpha_t \, \Delta \, \alpha_t^{'}] = R^{0^{'}}. \tag{41}$$

Following [3], the martingale approach in the statistical analysis for the stationary SE (37)–(41) involves a square variation of martingale (38):

$$\mathcal{E}_T(\mathbb{V}) := \frac{1}{T} \sum_{t=0}^{T-1} \Delta \, \mu_t \, \Delta \, \mu_t^{'} = \frac{1}{T} \sum_{t=1}^{T} (\Delta \, \alpha_t + \mathbb{V} \, \alpha_t)(\Delta \, \alpha_t + \mathbb{V} \, \alpha_t)^{'} = \mathbb{V} \, \mathbb{R}_T \, \mathbb{V}^{'} + \mathbb{V} \, \mathbb{R}_T^{0^{'}} + \mathbb{R}_T^0 \, \mathbb{V}^{'} + \mathbb{R}_T^{\Delta}, \tag{42}$$

where the following matrix statistics are used:

$$\mathbb{R}_T := \frac{1}{T}\sum_{t=1}^{T}\alpha_t\,\alpha_t^{'}, \quad \mathbb{R}_T^0 := \frac{1}{T}\sum_{t=1}^{T}\Delta\alpha_t\,\alpha_t^{'},$$

$$\mathbb{R}_T^{0^{'}} := \frac{1}{T}\sum_{t=1}^{T}\alpha_t\Delta\alpha_t^{'}, \quad \mathbb{R}_T^{\Delta} := \frac{1}{T}\sum_{t=1}^{T}\Delta\alpha_t\Delta\alpha_t^{'}. \tag{43}$$

Proposition 3. *The optimal estimator function given by Definition 3 has the representation*

$$\mathcal{E}_T(\mathbb{V}) = \mathcal{E}_T^0(\mathbb{V},\mathbb{V}_T^*) + \mathcal{B}_T, \tag{44}$$

where the following positive definite matrix is used:

$$\mathcal{E}_T^0(\mathbb{V},\mathbb{V}_T^*) = (\mathbb{V} - \mathbb{V}_T^*)\mathbb{R}_T(\mathbb{V} - \mathbb{V}_T^*)^{'}. \tag{45}$$

The statistics V_T^* has the following form:

$$\mathbb{V}_T^* = -\mathbb{R}_T^0\mathbb{R}_T^{-1}. \tag{46}$$

The residual term has the representation

$$\mathcal{B}_T = \mathbb{R}_T^{\Delta} - \mathbb{R}_T^0\mathbb{R}_T^{-1}\mathbb{R}_T^{0^{'}} \tag{47}$$

or, in another form,

$$\mathcal{B}_T = \mathbb{V}_T^*\mathbb{R}_T + \mathbb{R}_T\mathbb{V}_T^{*^{'}} - \mathbb{V}_T^*\mathbb{R}_T\mathbb{V}_T^{*^{'}}. \tag{48}$$

It is not difficult to deduce the following relation:

$$\mathcal{E}_T(\mathbb{V}) - \mathcal{E}_T^0(\mathbb{V},\mathbb{V}_T^*) = \mathcal{B}_T,$$

which yields (44).

Theorem 4. *The estimating statistics $\mathbb{V}_T^*$ for the predictable matrix $\mathbb{V}_0$ is derived from the optimality relation*

$$\min_V \mathcal{E}_T(\mathbb{V}) = \min_V[\mathcal{E}_T^0(\mathbb{V},\mathbb{V}_T^*) + \mathcal{B}_T] = \mathcal{B}_T, \tag{49}$$

and the estimating statistics $\mathbb{V}_T^*$ is given by relation (46).
The minimum of the optimality relation (44) is reached with (48).

Remark 6. It is easy to verify that the optimal statistics (46) is consistent in mean square:

$$l.i.m._{T\to\infty}\mathbb{V}_T^* = \mathbb{V}_0.$$

9.   Gaussian Markovian stationary multivariate SE

A *Gaussian Markovian stationary multivariate* SE (GM-SSE) $\alpha_t,\ t \in \mathcal{N}_0$ is determined by two moments:

$$E\,\alpha_t = a, \qquad E[\alpha_t \cdot \alpha_t^{'}] = R, \qquad t \geq 1. \tag{50}$$

The stationarity condition means that the initial value α_0 is a Gaussian random variable with

$$E\,\alpha_0 = a, \qquad E[\alpha_0 \cdot \alpha_0^{'}] = R. \tag{51}$$

Theorem 5. *The GM–SSE (50)–(51) can be defined by the difference stochastic equation*

$$\Delta \alpha_t^0 = -\mathbb{V}\alpha_t^0 + \boldsymbol{\sigma}\Delta W_{t+1}, \qquad t \in \mathcal{N}_0. \tag{52}$$

Here, by definition, $\alpha_t^0 := \alpha_t - a$, and $\boldsymbol{\sigma}^2 := B = \sigma \cdot \sigma'$.
The matrices $\mathbb{R}$, $\mathbb{V}$, and $\mathbb{B}$ are connected as follows:

$$\mathbb{B} = \mathbb{V}\mathbb{R} + \mathbb{R}\mathbb{V}' - \mathbb{V}\mathbb{R}\mathbb{V}' \tag{53}$$

and the Gaussian stationary sequence W_t, $t \geq 1$ is defined by the Gaussian martingale-differences:

$$W_{t+1} := \sum_{k=0}^{t} \Delta W_t, \quad E\,\Delta W_t = 0, \quad E[\Delta W_t \Delta W_t'] = \mathbb{I}. \tag{54}$$

The proof of Theorem 1 follows from the theorem of normal correlation [8, Th. 13.1 and Lem. 13.4].

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