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An analog of the Schwartz theorem on spectral analysis on a hyperbolic plane

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Abstract. Let $\mathbb{D}$ be an open unit disk in the complex plane. It is shown that every subspace in $C(\mathbb{D})$ invariant under weighted conformal shifts contains a radial eigenfunction of the corresponding invariant differential operator. This function can be expressed via the Gauss hypergeometric function and is a generalization of the spherical function on the disk $\mathbb{D}$ which is considered as a hyperbolic plane with the corresponding Riemannian structure.

Keywords. Spectral analysis, invariant subspace, hyperbolic plane.

1. Introduction

The well-known theorem by L. Schwartz on spectral analysis asserts that any nonzero shift-invariant subspace in $C(\mathbb{R}^1)$ contains the exponential function $e^{\lambda x}$ for some $\lambda \in \mathbb{C}$ (see [1]). In work [1], L. Schwartz assumed that, for $n \geq 2$, any nonzero subspace in $C(\mathbb{R}^n)$ with analogous property must contain the function

$$x \to e^{(\lambda, x)} \tag{1.1}$$

for some $\lambda \in \mathbb{C}^n$, where $(\cdot,\cdot)$ stands for a scalar product in $\mathbb{C}^n$. The question remained open for more than twenty five years. Finally, D.I. Gurevich [2] disproved the indicated hypothesis by L. Schwartz in 1975. More exactly, he proved the existence of six distributions $\mu_1, \dots, \mu_6$ with compact supports on $\mathbb{R}^n$, $n \geq 2$, such that the space $\mathcal{U} \subset C^\infty(\mathbb{R}^n)$ of solutions of the system of convolution equations

$$f * \mu_i = 0, \quad i = 1, \dots, 6,$$

is nonzero and does not contain a function of the form (1.1) for any $\lambda \in \mathbb{C}^n$.

The question about the additional conditions of validity of analogs of the Schwartz theorem for $C(\mathbb{R}^n)$ and other spaces of continuous functions was considered by many researchers (see, e.g., review [3] with large number of references and [4–6]). One of the most essential results in this direction was obtained by L. Brown, B. M. Schreiber, and B. A. Taylor [7]. They proved that any nonzero subspace $\mathcal{U}$ in $C(\mathbb{R}^n)$ that is invariant relative to shifts and rotations contains the radial function

$$x \to (\lambda\,|x|)^{1-\frac{n}{2}}\, J_{\frac{n}{2}-1}(\lambda\,|x|), \tag{1.2}$$

where λ is some complex number, $|x|$ is the Euclidean norm of a vector $x \in \mathbb{R}^n$, J_ν is the Bessel first-kind function with index ν. This is in agreement with the above-formulated Schwartz theorem, which can be seen in the following way. The exponential function $e^{\lambda x}$ is an eigenfunction of the differentiation operator $\dfrac{d}{dx}$ on the real line. This operator generates the algebra of all differential

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operators with constant coefficients on $\mathbb{R}^1$. Such operators commute, obviously, with shifts in $\mathbb{R}^1$. Analogously, function (1.2) is a radial eigenfunction of the operator

$$\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \tag{1.3}$$

in $\mathbb{R}^n$. This operator generates the algebra of all differential operators with constant coefficients in $\mathbb{R}^n$ that commute with all motions in $\mathbb{R}^n$. (We recall that a motion in $\mathbb{R}^n$ is called an inhomogeneous linear transformation conserving the distance between points of this space and its orientation.) In addition to the significant individual interest, the theorem proved by L. Brown, B. M. Schreiber, and B. A. Taylor has essential applications to integral geometry, for example, to the well-known Pompeiu problem (see [3]). Without any exaggeration, we can say that the new modern stage of studies of this old problem began after the publication of work [7]. Details can be found in reviews [3, 8, 9] and books [4–6] containing the proofs of the most strong results that are related to the Pompeiu transformation and its generalizations.

Some analogs of the theorem by L. Brown, B. M. Schreiber, and B. A. Taylor for noncompact symmetric spaces $X = G/K$ of rank 1 (see, e.g., [12]) were considered in works [10, 11]. There, it was shown that any nonzero subspace $\mathcal{U}$ in $C(X)$ that is invariant relative to the group G contains a spherical function

$$x \to \varphi_\lambda(x).$$

This function is invariant relative to the subgroup $K \subset G$ and is an eigenfunction of the Laplace–Beltrami operator L on X. We note that the operator L is a natural analog of operator (1.3) for the spaces X and, respectively, generates the whole algebra of differential operators with constant coefficients on X that is invariant relative to the group G. Similar results for the Damek–Ricci spaces and for the Heisenberg group were got in [13, 14].

The proofs of the main results in the above-cited works are based on the methods of classical harmonic analysis and use essentially the invariance of the problem under study relative to the corresponding group of transformations. In a number of cases where such invariance is broken, similar methods become inapplicable. This is referred, in particular, to the situation where the transformations with weight which appear frequently in problems of integral geometry are considered. We note that no analogs of the Schwartz theorem for shifts with weight multipliers on symmetric spaces are known till now.

In the last decade, the authors of the present work have developed a new approach to the above problems that is based on the application of operators with transmutation property (see [5,6,15]). This allows us to get the final results for some problems, in particular, local versions of the above-cited results (see [5,6]). In the present work, by using the developed technique of transmutation mappings, we will get the first weighty analog of the Schwartz theorem for the corresponding subspaces of continuous functions on the hyperbolic plane.

The exact formulation and the discussion of the main result are presented in § 2. In §§ 3–6, we will prove some auxiliary propositions and develop the necessary apparatus related to the generalized spherical transformation and transmutation mappings. The proof of the basic theorem 2.1 is given in § 7.

2. Formulation of the main result

In what follows, G is the group of conformal automorphisms of a unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For any $g \in G$, $z \in \mathbb{D}$, we denote the image of the point z at a mapping g by gz. Let $\alpha \in \mathbb{R}$. For

$z, \zeta \in \mathbb{D}$, we set

$$W(\zeta, z, \alpha) = \exp\big(2i\alpha \arg(1 - z\overline{\zeta})\big), \tag{2.4}$$

where the symbol arg stands for the principal value of the argument. We introduce an α-shift of a function $f \in C(\mathbb{D})$ by the rule

$$f_{g,\alpha}(z) = f(g^{-1}z)W(g0, z, \alpha), \quad z \in \mathbb{D}, g \in G. \tag{2.5}$$

Consider a differential operator $\mathfrak{L}_\alpha$ acting on the space $C^2(\mathbb{D})$ in the following way:

$$\mathfrak{L}_\alpha = 4(1-|z|^2)^2 \frac{\partial^2}{\partial z \partial \overline{z}} - 4\alpha(1-|z|^2)\left(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}\right) - 4\alpha^2|z|^2 Id, \tag{2.6}$$

where Id is the identity operator. Below, we will show that the operator $\mathfrak{L}_\alpha$ is invariant relative to α-shifts (2.5) (see § 4). Let

$$\mathcal{H}_{\lambda,\alpha}(z) = (1-|z|^2)^{\frac{1-i\lambda}{2}} F\left(\alpha + \frac{1-i\lambda}{2}, \frac{1-i\lambda}{2} - \alpha; 1; |z|^2\right), \quad z \in \mathbb{D}, \tag{2.7}$$

where $\lambda \in \mathbb{C}$, and F is the Gauss hypergeometric function. Using the formulas of differentiation for the hypergeometric function (see [16, formulas 2.8(25), 2.8(26)]) and relations (2.6) and (2.7), we obtain

$$(\mathfrak{L}_\alpha \mathcal{H}_{\lambda,\alpha})(z) = -(\lambda^2 + 4\alpha^2 + 1)\mathcal{H}_{\lambda,\alpha}(z). \tag{2.8}$$

The main result of the present work is the following theorem.

Theorem 2.1. *Let $\alpha \in \mathbb{R}$, and let $\mathcal{U}$ be a nonzero subspace in $C(\mathbb{D})$ invariant relative to all shifts*

$$f \to f_{g,\alpha}, \quad g \in G.$$

Then $\mathcal{H}_{\lambda,\alpha} \in \mathcal{U}$ for some $\lambda \in \mathbb{C}$.

The proof of Theorem 2.1 is based on the development of authors' methods proposed in [15] and consists in the following. The group G is a group of motions of the hyperbolic plane $\mathbb{H}^2$ realized in the form of a disk $\mathbb{D}$ with corresponding Riemannian structure (see [17, Introduction, § 4]). The problem is reduced to the study of the operators of generalized convolution of the form

$$f \to \int_{\mathbb{H}^2} f(z)\, K(g^{-1}z)\, W(g0, z, \alpha)\, d\mu(z), \quad g \in G, \tag{2.9}$$

where $K \in C(\mathbb{H}^2)$ is a radial function with compact support, and $d\mu$ is the measure on $\mathbb{H}^2$ that is invariant relative to the group G.

To study operators (2.9), we introduce transmutation operators that establish a homeomorphism between the space of smooth radial functions on $\mathbb{H}^2$ and the space of even functions from $C^\infty(\mathbb{R}^1)$. In some generalized meaning, those operators commute with the operator of generalized convolution, which allows us to perform a further reduction of the problem to a one-dimensional case. Moreover, we will use the known possibility to apply the spectral analysis in $C^\infty(\mathbb{R}^1)$ (see [1]).

A realization of the indicated approach requires the development of the apparatus related to the study of generalized spherical functions on $\mathbb{H}^2$ and corresponding spherical transformations on the space of radial functions from $C^\infty(\mathbb{H}^2)$ with compact supports. The necessary auxiliary material and corresponding results are presented in §§ 3–6.

Other aspects of the theory of convolution operators on groups can be found, for example, in [6, Part 2, Chapt. 8], [11].

3. Basic notations

As known, for any $g \in G$, the numbers $\tau, z \in \mathbb{C}$ exist, are uniquely determined, and are such that $|\tau| = 1$, $|z| < 1$, and

$$gw = \tau \frac{w - z}{1 - \overline{z}w} \tag{3.10}$$

for all $w \in \mathbb{D}$. Mappings (3.10) are motions in the Poincaré model of a hyperbolic plane $\mathbb{H}^2$ realized in the form of a disk$\mathbb{D}$ (see, e.g., [17, Introduction, § 4]). The hyperbolic distance d between the points $z_1, z_2 \in \mathbb{H}^2$ in this model is defined by the equality

$$d(z_1, z_2) = \frac{1}{2} \ln \frac{|1 - \overline{z}_1 z_2| + |z_2 - z_1|}{|1 - \overline{z}_1 z_2| - |z_2 - z_1|}.$$

In particular,

$$d(z, 0) = \frac{1}{2} \ln \frac{1 + |z|}{1 - |z|} \quad \text{and} \quad |z| = \operatorname{th} d(z, 0), \quad z \in \mathbb{H}^2.$$

The distance d and the hyperbolic measure $d\mu$ on $\mathbb{H}^2$ defined by the equality

$$d\mu(z) = \frac{i}{2} \frac{dz \wedge \overline{dz}}{(1 - |z|^2)^2}$$

are invariant relative to the group G.

For $r > 0$, we denote, by the symbol B_r, an open hyperbolic disk with radius r centered at zero, i.e.,

$$B_r = \{z \in \mathbb{H}^2 : d(0, z) < r\}.$$

For $r \geq 0$, we denote $\overline{B}_r = \{z \in \mathbb{H}^2 : d(0, z) \leq r\}$.

Let $\mathcal{D}(\mathbb{H}^2)$ (respectively, $\mathcal{D}(\mathbb{R}^1)$) be a set of all functions with compact supports from $C^\infty(\mathbb{H}^2)$ (respectively, $C^\infty(\mathbb{R}^1)$) with standard topology (see, e.g., [17, Chapt. 2, § 2, i. 2]). For a function $f \in \mathcal{D}(\mathbb{H}^2)$, we set

$$r(f) = \min \{r > 0 : \operatorname{supp} f \subset \overline{B}_r\},$$

where $\operatorname{supp} f$ is the support of f. For $f \in \mathcal{D}(\mathbb{R}^1)$, we define the quantity $r(f)$ by the equality

$$r(f) = \min \{r > 0 : \operatorname{supp} f \subset [-r, r]\}.$$

The symbols $C_\natural(\mathbb{H}^2)$, $C_\natural^\infty(\mathbb{H}^2)$, and $\mathcal{D}_\natural(\mathbb{H}^2)$ stand for, respectively, the spaces of radial functions from $C(\mathbb{H}^2)$, $C^\infty(\mathbb{H}^2)$, and $\mathcal{D}(\mathbb{H}^2)$ with induced topology. Analogously, the symbols $C_\natural^\infty(\mathbb{R}^1)$ and $\mathcal{D}_\natural(\mathbb{R}^1)$ stand for the spaces of even functions from $C^\infty(\mathbb{R}^1)$ and $\mathcal{D}(\mathbb{R}^1)$, respectively. For $f \in C_\natural(\mathbb{H}^2)$, we define a function f_0 on $[0, +\infty)$ by the equality

$$f_0(|z|) = f(z), \quad z \in \mathbb{H}^2. \tag{3.11}$$

As usual, we denote, by the symbol $\widehat{h}$, the Fourier transformation of a function $h \in L^1(\mathbb{R}^1)$, i.e.,

$$\widehat{h}(\lambda) = \int_{-\infty}^{+\infty} h(t) e^{-i\lambda t} dt, \quad \lambda \in \mathbb{R}^1.$$

If $h_1, h_2 \in L^1(\mathbb{R}^1)$, then the convolution $h_1 * h_2 \in L^1(\mathbb{R}^1)$ is defined, and

$$\widehat{h_1 * h_2} = \widehat{h}_1 \widehat{h}_2. \tag{3.12}$$

We now give some identities for the function $W(\zeta, z, \alpha)$ which will be necessary in what follows.

Lemma 3.1. *Let $\zeta, z \in \mathbb{D}$, $g \in G$. Then*

$$W(\zeta, z, \alpha) = W(-\zeta, -z, \alpha) = W(z, \zeta, -\alpha) = W(\overline{z}, \overline{\zeta}, \alpha), \tag{3.13}$$

$$W(g0, \zeta, \alpha) W(z, g0, \alpha) = W(g^{-1}\zeta, g^{-1}z, \alpha) W(z, \zeta, \alpha). \tag{3.14}$$

In particular,

$$W(g0, gz, \alpha) = W(z, g^{-1}0, \alpha). \tag{3.15}$$

Proof. The equalities in (3.13) follow directly from (2.4). To prove (3.14), we write the action of g in the form

$$gz = \frac{az + b}{\overline{b}z + \overline{a}}, \quad \text{where } a, b \in \mathbb{C}, \quad |a|^2 - |b|^2 = 1. \tag{3.16}$$

Then $g0 = b/\overline{a}$,

$$g^{-1}z = \frac{\overline{a}z - b}{-\overline{b}z + a},$$

and (3.14) is obtained by the dirct calculation. Setting $\zeta = 0$ in (3.14) and replacing g by g^{-1}, we arrive at (3.15). □

Relation (3.14) indicates that a superposition of α-shifts is transformed according to the formula

$$(f_{g,\alpha})_{h,\alpha} = W(g0, h^{-1}0, \alpha) f_{hg,\alpha}, \quad h, g \in G.$$

4. Invariant operator $\mathfrak{L}_\alpha$

Our subsequent target is to prove the invariance of the operator $\mathfrak{L}_\alpha$ relative to α-shifts (2.5). It is convenient to divide the proof of this assertion into several lemmas. We consider that the action of an element $g \in G$ on a point $z \in \mathbb{H}^2$ can be written in the form (3.16). For convenience, we set

$$u_\alpha(z) = W(g0, z, \alpha) = \exp\left(2i\alpha \arg\left(1 - \frac{\overline{a}\,\overline{b}}{|a|^2}\, z\right)\right).$$

We recall that the Laplace–Beltrami operator $L_{\mathbb{H}^2}$ on a hyperbolic plane takes the form

$$L_{\mathbb{H}^2} = 4(1 - |z|^2)^2 \frac{\partial^2}{\partial z\, \partial \overline{z}} \tag{4.17}$$

(see [17, Introduction]).

Lemma 4.1. *The equality*

$$L_{\mathbb{H}^2}(u_\alpha)(z) = -4\alpha^2 |a|^2 |b|^2 \left(\frac{1 - |z|^2}{|a|^2 - ab\overline{z}}\right)^2 u_{\alpha-1}(z) \tag{4.18}$$

holds.

Proof. Using the relations

$$\frac{\partial u_\alpha}{\partial z} = -\frac{\alpha\, \overline{a}\, \overline{b}}{|a|^2 - ab\overline{z}} u_{\alpha-1}(z), \quad \frac{\partial u_\alpha}{\partial \overline{z}} = \frac{\alpha\, a\, b}{|a|^2 - ab\overline{z}} u_\alpha(z), \tag{4.19}$$

we find

$$\frac{\partial^2 u_\alpha}{\partial z \partial \overline{z}} = \frac{\alpha\, a\, b}{|a|^2 - ab\overline{z}} \frac{\partial u_\alpha}{\partial z} = -\frac{\alpha^2 |a|^2 |b|^2}{(|a|^2 - ab\overline{z})^2} u_{\alpha-1}(z).$$

This result and relation (4.17) yield (4.18). □

Lemma 4.2. *Let* $\Phi(z) = f(g^{-1}z)\, u_\alpha(z)$. *Then*

$$\frac{\partial \Phi}{\partial z} = \frac{\partial f}{\partial z}(g^{-1}z)\frac{u_\alpha(z)}{(a-\bar{b}z)^2} - \alpha\bar{a}\bar{b}\, f(g^{-1}z)\,\frac{u_{\alpha-1}(z)}{|a|^2 - ab\bar{z}}, \tag{4.20}$$

$$\frac{\partial \Phi}{\partial \bar{z}} = \frac{\partial f}{\partial \bar{z}}(g^{-1}z)\frac{u_\alpha(z)}{(\bar{a}-b\bar{z})^2} + \alpha ab\, f(g^{-1}z)\,\frac{u_\alpha(z)}{|a|^2 - ab\bar{z}}. \tag{4.21}$$

Proof. Since g^{-1} is a holomorphic mapping, we have

$$\frac{\partial}{\partial z}(f\circ g^{-1}) = \frac{\partial f}{\partial z}(g^{-1}z)\,\frac{\partial g^{-1}}{\partial z} = \frac{\partial f}{\partial z}(g^{-1}z)\,\frac{1}{(a-\bar{b}z)^2}, \tag{4.22}$$

$$\frac{\partial}{\partial \bar{z}}(f\circ g^{-1}) = \frac{\partial f}{\partial \bar{z}}(g^{-1}z)\,\frac{\partial \overline{g^{-1}}}{\partial \bar{z}} = \frac{\partial f}{\partial \bar{z}}(g^{-1}z)\,\frac{1}{(\bar{a}-b\bar{z})^2}. \tag{4.23}$$

Equalities (4.22), (4.23), and (4.19) yield (4.20) and (4.21). □

Lemma 4.3. *Let* $a_1(z) = -4\alpha^2|z|^2$, *and let* $a_2(z) = -4\alpha(1-|z|^2)$. *Then*

$$a_1(g^{-1}z) = a_1(z) + \frac{4\alpha^2|a|^2(1-|z|^2)}{\big||a|^2 - ab\bar{z}\big|^2}\Big(2z(\mathrm{Re}(ab) - |b|^2\mathrm{Re}z) - |b|^2(1-|z|^2)\Big), \tag{4.24}$$

$$(a-\bar{b}z)^2 a_2(g^{-1}z)g^{-1}(z) = \frac{4\alpha\, ab(1-|z|^2)^2}{|a|^2 - ab\bar{z}} + a_2(z)z, \tag{4.25}$$

$$(\bar{a}-b\bar{z})^2 a_2(g^{-1}z)\overline{g^{-1}(z)} = \frac{4\alpha\, \bar{a}\bar{b}(1-|z|^2)^2}{|a|^2 - \bar{a}\bar{b}z} + a_2(z)\bar{z}. \tag{4.26}$$

Proof. Relations (4.24)–(4.26) are obtained by the direct calculation with the use of the equality $|a|^2 - |b|^2 = 1$. □

The following proposition concerns the above-mentioned invariance of $\mathfrak{L}_\alpha$ relative to α-shifts.

Lemma 4.4. *The operator* $\mathfrak{L}_\alpha$ *possesses the generalized property of the invariance relative to the action of the group* G:

$$\mathfrak{L}_\alpha(f(g^{-1}z)\, u_\alpha(z)) = (\mathfrak{L}_\alpha f)(g^{-1}z)\, u_\alpha(z). \tag{4.27}$$

Proof. By the formula describing the action of the Laplace–Beltrami operator on a product of functions (see [17, Chapt. 2, § 2, formula (17)]), we have

$$L_{\mathbb{H}^2}((f\circ g^{-1})\, u_\alpha) = (f\circ g^{-1})L_{\mathbb{H}^2}(u_\alpha) + u_\alpha L_{\mathbb{H}^2}(f\circ g^{-1})$$
$$+ 4(1-|z|^2)^2\left(\frac{\partial u_\alpha}{\partial z}\,\frac{\partial}{\partial \bar{z}}(f\circ g^{-1}) + \frac{\partial u_\alpha}{\partial \bar{z}}\,\frac{\partial}{\partial z}(f\circ g^{-1})\right).$$

Since g^{-1} is a motion in $\mathbb{H}^2$, we have

$$L_{\mathbb{H}^2}(f\circ g^{-1}) = (L_{\mathbb{H}^2} f)\circ g^{-1}.$$

Then relations (4.18)–(4.21) yield

$$L_{\mathbb{H}^2}((f\circ g^{-1})u_\alpha) = u_\alpha(L_{\mathbb{H}^2} f)\circ g^{-1} - 4\alpha^2|a|^2|b|^2\left(\frac{1-|z|^2}{|a|^2 - ab\bar{z}}\right)^2\cdot u_{\alpha-1}(z)\, f(g^{-1}z) + 4(1-|z|^2)^2$$
$$\times\left(\frac{\alpha\, ab\, u_\alpha(z)}{(|a|^2 - ab\bar{z})(a-\bar{b}z)^2}\,\frac{\partial f}{\partial z}(g^{-1}z) - \frac{\alpha\,\bar{a}\bar{b}\, u_{\alpha-1}(z)}{(|a|^2 - ab\bar{z})(\bar{a}-b\bar{z})^2}\,\frac{\partial f}{\partial \bar{z}}(g^{-1}z)\right). \tag{4.28}$$

Setting

$$A_1 = -4\alpha^2|z|^2\mathrm{Id}, \quad A_2 = -4\alpha(1-|z|^2)\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right), \tag{4.29}$$

and using Lemma 4.2, we get

$$A_1((f\circ g^{-1})u_\alpha)(z) = a_1(z)\,u_\alpha(z)\,f(g^{-1}z), \tag{4.30}$$

$$A_2((f\circ g^{-1})u_\alpha)(z) = a_2(z)\Bigg(\frac{z\,u_\alpha(z)}{(a-\bar{b}z)^2}\,\frac{\partial f}{\partial z}(g^{-1}z) - \frac{\bar{z}\,u_\alpha(z)}{(\bar{a}-b\bar{z})^2}\,\frac{\partial f}{\partial \bar{z}}(g^{-1}z)$$
$$-\frac{\alpha\,z}{|a|^2-ab\,\bar{z}}\left(\bar{a}\bar{b}u_{\alpha-1}(z)+ab\,u_\alpha(z)\right)f(g^{-1}z)\Bigg). \tag{4.31}$$

Relations (4.28)–(4.31) yield

$$\mathfrak{L}_\alpha\left((f\circ g^{-1})u_\alpha\right) = u_\alpha(L_{\mathbb{H}^2}f)\circ g^{-1} + c_1(z)\,f(g^{-1}z) + c_2(z)\,\frac{\partial f}{\partial z}(g^{-1}z) + c_3(z)\,\frac{\partial f}{\partial \bar{z}}(g^{-1}z), \tag{4.32}$$

where

$$c_1(z) = a_1(z)\,u_\alpha(z) - 4\alpha^2|a|^2|b|^2\left(\frac{1-|z|^2}{|a|^2-ab\bar{z}}\right)^2 u_{\alpha-1}(z) - \frac{\alpha\,z\,a_2(z)}{|a|^2-ab\bar{z}}(\bar{a}\bar{b}\,u_{\alpha-1}(z)+ab\,u_\alpha(z)),$$

$$c_2(z) = \frac{4\alpha\,ab(1-|z|^2)^2u_\alpha(z)}{(|a|^2-ab\bar{z})(a-\bar{b}z)^2} + \frac{z\,a_2(z)\,u_\alpha(z)}{(a-\bar{b}z)^2},$$

$$c_3(z) = \frac{-4\alpha\,\bar{a}\bar{b}(1-|z|^2)^2u_{\alpha-1}(z)}{(|a|^2-ab\bar{z})(\bar{a}-b\bar{z})^2} - \frac{\bar{z}\,a_2(z)\,u_\alpha(z)}{(\bar{a}-b\bar{z})^2}.$$

On the other hand,

$$\begin{aligned}(\mathfrak{L}_\alpha f)(g^{-1}z)\,u_\alpha(z) &= u_\alpha(z)\,(L_{\mathbb{H}^2}f)(g^{-1}z) + u_\alpha(z)\,a_1(g^{-1}z)\,f(g^{-1}z)\\ &+u_\alpha(z)\,a_2(g^{-1}z)g^{-1}(z)\,\frac{\partial f}{\partial z}(g^{-1}z) - u_\alpha(z)\,a_2(g^{-1}z)\overline{g^{-1}(z)}\,\frac{\partial f}{\partial \bar{z}}(g^{-1}z).\end{aligned} \tag{4.33}$$

Comparing (4.32) with (4.33) and taking Lemma 4.3 into account, we see that relation (4.27) is proper. □

5. "α"-convolution and a generalized spherical transformation on $\mathbb{H}^2$

Assume that $f_1, f_2 \in C_\natural(\mathbb{H}^2)$ and at least one of the functions f_1, f_2 has a compact support. We introduce an "α"-convolution $f_1 \times f_2$ in the following way:

$$(f_1\times f_2)(z) = \int\limits_{\mathbb{H}^2} f_2(\zeta)\,f_1\left(\frac{z-\zeta}{1-\zeta\bar{z}}\right)\,W(z,\zeta,\alpha)\,d\mu(\zeta), \quad z\in\mathbb{H}^2. \tag{5.34}$$

Relation (5.34) and the invariance of the measure $d\mu$ imply that $f_1\times f_2 \in C_\natural(\mathbb{H}^2)$ and

$$f_1\times f_2 = f_2\times f_1.$$

In addition, using (3.14), we get

$$(f_1)_{g,\alpha}\times f_2 = (f_1\times f_2)_{g,\alpha}$$

for any $g \in G$.

If $f_i \in C_\natural(\mathbb{H}^2)$, $i = 1, 2, 3$, and if at least two of the functions f_i have compact supports, then relation (5.34) yields

$$(f_1 \times f_2) \times f_3 = f_1 \times (f_2 \times f_3).$$

Lemma 5.1. *Let $f_1, f_2 \in C_\natural \cap C^2(\mathbb{H}^2)$, and let at least one of the functions f_1, f_2 has a compact support. Then*

$$\mathfrak{L}_\alpha(f_1 \times f_2) = f_1 \times \mathfrak{L}_\alpha f_2 = (\mathfrak{L}_\alpha f_1) \times f_2. \tag{5.35}$$

Proof. Relation (5.34) can be rewritten in the form

$$(f_1 \times f_2)(z) = \int\limits_G f_1(g0)\, f_2(g^{-1}z)\, W(g0, z, \alpha)\, dg, \tag{5.36}$$

where dg is the Haar measure on G normalized by the condition

$$\int\limits_G f(g0)dg = \int\limits_{\mathbb{D}} f(z)d\mu(z), \quad f \in \mathcal{D}(\mathbb{D})$$

(see [17, Introduction, § 4, i. 3]). Then the first equality in (5.35) is verified by the direct calculation with the use of (5.36) and (2.6). The second equality in (5.35) follows from the first one by virtue of the commutativity of the "α''-convolution. □

Let the function $f \in C_\natural(\mathbb{H}^2)$ has a compact support. For $\lambda \in \mathbb{C}$, we introduce the generalized spherical transformation

$$\mathcal{F}(f)(\lambda) = \int\limits_{\mathbb{H}^2} f(z)\mathcal{H}_{\lambda,\alpha}(z)d\mu(z), \tag{5.37}$$

where, as above, $\alpha \in \mathbb{R}$ is fixed. For $\alpha = 0$, this transformation coincides with the classical spherical transformation $\widetilde{f}$ of a function f on $\mathbb{H}^2$ (see [17, Introduction, § 4, i. 2]).

Lemma 5.2. *Let functions f_1 and f_2 of the class $C_\natural(\mathbb{H}^2)$ have compact supports. Then*

$$\mathcal{F}(f_1 \times f_2) = \mathcal{F}(f_1)\, \mathcal{F}(f_2). \tag{5.38}$$

Proof. Equality (5.37) and the associativity of the "α''-convolution yield

$$\mathcal{F}(f_1 \times f_2)(\lambda) = \int\limits_{\mathbb{H}^2} (f_1 \times f_2)(z)\mathcal{H}_{\lambda,\alpha}(z)\, d\mu(z) = \int\limits_{\mathbb{H}^2} f_1(z)F(z)\, d\mu(z), \tag{5.39}$$

where

$$F(z) = \mathcal{H}_{\lambda,\alpha} \times f_2. \tag{5.40}$$

Using Lemma 5.1 and (2.8), we get

$$\mathfrak{L}_\alpha F = (\mathfrak{L}_\alpha \mathcal{H}_{\lambda,\alpha}) \times f_2 = -(\lambda^2 + 4\alpha^2 + 1)F. \tag{5.41}$$

Equality (5.40) indicates that the function F is continuous at zero. Comparing this fact with relation (5.41), we may conclude that

$$F(z) = F(0)\mathcal{H}_{\lambda,\alpha}(z) = \mathcal{F}(f_2)\mathcal{H}_{\lambda,\alpha}(z) \tag{5.42}$$

(see [17, Introduction, the proof of Lemma 3.7]). Then relation (5.39) yields

$$\mathcal{F}(f_1{\times}f_2)(\lambda) = \mathcal{F}(f_2) \int\limits_{\mathbb{H}^2} f_1(z)\mathcal{H}_{\lambda,\alpha}(z)\, d\mu(z) = \mathcal{F}(f_1)\, \mathcal{F}(f_2),$$

which was to be proved. □

Consider the function

$$a(\lambda) = \frac{\Gamma\left(\frac{i\lambda-2\alpha+1}{2}\right)\Gamma\left(\frac{i\lambda+2\alpha+1}{2}\right)}{2^{2\alpha+1-i\lambda}\,\Gamma(i\lambda)}, \quad \lambda \in \mathbb{C}. \tag{5.43}$$

The properties of the gamma-function imply that the function $a(\lambda)$ is meromorphic in $\mathbb{C}$. Let $\mathcal{P}$ be a set of singular points of the function $a(-\lambda)$ in the half-plane $\{\lambda \in \mathbb{C} : \mathrm{Im}\lambda > 0\}$. We note that, for $\alpha = 0$, the set $\mathcal{P}$ is empty. We set

$$\tau(\lambda) = -i \operatorname*{res}_{z=\lambda}(a(z)\, a(-z)), \quad \lambda \in \mathcal{P}.$$

We now obtain the inversion formula for the transformation $\mathcal{F}$. Below, all sums with the empty set of summation indices are considered to be zero.

Lemma 5.3. *Let $f \in \mathcal{D}_\natural(\mathbb{H}^2)$. Then*

$$f(z) = \frac{2^{4\alpha}}{\pi^2} \int\limits_0^\infty \mathcal{F}(f)(\lambda)\, \mathcal{H}_{\lambda,\alpha}(z)\, |a(\lambda)|^2 d\lambda + \frac{2^{4\alpha+1}}{\pi} \sum_{\lambda\in\mathcal{P}} \tau(\lambda)\, \mathcal{F}(f)(\lambda)\, \mathcal{H}_{\lambda,\alpha}(z). \tag{5.44}$$

In this case, the integral in equality (5.44) *converges absolutely.*

Proof. Let $\alpha \geq 0$. The Stirling formula and relation (5.43) yield

$$|a(\lambda)| \leq c_1(1 + |\lambda|)^{1/2}, \tag{5.45}$$

where $c_1 > 0$ is independent of λ. In addition, for $z \in \mathbb{H}^2$, $t = \operatorname{arth}|z|$, we have

$$\mathcal{H}_{\lambda,\alpha}(z) = \frac{2^{\,3/2}}{\pi} \int\limits_0^t (\mathrm{ch}2t - \mathrm{ch}2\xi)^{-\frac{1}{2}} F\left(2\alpha, -2\alpha; \frac{1}{2}; \frac{\mathrm{ch}t - \mathrm{ch}\xi}{2\mathrm{ch}t}\right) \cos\lambda\xi\, d\xi \tag{5.46}$$

(see [5, Proposition 7.3]). We set $h(t,\xi) = \mathrm{ch}2t - \mathrm{ch}t$ for $0 < \xi < t/2$ and $h(t,\xi) = 2(t-\xi)\,\mathrm{sh}\,t$ for $t/2 \leq \xi < t$. Applying the Lagrange mean-value theorem, we get the estimate $h(t,\xi) \leq \mathrm{ch}2t - \mathrm{ch}2\xi$. Then the expansion of the function F in a hypergeometric series yields

$$|\mathcal{H}_{\lambda,\alpha}(z)| \leq c_2 \int\limits_0^t (h(t,\xi))^{-1/2} d\xi \leq c_3, \tag{5.47}$$

where c_2 and $c_3 > 0$ depend only on α. Further, we have

$$\mathcal{F}(f)(\lambda) = 2\pi \int\limits_0^{r(f)} \frac{\rho}{(1-\rho^2)^2}\, f_0(\rho)\, H_{\lambda,\alpha}(\rho)\, d\rho, \tag{5.48}$$

where $H_{\lambda,\alpha}(\rho) = \mathcal{H}_{\lambda,\alpha}(z)$ for $\rho = |z|$, $z \in \mathbb{H}^2$ (see (3.11)). Integrating in (5.48) by parts with the use of [16, formulas 2.8(25), 2.8(26)], we obtain the equality

$$\mathcal{F}(f)(\lambda) = \big((1 - i\lambda - 2\alpha)(1 - i\lambda + 2\alpha - 2)\big)^{-m}\mathcal{F}\big((d\,D)^m f_0(\rho)\big)(\lambda) \tag{5.49}$$

for any $m \in \mathbb{N}$. Here, the differential operators d and D act on the function $h \in C^1(0,1)$ in the following way:

$$(d\,h)(\rho) = \frac{(1-\rho^2)^{2-\alpha}}{\rho}\frac{d}{d\rho}\left(\frac{\rho h(\rho)}{(1-\rho^2)^{1-\alpha}}\right),$$

$$(Dh)(\rho) = (1-\rho^2)^{\alpha+1}\frac{d}{d\rho}\left(\frac{h(\rho)}{(1-\rho^2)^{\alpha}}\right).$$

Since $f \in \mathcal{D}_\natural(\mathbb{H}^2)$, relations (5.47) and (5.49) imply that the function $\mathcal{F}(f)(\lambda)$ decreases, as $\lambda \to +\infty$, faster than any negative power of λ. It follows from whence and estimates (5.45) and (5.47) that the integral in (5.44) converges absolutely. Let

$$\varphi_{\lambda,\alpha}(t) = F\left(\frac{2\alpha+1-i\lambda}{2}, \frac{2\alpha+1+i\lambda}{2}; 1; -\mathrm{sh}^2 t\right),$$

$$\triangle_\alpha(t) = 2^{4\alpha+2}\,\mathrm{sh}t\,\mathrm{ch}t.$$

In view of [16, formula 2.9(3)], equality (5.37) for the function f can be written in the form

$$\mathcal{F}(f)(\lambda) = \int_0^\infty \Phi(t)\,\varphi_{\lambda,\alpha}(t)\,\triangle_\alpha(t)\,dt,$$

where

$$\Phi(t) = 2^{-4\alpha-1}\pi\, f_0(\mathrm{th}t)(\mathrm{ch}t)^{-2\alpha}.$$

From whence, using [18, Theorem 2.3], we obtain (5.44). □

Corollary 5.1. *Let $f \in C_\natural(\mathbb{H}^2)$ and let it have a compact support. Then if $\mathcal{F}(f)(\lambda) = 0$ for all $\lambda > 0$, then $f = 0$.*

Proof. Let $\varphi \in \mathcal{D}_\natural(\mathbb{H}^2)$. Using Lemma 5.2, we have

$$\mathcal{F}(f \times \varphi)(\lambda) = \mathcal{F}(f)(\lambda)\mathcal{F}(\varphi)(\lambda) = 0$$

for all $\lambda > 0$. By virtue of the analyticity of the function $\mathcal{F}(f \times \varphi)$ in λ, the last equality is valid for all $\lambda \in \mathbb{C}$. Since $f \times \varphi \in \mathcal{D}_\natural(\mathbb{H}^2)$, Lemma 5.3 yields $f \times \varphi = 0$ in $\mathbb{H}^2$. From whence and from the arbitrariness of φ, we conclude that $f = 0$. □

Let $f \in \mathcal{D}_\natural(\mathbb{H}^2)$. As was mentioned above in the proof of Lemma 5.3, $\mathcal{F}(f)(\lambda) = O(\lambda^{-\gamma})$ as $\lambda \to +\infty$ for any fixed $\gamma > 0$. In addition, relation (5.46) implies that the function $\mathcal{F}(f)$ is entire and satisfies the estimate

$$|\mathcal{F}(f)(\lambda)| \le ce^{r(f)|\mathrm{Im}\lambda|}$$

with a constant $c > 0$ independent of λ. Then the Paley–Wiener theorem implies that (see [19, Theorem 7.3.1]) there exists a function $\Lambda(f) \in \mathcal{D}_\natural(\mathbb{R}^1)$ such that $\widehat{\Lambda(f)} = \mathcal{F}(f)$ and $\Lambda(f) = 0$ outside the interval $[-r(f), r(f)]$.

6. Transmutation mapping $\mathfrak{A}$

For $f \in \mathcal{D}_\natural(\mathbb{H}^2)$, $t \in \mathbb{R}^1$, we set

$$\mathfrak{A}(f)(t) = \frac{2^{2\alpha}}{\pi^2} \int\limits_0^\infty \mathcal{F}(f)(\lambda)|a(\lambda)|^2 \cos(\lambda t) d\lambda + \frac{2^{4\alpha+1}}{\pi} \sum_{\lambda \in \mathcal{P}} \tau(\lambda)\, \mathcal{F}(f)(\lambda) \cos(\lambda t). \tag{6.50}$$

It is seen from the proof of Lemma 5.3 that the function $\mathfrak{A}(f)$ belongs to $C^\infty_\natural(\mathbb{R}^1)$. In order to extend the mapping $\mathfrak{A} : \mathcal{D}_\natural(\mathbb{H}^2) \to C^\infty(\mathbb{R}^1)$ onto the space $C^\infty_\natural(\mathbb{H}^2)$, we need the following lemma.

Lemma 6.1. *Let $f \in \mathcal{D}_\natural(\mathbb{H}^2)$, $r \in (0, +\infty)$. Then the following assertions are equivalent:*

(i) $f = 0$ *in* B_r;

(ii) $\mathfrak{A}(f) = 0$ *on* $(-r; r)$.

Proof. Relations (5.44), (6.50), and (5.46) yield

$$f_0(\text{th}\rho) = \frac{2^{\,3/2}}{\pi} \int\limits_0^\rho \mathfrak{A}(f)(t)\, (\text{ch}2\rho - \text{ch}2t)^{-\frac{1}{2}} F\left(2\alpha, -2\alpha; \frac{1}{2}; \frac{\text{ch}\rho - \text{ch}t}{2\text{ch}\rho}\right) dt \tag{6.51}$$

for any $\rho > 0$. If $f = 0$ in B_r, we get

$$\int\limits_0^\rho \mathfrak{A}(f)(t)\, K(\rho, t)(\rho - t)^{-\frac{1}{2}} dt = 0$$

for some function $K \in C^\infty(\mathbb{R}^2)$. By virtue of the evenness of the function $\mathfrak{A}(f)(t)$, the given integral equation (see [20, Chapt. 3, §4, Theorem 4.6]) yields $\mathfrak{A}(f)(t) = 0$ on $(-r, r)$. Therefore, we have shown the validity of the implication $(i) \Rightarrow (ii)$. The inverse implication is obvious in view of equality (6.51). □

The assertion of Lemma 6.1 allows us to continue the operator $\mathfrak{A}$ onto the space $C^\infty_\natural(\mathbb{H}^2)$ by the formula

$$\mathfrak{A}(f)(t) = \mathfrak{A}(f\eta)(t), \quad f \in C^\infty_\natural(\mathbb{H}^2),\ t \in \mathbb{R}^1, \tag{6.52}$$

where η is any function of the class $\mathcal{D}_\natural(\mathbb{H}^2)$ that is equal to 1 in $B_{|t|+\varepsilon}$ for some $\varepsilon > 0$. Then $\mathfrak{A}(f) \in C^\infty_\natural(\mathbb{R}^1)$, and

$$\mathfrak{A}(f|_{B_r}) = \mathfrak{A}(f)|_{(-r,r)} \quad \text{for any} \quad r > 0.$$

Theorem 6.1. *The following assertions are valid:*

(i) If $f_1 \in C^\infty_\natural(\mathbb{H}^2), f_2 \in \mathcal{D}_\natural(\mathbb{H}^2)$, then the transmutation relation

$$\mathfrak{A}(f_1 \times f_2) = \mathfrak{A}(f_1) * \Lambda(f_2)$$

holds.

(ii) If $\lambda \in \mathbb{C}$, then

$$\mathfrak{A}(\mathcal{H}_{\lambda,\alpha})(t) = \cos \lambda t.$$

(iii) The transformation $\mathfrak{A}$ realizes the homeomorphism between $C_\natural^\infty(\mathbb{H}^2)$ and $C_\natural^\infty(\mathbb{R}^1)$.

Proof. The definition of the operator $\mathfrak{A}$ on the space $C_\natural^\infty(\mathbb{H}^2)$ shows (see (6.52)) that it is sufficient to prove assertion (i) for $f_1 \in \mathcal{D}_\natural(\mathbb{H}^2)$. However, in this case, it is a simple consequence of relations (6.50), (5.38), and (3.12). Then relation (6.52) implies that equality (6.51) holds for any function $f \in C_\natural^\infty(\mathbb{H}^2)$. Relation (6.51) and the integral representation (5.46) yield

$$\int_0^\rho (\mathfrak{A}(\mathcal{H}_{\lambda,\alpha})(t) - \cos(\lambda t))\,(\mathrm{ch}2\rho - \mathrm{ch}2t)^{-\frac{1}{2}} \cdot F\left(2\alpha, -2\alpha; \frac{1}{2}; \frac{\mathrm{ch}\rho - \mathrm{ch}t}{2\mathrm{ch}\rho}\right) dt = 0$$

for any $\rho > 0$. Like the proof of Lemma 6.1, this yields (ii).

To prove (iii), we will find the inverse operator $\mathfrak{A}^{-1}$. For $F \in \mathcal{D}_\natural(\mathbb{R}^1)$, we set

$$\mathfrak{B}(F)(z) = \frac{1}{\pi}\int_0^\infty \widehat{F}(\lambda)\,\mathcal{H}_{\lambda,\alpha}(z)\,d\lambda, \quad z \in \mathbb{H}^2. \tag{6.53}$$

Let $f_1 \in \mathcal{D}_\natural(\mathbb{R}^1)$ and $f_2 \in \mathcal{D}_\natural(\mathbb{H}^2)$. Using relations (6.53), (3.12), and (5.42), we get

$$\begin{aligned}(\mathfrak{B}(f_1)\times f_2)(z) &= \frac{1}{\pi}\int_0^\infty \widehat{f_1}(\lambda)\,(\mathcal{H}_{\lambda,\alpha}\times f_2)(z)\,d\lambda = \frac{1}{\pi}\int_0^\infty \widehat{f_1}(\lambda)\,\mathcal{F}(f_2)(\lambda)\mathcal{H}_{\lambda,\alpha}(z)\,d\lambda \\ &= \frac{1}{\pi}\int_0^\infty \widehat{f_1}(\lambda)\,\widehat{\Lambda(f_2)}(\lambda)\,\mathcal{H}_{\lambda,\alpha}(z)\,d\lambda = \mathfrak{B}(f_1 * \Lambda(f_2))(z).\end{aligned} \tag{6.54}$$

Then relations (6.53) and (5.46) and the inversion formula for the Fourier cosine transformation yield the equality

$$\mathfrak{B}(F)(z) = \frac{2^{3/2}}{\pi}\int_0^{\mathrm{arth}\,\rho} F(t)\,(\mathrm{ch}2\rho - \mathrm{ch}2t)^{-\frac{1}{2}} \cdot F\left(2\alpha, -2\alpha; \frac{1}{2}; \frac{\mathrm{ch}\rho - \mathrm{ch}t}{2\mathrm{ch}\rho}\right) dt.$$

This equality implies that if $F \in \mathcal{D}_\natural(\mathbb{R}^1)$ and $r > 0$, then $F = 0$ on $(-r, r)$ iff $\mathfrak{B}(F) = 0$ in B_r (see the proof of Lemma 6.1). We now continue the operator $\mathfrak{B}$ onto the space $C_\natural^\infty(\mathbb{R}^1)$ by the formula

$$\mathfrak{B}(F)(z) = \mathfrak{B}(F\eta)(z), \quad F \in C_\natural^\infty(\mathbb{R}^1), \quad z \in \mathbb{H}^2,$$

where η is any function from $\mathcal{D}_\natural(\mathbb{R}^1)$ that is equal to 1 in some vicinity of the interval $[-\mathrm{arth}\,|z|, \mathrm{arth}\,|z|]$. The above consideration implies that such continuation is independent of η. In this case, $\mathfrak{B}(F) \in C_\natural^\infty(\mathbb{H}^2)$, $\mathfrak{B}(F|_{(-r,r)}) = \mathfrak{B}(F)|_{B_r}$ for any $r > 0$, and equality (6.54) holds for $f_1 \in C_\natural^\infty(\mathbb{R}^1), f_2 \in \mathcal{D}_\natural(\mathbb{H}^2)$. Repeating the reasoning from [5, proof of Theorem 9.5], we get $\mathfrak{B} = \mathfrak{A}^{-1}$. Hence, assertion (iii) is valid. □

7. Proof of the main result

We now prove Theorem 2.1. Let $f \in \mathcal{U}$, $f \neq 0$. For any $g \in G$, let us consider the function

$$\Phi_g(z) = \int_{-\pi}^{\pi} f_{g,\alpha}(ze^{i\varphi})d\varphi, \quad z \in \mathbb{H}^2. \tag{7.55}$$

Approximating the integral in (7.55) by Riemannian sums locally uniformly in the variable z and using the invariance of $\mathcal{U}$ relative to the group of rotations, we conclude that $\Phi_g \in \mathcal{U}$. Relation (7.55) yields

$$\Phi_g(ze^{it}) = \Phi_g(z) \tag{7.56}$$

for any $t \in \mathbb{R}^1$. In addition, since $f \neq 0$ and

$$\Phi_g(0) = \int_{-\pi}^{\pi} f_{g,\alpha}(0)d\varphi = 2\pi f(g^{-1}0),$$

we obtain that $\Phi_g(0) \neq 0$ for some $g \in G$. This result and relation (7.56) imply that there exists a nonzero radial function $u \in \mathcal{U}$. It follows from the definition of α-convolution that there exists a radial function $v \in \mathcal{D}(\mathbb{H}^2)$ such that $u \times v \neq 0$. As above, by approximating the integral in the equality

$$(u \times v)(z) = \int_{\mathbb{H}^2} v(\zeta)\, u\left(\frac{z-\zeta}{1-\zeta\overline{z}}\right) W(z,\zeta,\alpha)\, d\mu(\zeta) \tag{7.57}$$

by Riemannian sums, we have $u \times v \in \mathcal{U}$. In addition, relation (7.57) implies that the function $u \times v$ is radial and belongs to the class $C^\infty(\mathbb{H}^2)$. Thus, there exists a nonzero function $w \in \mathcal{U} \cap C^\infty_\natural(\mathbb{H}^2)$. By Theorem 6.1 (iii), $\mathfrak{A}(w) \in C^\infty_\natural(\mathbb{R}^1)$ and $\mathfrak{A}(w) \neq 0$. In addition, the closure of the set

$$\{\mathfrak{A}(w) * \psi, \quad \psi \in \mathcal{D}_\natural(\mathbb{R}^1)\}$$

in the space $C^\infty(\mathbb{R}^1)$ coincides with the image of the closure of the set

$$\{w \times \varphi, \quad \varphi \in \mathcal{D}_\natural(\mathbb{H}^2)\}$$

in $C^\infty(\mathbb{H}^2)$ under the mapping $\mathfrak{A}$. We now prove that this image contains the function $\cos \lambda t$ for some $\lambda \in \mathbb{C}$. In view of the evenness of a cosine and the function $\mathfrak{A}(w)$, it is sufficient to prove that $\cos \lambda t \in A$, where A is the closure of the set

$$\{\mathfrak{A}(w) * \psi, \quad \psi \in \mathcal{D}(\mathbb{R}^1)\}$$

in the space $C^\infty(\mathbb{R}^1)$. By the Schwartz theorem on the spectral analysis in $C^\infty(\mathbb{R}^1)$ (see [1]), the set A contains the exponential function $e^{i\lambda t}$ for some $\lambda \in \mathbb{C}$. In view of the invariance of A relative to the reflections $t \to -t$, we have an analogous assertion also for the function $\cos \lambda t$. Using Theorem 6.1 (ii), (iii), we get $\mathcal{H}_{\lambda,\alpha} \in \mathcal{U}$. Thus, Theorem 2.1 is proved. □

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