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A NOTE ON CHARACTERIZATIONS OF THE EXPONENTIAL DISTRIBUTION*

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The following classical characterization of the exponential distribution is well known. Let $X_1, X_2, \dots X_n$ be independent and identically distributed random variables. Their common distribution is exponential if and only if random variables X_1 and $n\min(X_1, \dots, X_n)$ have the same distribution. In this note we show that the characterization can be substantially simplified if the exponentiality is characterized within a broad family of distributions that includes, in particular, gamma, Weibull and generalized exponential distributions. Then the necessary and sufficient condition is the equality only expectations of these variables. A similar characterization holds for the maximum.

1. Introduction and main results

The exponential distribution plays an important role in many areas of statistics, including reliability and survival analysis; therefore there is an extensive literature on this distribution. In particular, many works are devoted to characterizations of exponentiality. A detailed survey of these characterizations can be found in [1]. References to more recent publications are in [2].

The three main generalizations of the exponential distribution are the gamma distribution, the Weibull distribution, and the generalized exponential distribution. Denote these three families by $\mathcal{G}$, $\mathcal{W}$, and $\mathcal{E}_g$ respectively. Thus, $\mathcal{G}$ is the set of all distributions with the probability density function

$$\frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta},\ x > 0,\ \alpha > 0,\ \beta > 0, \tag{1}$$

$\mathcal{W}$ is the set of all distributions with the cumulative distribution function

$$1 - e^{-(x/\beta)^\alpha},\ x \geqslant 0,\ \alpha > 0,\ \beta > 0, \tag{2}$$

and $\mathcal{E}_g$ is the set of all distributions with the cumulative distribution function

$$\left(1 - e^{-x/\beta}\right)^\alpha,\ x \geqslant 0,\ \alpha > 0,\ \beta > 0. \tag{3}$$

Parameters α and β are called (for all these three families) the shape parameter and the scale parameter respectively. The gamma distribution is widely used in various scientific fields including reliability, survival analysis, and financial mathematics. The Weibull distribution was introduced by Frćhet and is named after Weibull who used it for the statistical analysis of the strength of materials and studied properties of this distribution. The generalized exponential distribution (let us call it g-exponential) was introduced relatively recently in [3] as an alternative to the gamma and Weibull distributions for analyzing lifetime data. We denote the family of all exponential distributions by $\mathcal{E}$, that is, $\mathcal{E}$ is the set of all distributions with the cumulative distribution function

$$1 - e^{-x/\beta},\ x \geqslant 0,\ \alpha > 0,\ \beta > 0.$$

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Thus $\mathcal{E} = \mathcal{G} \cap \mathcal{W} \cap \mathcal{E}_g$, and an exponential distribution is a gamma/Weibull/g-exponential with the shape parameter 1.

Let $F(x)$ and $f(x)$ be the distribution function and the probability density function of a nonnegative random variable, $F(x) < 1$. The function

$$h(x) = \frac{f(x)}{1 - F(x)}$$

is called the hazard rate. The hazard rate is used in many theoretical and applied problems, and distributions with the monotone hazard rate are of special interest. The hazard rate of the gamma distribution, the Weibull distribution and the g-exponential distribution decreases when the shape parameter $0 < \alpha < 1$ and increases when $\alpha > 1$, that is all distributions from these families have the monotone hazard rate.

Let $X_1, \ldots, X_n$ be independent and identically distributed random variables. The following classical characterization of the exponential distribution was obtained by Desu: if X_1 is not degenerate, then $X_1 \stackrel{d}{=} n \min(X_1, \ldots, X_n)$ ($\stackrel{d}{=}$ means equal in distribution) if and only if X_1 has an exponential distribution. In this note, we prove that the characterization can be substantially simplified if the exponentiality is characterized within the class of distributions with the monotone hazard rate. In this case the necessary and sufficient condition is the equality only expectations of the variables X_1 and $n \min(X_1, \ldots, X_n)$. We obtain also a similar characterization for the maximum. The families $\mathcal{G}$, $\mathcal{W}$, $\mathcal{E}_g$ will be studied separately.

Theorem 1. *Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with the distribution F having the monotone (non-increasing or non-decreasing) hazard rate. Then $F \in \mathcal{E}$ if and only if for at least one $k \geqslant 2$ either*

$$\mathsf{E}X_1 = k\mathsf{E}\min(X_1, \ldots, X_k) \tag{4}$$

or

$$\mathsf{E}X_1 = \left(1 + \frac{1}{2} + \ldots + \frac{1}{k}\right)^{-1} \mathsf{E}\max(X_1, \ldots, X_k). \tag{5}$$

Note that Eq. (4) and (5) do not characterize the exponential distribution in the set of all distributions (and even in the set of all distributions concentrated on the positive half-line). Indeed, let X_1 have the probability density function

$$f(x) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{for } \ 0 \leqslant x \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\max(X_1, X_2)$ has the uniform on $[0, 1]$ distribution and therefore

$$\mathsf{E}X_1 = \frac{1}{3} = \frac{2}{3}\mathsf{E}\max(X_1, X_2).$$

Theorem 2. *Let X and Y be independent and identically distributed random variables having one of distributions* (1)–(3). *Then the ratio*

$$R_1(\alpha) = \frac{\mathsf{E}\min(X, Y)}{\mathsf{E}X}$$

strictly increases in α, and the ratio

$$R_2(\alpha) = \frac{\mathsf{E}\max(X, Y)}{\mathsf{E}X}$$

strictly decreases in α.

2. Proofs

Proof of the Theorem 1 is based on the following lemma.

Lemma 1. *If $F(x)$ is a distribution function with the strictly decreasing hazard rate, then*

$$n\mathsf{E}\min(X_1,\ldots,X_n) < \mathsf{E}X_1 < \left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)^{-1}\mathsf{E}\max(X_1,\ldots,X_n);$$

if $F(x)$ is a distribution function with the strictly increasing hazard rate, then

$$n\mathsf{E}\min(X_1,\ldots,X_k) > \mathsf{E}X_1 > \left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)^{-1}\mathsf{E}\max(X_1,\ldots,X_n).$$

Proof. Consider the difference $\mathsf{E}X_1 - n\mathsf{E}\min(X_1,\ldots,X_n)$.

Since the distribution function of the random variable $\min(X_1,\ldots,X_n)$ is $1-(1-F(x))^n$, the following equality holds:

$$\mathsf{E}X_1 - n\mathsf{E}\min(X_1,\ldots,X_n) = \int_0^\infty (1-F(x))dx - n\int_0^\infty (1-F(x))^n dx =$$

$$= \int_0^\infty \frac{1-F(x)}{f(x)}(1-n(1-F(x))^{n-1})f(x)dx. \quad (6)$$

We have

$$\int_0^\infty (1-n(1-F(x))^{n-1})f(x)dx = \int_0^\infty (1-n(1-F(x))^{n-1})dF(x) =$$

$$= \int_0^1 (1-n(1-u)^{n-1})du = \int_0^1 (1-nv^{n-1})dv = (v-v^n)\,\big|_0^1 = 0.$$

Consider the function $\psi(u) = 1-n(1-u)^{n-1}$. It increases in the interval $[0,1]$ and $\psi(0) = 1-n < 0$, $\psi(1) = 1 > 0$. Hence there exists a unique $u_0 \in (0,1)$ such that $\psi(u_0) = 0$, $\psi(u) < 0$, $u \in [0,u_0)$, $\psi(u) > 0$, $u \in (u_0,1]$. Let x_0 be such that $F(x_0) = u_0$. The integral in (6) is represented as a difference of two integrals of nonnegative functions:

$$\int_0^\infty \frac{1-F(x)}{f(x)}(1-n(1-F(x))^{n-1})f(x)dx =$$

$$= \int_{x_0}^\infty \frac{1-F(x)}{f(x)}(1-n(1-F(x))^{n-1})f(x)dx - \int_0^{x_0} \frac{1-F(x)}{f(x)}(n(1-F(x))^{n-1}-1)f(x)dx.$$

Let $F(x)$ be a distribution function with the decreasing hazard rate. Then $\dfrac{1-F(x)}{f(x)}$ increases. Therefore

$$\int_{x_0}^\infty \frac{1-F(x)}{f(x)}(1-n(1-F(x))^{n-1})f(x)dx > \frac{1-F(x_0)}{f(x_0)}\int_{x_0}^\infty (1-n(1-F(x))^{n-1})f(x)dx,$$

and

$$\int_0^{x_0} \frac{1-F(x)}{f(x)}(n(1-F(x))^{n-1}-1)f(x)dx < \frac{1-F(x_0)}{f(x_0)} \int_0^{x_0} (n(1-F(x))^{n-1}-1)f(x)dx.$$

This implies

$$\begin{aligned} &\mathsf{E}X_1 - n\mathsf{E}\min(X_1,\dots,X_n) > \\ &> \frac{1-F(x_0)}{f(x_0)} \left(\int_{x_0}^{\infty} (1-n(1-F(x))^{n-1})f(x)dx - \int_0^{x_0} (n(1-F(x))^{n-1}-1)f(x)dx \right) = \\ &= \frac{1-F(x_0)}{f(x_0)} \int_0^{\infty} (1-n(1-F(x))^{n-1})f(x)dx = 0. \end{aligned}$$

If $F(x)$ is a distribution function with the increasing hazard rate, then we obtain the inverse inequalities. Thus the lemma is proved for $\min(X_1,\dots,X_n)$.

Consider now $\left(1+\frac{1}{2}+\dots+\frac{1}{n}\right)\mathsf{E}X_1 - \mathsf{E}\max(X_1,\dots,X_n)$.

Since the distribution function of $\max(X_1,\dots,X_n)$ is $F^n(x)$, we obtain

$$\begin{aligned} &\left(1+\frac{1}{2}+\dots+\frac{1}{n}\right)\mathsf{E}X_1 - \mathsf{E}\max(X_1,\dots,X_n) = \\ &= \left(1+\frac{1}{2}+\dots+\frac{1}{n}\right)\int_0^{\infty}(1-F(x))dx - \int_0^{\infty}(1-F^n(x))dx = \\ &= \int_0^{\infty} \frac{1-F(x)}{f(x)} \left(1+\frac{1}{2}+\dots+\frac{1}{n} - (1+F(x)+F^2(x)+\dots+F^{n-1}(x))\right) f(x)dx. \end{aligned}$$

We have

$$\begin{aligned} &\int_0^{\infty} \left(1+\frac{1}{2}+\dots+\frac{1}{n} - (1+F(x)+F^2(x)+\dots+F^{n-1}(x))\right) f(x)dx = \\ &= \int_0^1 \left(1+\frac{1}{2}+\dots+\frac{1}{n} - (1+u+u^2+\dots+u^{n-1})\right) du = \\ &= \left(1+\frac{1}{2}+\dots+\frac{1}{n}\right) u\Big|_0^1 - \left(u+\frac{u^2}{2}+\dots+\frac{u^n}{n}\right)\Big|_0^1 = 0. \end{aligned}$$

The function

$$\chi(u) = \left(1+\frac{1}{2}+\dots+\frac{1}{n} - (1+u+u^2+\dots+u^{n-1})\right)$$

satisfies conditions

$$\chi(0) = \frac{1}{2}+\dots+\frac{1}{n} > 0, \ \chi(1) = 1+\frac{1}{2}+\dots+\frac{1}{n} - n < 0,$$

$\chi(u)$ decreases in the interval $[0,1]$. Let u_0 be the unique zero of the function $\chi(u)$ in the interval $(0,1)$, and $F(x_0)=u_0$. We have

$$\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)\mathsf{E}X_1-\mathsf{E}\max(X_1,\ldots,X_n)=$$
$$=\int_0^{x_0}\frac{1-F(x)}{f(x)}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-(1+F(x)+F^2(x)+\ldots+F^{n-1}(x))\right)f(x)dx-$$
$$-\int_{x_0}^{\infty}\frac{1-F(x)}{f(x)}\left((1+F(x)+F^2(x)+\ldots+F^{n-1}(x))-1-\frac{1}{2}-\ldots-\frac{1}{n}\right)f(x)dx.$$

For the distribution function $F(x)$ with the strictly decreasing hazard rate we have

$$\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)\mathsf{E}X_1-\mathsf{E}\max(X_1,\ldots,X_n)<$$
$$<\frac{1-F(x_0)}{f(x_0)}\int_0^{x_0}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-(1+F(x)+F^2(x)+\ldots+F^{n-1}(x))\right)f(x)dx-$$
$$-\frac{1-F(x_0)}{f(x_0)}\int_{x_0}^{\infty}\left((1+F(x)+F^2(x)+\ldots+F^{n-1}(x))-1-\frac{1}{2}-\ldots-\frac{1}{n}\right)f(x)dx=$$
$$=\frac{1-F(x_0)}{f(x_0)}\int_0^{\infty}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-(1+F(x)+F^2(x)+\ldots+F^{n-1}(x))\right)f(x)dx=0.$$

For the distribution function $F(x)$ with the increasing hazard rate, the inverse inequality holds. The lemma is proved.

For the proof of Theorem 2, we need the following proposition.

Lemma 2. *Let X and Y be independent and identically distributed random varables with the gamma distribution with the shape parameter α and the scale parameter β. Then*

$$\mathsf{E}\min(X,Y)=\alpha\beta\left(1-\frac{\Gamma(2\alpha)}{2^{2\alpha-1}\Gamma(\alpha)\Gamma(\alpha+1)}\right)=\alpha\beta\left(1-\frac{\Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}\Gamma(\alpha+1)}\right),\qquad(7)$$

$$\mathsf{E}\max(X,Y)=\alpha\beta\left(1+\frac{\Gamma(2\alpha)}{2^{2\alpha-1}\Gamma(\alpha)\Gamma(\alpha+1)}\right)=\alpha\beta\left(1+\frac{\Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}\Gamma(\alpha+1)}\right).\qquad(8)$$

Proof. Since $\mathsf{E}\min(cX,cY)=c\mathsf{E}\min(X,Y)$, we can suppose without loss of generality that $\beta=1$. Denote the cumulative distribution function and the probability density function of the gamma distribution with parameters α and 1 by $G_\alpha(x)$ and $g_\alpha(x)$ respectively. That is,

$$g_\alpha(x)=\frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x},\ x>0.$$

Let $p_\alpha(x)$ be the density of $\min(X,Y)$. Then $p_\alpha(x)=2(1-G_\alpha(x))g_\alpha(x)$. Let us show first that

$$\mathsf{E}\min(X,Y)=2\alpha\int_0^{\infty}g_\alpha(u)G_{\alpha+1}(u)du.\qquad(9)$$

Indeed, since $xg_\alpha(x) = \alpha g_{\alpha+1}(x)$, we have

$$\mathsf{E}\min(X,Y) = 2\int_0^\infty x(1-G_\alpha(x))g_\alpha(x)dx = 2\alpha\int_0^\infty \left(\int_x^\infty g_\alpha(u)du\right) g_{\alpha+1}(x)dx =$$
$$= 2\alpha\int_0^\infty g_\alpha(u)\left(\int_0^u g_{\alpha+1}(x)dx\right)du = 2\alpha\int_0^\infty g_\alpha(u)G_{\alpha+1}(u)du,$$

i.e., (9) holds. Use the following equality:

$$G_{\alpha+1}(u) = G_\alpha(u) - \frac{u^\alpha e^{-u}}{\Gamma(\alpha+1)}, \tag{10}$$

see for example [4]. From (9) and (10) we obtain

$$\mathsf{E}\min(X,Y) = 2\alpha\int_0^\infty g_\alpha(u)\left(G_\alpha(u) - \frac{u^\alpha e^{-u}}{\Gamma(\alpha+1)}\right)du = 2\alpha\int_0^\infty G_\alpha(u)dG_\alpha(u) -$$
$$- 2\alpha\int_0^\infty g_\alpha(u)\frac{u^\alpha e^{-u}}{\Gamma(\alpha+1)}du = \alpha - 2\alpha\int_0^\infty \frac{u^{2\alpha-1}e^{-2u}}{\Gamma(\alpha)\Gamma(\alpha+1)}du = \alpha - \frac{2\alpha\Gamma(2\alpha)}{2^{2\alpha}\Gamma(\alpha)\Gamma(\alpha+1)}.$$

Thus the first equality in (7) is proved.

To prove the second equality in (7), it is suffisient to use the following identity:

$$\Gamma(2z) = (2\pi)^{-1/2}2^{2z-1/2}\Gamma(z)\Gamma(z+1/2);$$

see [4].

To obtain Eq. (8) note that

$$\min(X,Y) + \max(X,Y) = X + Y;$$

therefore

$$\mathsf{E}\max(X,Y) = \mathsf{E}(X+Y) - \mathsf{E}\min(X,Y).$$

Now (8) follows from (7) and $\mathsf{E}(X+Y) = 2\alpha\beta$.

Proof of Theorem 2.

1). First suppose that X and Y have a gamma distribution. Since $\mathsf{E}X = \alpha\beta$, due to Lemma 2, the two ratios are

$$R_1(\alpha) = 1 - \frac{\Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}\Gamma(\alpha+1)}$$

and

$$R_2(\alpha) = 1 + \frac{\Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}\Gamma(\alpha+1)}.$$

Thus it is sufficient to show that the function

$$\frac{\Gamma(z+\frac{1}{2})}{\sqrt{\pi}\Gamma(z+1)}$$

decreases in z for $z > 0$. Let us use the Euler formula for the gamma function

$$\Gamma(z) = \lim_{n\to\infty}\frac{n!n^z}{z(z+1)\cdot\ldots\cdot(z+n)}.$$

Then we obtain

$$\frac{\Gamma(z+1/2)}{\Gamma(z+1)} = \lim_{n\to\infty} \frac{z+1}{z+\frac{1}{2}} \cdot \frac{z+2}{z+\frac{3}{2}} \cdot \ldots \cdot \frac{z+n+1}{z+n+\frac{1}{2}} \cdot \frac{1}{\sqrt{n}} =$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{2z+1}\right) \cdot \ldots \cdot \left(1+\frac{1}{2z+2n+1}\right) \cdot \frac{1}{\sqrt{n}}.$$

Each factor on the right-hand side (apart from $1/\sqrt{n}$) is a decreasing function of z; therefore the left-hand side decreases in z.

2) Let X and Y be independent and identically distributed random variables having a Weibull distribution with the shape parameter α and the scale parameter β. Then

$$\mathsf{E}\min(X,Y) = 2^{-1/\alpha}\mathsf{E}X$$

and

$$\mathsf{E}\max(X,Y) = (2-2^{-1/\alpha})\mathsf{E}X.$$

Similiary,

$$\mathrm{P}(2^{1/\alpha}\min(X,Y) \leqslant x) = 1-(1-F(x/2^{1/\alpha}))^2 = 1-e^{-(x/\beta)^\alpha} = \mathrm{P}(X \leqslant x),$$

that is, $2^{1/\alpha}\min(X,Y)$ has the same distribution as X, which implies the first equality. To obtain the second equality, note that

$$\mathsf{E}\max(X,Y) + \mathsf{E}\min(X,Y) = 2\mathsf{E}X.$$

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