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LIMIT THEOREMS FOR QUEUING SYSTEMS WITH REGENERATIVE DOUBLY STOCHASTIC INPUT FLOW*

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This article focuses on queuing systems with doubly stochastic Poisson regenerative input flow and an infinite number of servers. Service times have the heavy-tailed distribution. The analogs of the law of large numbers and the central limit theorem for the number of occupied servers are obtained. These theorems follow from results for systems with general doubly stochastic Poisson processes [1]. As examples, we consider systems in which the input flow is controlled by a semi-Markov modulated and Markov modulated processes.

1. Introduction

An extensive literature is devoted to queuing systems with an infinite number of servers. Since there is no opportunity to mention all of the authors, we note only articles closest to the problem studied in the article. In this regard, we note that different infinite channel systems are considered, for example, systems with restrictions [12, 13], systems in a random environment [14], infinite network systems [15], and others. This is due to the wide range of practical issues in which these models are useful, and a number of emerging interesting mathematical problems.

At first glance, infinite channel systems seem to be unrealistic, but in fact, they can be considered as models of many real-world objects, for example, in communication theory, in the study of total flow of impulsesin the description of the formation of queues at the crossroads of unmanaged highways [16], and in some problems of security (see [17] about the relation between the risk model of Cramer–Lundberg and the queuing system $G/G/\infty$). In addition, these models can be considered as approximations of systems with a large number of servers. Note that approaches used for studying such systems are useful for queuing problems in the case of the high load.

We consider an infinite server queuing system with doubly stochastic Poisson process (DSPP) input flow and random intensity that is assumed to be a stationary regenerative process. If service time has a finite mathematical expectation then there exists a proper limit distribution for the number of customers $q(t)$ in the system at time t as $t \to \infty$. But this is not the case when the distribution function of service times has a heavy tail, i.e., there is no mean of service times. In this situation $q(t)$ goes to infinity as $t \to \infty$ and the problem of asymptotic analysis of its behavior occurs. This problem was studied in [2] for queuing system $GI/G/\infty$. Our results are similar to those obtained there, but we consider another class of input flows.

The main purpose of this work is the asymptotic analysis of the number of occupied servers. Analogues of the law of large numbers and the central limit theorem for the number of occupied servers for the system with DSPP input flow were given in [1]. Here we consider a doubly stochastic Poisson regenerative (DSPR) input flow, which is a particular case of the DSPP. Limit theorems for the number of occupied servers in the system with such flow are obtained. This becomes possible due to the estimation of the covariance for the intensity of DSRP and basic results from [1].

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2. Previous results

Consider a system with an infinite number of servers. Arriving customers form a DSPP $A(t)$, which is defined as follows [3]:

$$A(t) = A^*(\Lambda(t))$$

where $\{A^*(t), t \geqslant 0\}$ is a standard Poisson process, and $\{\Lambda(t), t \geqslant 0\}$ is a stochastic process with non-decreasing right-continuous trajectories not depending on $A^*(t)$, $\Lambda(0) = 0$.

Condition 1. *The process $\Lambda(t)$ has the following form:*

$$\Lambda(t) = \int_0^t \lambda(y, \omega) dy,$$

where $\lambda(y)$ is a non-negative bounded stationary stochastic process such that

$$|r(x)| = |\mathrm{cov}(\lambda(0), \lambda(x))| \leqslant cx^{-\alpha} \text{ for } x \text{ sufficiently large.} \tag{1}$$

Here α and c are certain positive constants.

We denote $E\lambda(t) = \lambda$.

The process $\Lambda(t)$ is called the leading process and $\lambda(t)$ is the intensity of $A(t)$.

We assume that service times $\{\eta_i\}_{i=1}^{\infty}$ are independent identically distributed (i.i.d.) random variables with a distribution function $B(x)$, $\overline{B}(x) = 1 - B(x)$. Suppose that this function satisfies the following

Condition 2. *For some positive constants c_1, c_2*

$$c_1 t^{-\Delta} \leqslant \overline{B}(t) \leqslant c_2 t^{-\Delta}, 0 < \Delta < 1, \tag{2}$$

for t sufficiently large.

Denote $\beta(t) = \int_0^t \overline{B}(x)dx$. Let us formulate limit theorems for the process $q(t)$ from [1].

Theorem 1. *If Conditions 1 and 2 are fulfilled and $\left|\Delta - \frac{1}{2}\right| < \alpha - \frac{1}{2}$, then weak convergence takes place*

$$\frac{q(t) - \lambda\beta(t)}{\sqrt{\lambda\beta(t)}} \xrightarrow{w} \mathcal{N}(0, 1) \text{ as } t \to \infty. \tag{3}$$

Theorem 2. *If Conditions 1 and 2 are fulfilled and $\alpha > 2\Delta - 1$ then*

$$\frac{q(t)}{\lambda\beta(t)} \xrightarrow{p} 1 \text{ as } t \to \infty. \tag{4}$$

An essential part of DSPP flows are processes with intensity $\lambda(t)$ that is a regenerative process. It includes Markov modulated process (MMP), semi-Markov modulated process (SMMP), and some others often used in queuing theory. The purpose of this article is to establish conditions for the validity of the central limit theorem (CLT) and the law of large numbers (LLN), i.e. (3) and (4) in terms of the distribution function of the regeneration period for such processes.

3. Model description

We consider an infinite server queuing system with DSPP as input flow $A(t)$ with random intensity $\lambda(t)$. Suppose that $\lambda(t)$ is a regenerative process. This means that there is an increasing sequence of random variables $\{\theta_n\}_{n=0}^{\infty}$ such that the sequence $\{\kappa_n\}_{n=0}^{\infty}$, where

$$\{\kappa_n\}_{n=1}^{\infty} = \{\lambda(t - \theta_{n-1}),\ \theta_n - \theta_{n-1},\ t \in (\theta_{n-1}, \theta_n)\}_{n=1}^{\infty},$$

$$\kappa_0 = \{\lambda(t), \theta_0, t \in (0, \theta_0)\},$$

consists of independent random elements and $\{\kappa_n\}_{n=1}^{\infty}$ are identically distributed.

Then $\tau_n = \theta_n - \theta_{n-1}$ is the nth period of regeneration for $n \geqslant 1$ and $\tau_0 = \theta_0$. We denote $F(t) = P(\tau_1 \leqslant t)$, $\overline{F}(t) = 1 - F(t)$, and $a = E\tau_1 < \infty$. In order to use Theorem 1 and 2, in addition, we assume that $\lambda(t)$ is a stationary process. It follows from the renewal theory that in this case

$$F_0(t) = P(\tau_0 \leqslant t) = \frac{1}{a}\int\limits_0^t F(y)dy.$$

Condition 3. $\sup_t \lambda(t,\omega) \leqslant \lambda_M < \infty$ *with probability* 1.

We assume that service times are i.i.d. random variables with a distribution function $B(x)$. Below, we consider the infinite server queuing system S for which Conditions 2 and 3 are always fulfilled and we will not specify this fact.

Theorem 3. *Let for system* S

1) $\overline{F}(t) \leqslant ce^{-\alpha t}$ *for some* $\alpha > 0$, $0 < c < \infty$, *and* t *sufficiently large*; *then convergences* (3) *and* (4) *take place*;

2) $\overline{F}(t) \leqslant Ct^{-d}$ *for* $d > 1$, $C < \infty$ *and* t *sufficiently large*; *then convergence* (3) *takes place if* $\left|\Delta - \frac{1}{2}\right| < d - \frac{3}{2}$, *and convergence* (4) *takes place if* $d > 2\Delta$.

Proof. First, we state an auxiliary result.

Lemma 1. *Assume that a stationary regenerative process* $\lambda(t)$ *satisfies Condition* 3. *Then*

$$|r(t)| = |\mathrm{cov}(\lambda(0), \lambda(t))| \leqslant 4\lambda_M^2 P(\tau_0 > t) = 4\lambda_M^2 \overline{F}_0(t) \text{ for } t \geqslant 0. \quad (5)$$

Proof. For $t \geqslant 0$ we have

$$r(t) = \mathrm{cov}(\lambda(0), \lambda(t)) = E\lambda(0)\lambda(t) - E\lambda(0)E\lambda(t) =$$

$$= E\big(\lambda(0)\lambda(t)\,(\mathbb{I}_{\tau_0>t} + \mathbb{I}_{\tau_0\leqslant t})\,\big) - E\big(\lambda(0)\,(\mathbb{I}_{\tau_0>t} + \mathbb{I}_{\tau_0\leqslant t})\,\big)E\big(\lambda(t)\,(\mathbb{I}_{\tau_0>t} + \mathbb{I}_{\tau_0\leqslant t})\,\big) = I - II.$$

Here $\mathbb{I}_A$ is an indicator function of the event A.

Since $\lambda(t)$ is a regenerative process, then $\lambda(0)$ and $\lambda(t)$ are conditionally independent, provided $\{\tau_0 \leqslant t\}$. Therefore

$$E(\lambda(0)\lambda(t)|\tau_0 \leqslant t) = E(\lambda(0)|\tau_0 \leqslant t)E(\lambda(t)|\tau_0 \leqslant t).$$

Applying the last equality we get

$$I = E(\lambda(0)\lambda(t)|\tau_0 > t)P(\tau_0 > t) + E(\lambda(0)|\tau_0 \leqslant t)E(\lambda(t)|\tau_0 \leqslant t)P(\tau_0 \leqslant t),$$

$$II = P(\tau_0 > t)\,[E(\lambda(0)|\tau_0 > t)E\lambda(t) + E(\lambda(0)|\tau_0 \leqslant t)E(\lambda(t)|\tau_0 > t)P(\tau_0 \leqslant t)] +$$
$$+ E(\lambda(0)|\tau_0 \leqslant t)E(\lambda(t)|\tau_0 \leqslant t)P^2(\tau_0 \leqslant t).$$

So, we have

$$r(t) = P(\tau_0 > t)\big[E(\lambda(0)\lambda(t)|\tau_0 > t) - E(\lambda(0)|\tau_0 > t)E\lambda(t) -$$
$$- E(\lambda(0)|\tau_0 \leqslant t)E(\lambda(t)|\tau_0 > t)P(\tau_0 \leqslant t)\big] +$$
$$+ E(\lambda(0)|\tau_0 \leqslant t)E(\lambda(t)|\tau_0 \leqslant t)\big[P(\tau_0 \leqslant t) - P^2(\tau_0 \leqslant t)\big].$$

Thus $|r(t)| \leqslant P(\tau_0 > t)\left(\lambda_M^2 + \lambda_M^2 + \lambda_M^2 + \lambda_M^2\right) = 4\lambda_M^2 P(\tau_0 > t)$.

If $\overline{F}(t) \leqslant ce^{-\alpha t}$ for some $\alpha > 0$ and $0 < c < \infty$ and t sufficiently large then

$$|r(t)| \leqslant c_1 e^{-\alpha t} \text{ for some positive } c_1 < \infty.$$

So, Theorems 1 and 2 hold for any $0 < \Delta < 1$ in this case.

If for t sufficiently large $\overline{F}(t) \leqslant Ct^{-d}$ for $d > 1$ and $C < \infty$ then

$$|r(t)| \leqslant c_2 t^{1-d} \text{ for some positive } c_2 < \infty.$$

Therefore (3) is valid if $\left|\Delta - \frac{1}{2}\right| < d - \frac{3}{2}$ and convergence (4) takes place if $d > 2\Delta$.

Now we present an example of the stationary regenerative DSPP. The other examples will be given in the next sections. We see that the asymptotic behavior of $q(t)$ for system S with this input flow is determined by the asymptotics of $1 - F(t)$ as $t \to \infty$.

Example 1. Let $\{x_j(t), t \geqslant 0\}_{j=0}^{\infty}$ be a sequence of independent identically distributed stationary stochastic processes taking values in the interval $[0, \lambda_M)$ and $\{\tau_j\}_{j=2}^{\infty}$ be a sequence of positive i.i.d. random variables with distribution function $F(t)$ and finite mean a. Let τ_1 be a random variable not depending on $\{\tau_j\}_{j=2}^{\infty}$ and $P\{\tau_1 \leqslant t\} = F_1(t) = \frac{1}{a}\int_0^t F(y)dy$. Moreover, sequences $\{\tau_j\}_{j=1}^{\infty}$ and $\{x_j(t), t \geqslant 0\}_{j=0}^{\infty}$ are independent. We put $\theta_j = \tau_1 + \ldots + \tau_j$, $j \geqslant 1$ and $\theta_0 = 0$. Denote by $N(t)$ the counting process for the renewal process $\{\theta_j\}_{j=0}^{\infty}$ i.e.

$$N(t) = \sup\{n \geqslant 0 : \theta_n \leqslant t\}.$$

One can easily see that the process $\lambda(t) = x_{N(t)}(t - \theta_{N(t)})$ is a regenerative stationary stochastic process.

4. System with Markov modulated input flow

Here we consider an infinite server queuing system S with Markov modulated input flow. The Markov modulated process is DSPP with intensity $\lambda(t)$ defined by the relation

$$\lambda(t) = \sum_{j=0}^{\infty} \lambda_j \mathbb{I}(U(t) = j), \tag{6}$$

where $U(t)$ is a continuous-time Markov chain with the set of states $\{0, 1, 2, \ldots\}$ [19]. We assume that $U(t)$ is an ergodic and stationary process. Then $\lambda(t)$ is a stationary regenerative process. As points of regeneration one may take the moments when $U(t)$ gets into a fixed state, for example, zero. Then $F(t)$ is the distribution function of the first passage time to the zero state after exit from it.

Throughout what follows, we assume that for the distribution function of service times $\overline{B}(t)$ Condition 2 holds. Let τ_{00} be the return time to state (0).

Corollary 1. *For a queuing system S with a stationary MMP input flow that has a finite set of states $(0, 1, \ldots, N)$ convergences (4) and (3) hold.*

Proof. Let $Q = \{q_{ij}\}_{i,j=0}^{N}$ be the infinitesimal matrix for $U(t)$. Since $U(t)$ is ergodic, then $\min\limits_{i=\overline{1,N}}(-q_{ij}) = q > 0$. Therefore, the distribution function $G_i(t)$ of the sojourn time in the state i satisfies the following inequality (see, e.g., [4])

$$\overline{G}_i(t) \leqslant e^{-qt}, \tag{7}$$

for $t \geqslant 0$, $i = \overline{1,N}$. Let us consider the embedded Markov chain $U_n = U(t_n + 0)$, where t_n is the moment of the nth jump of $U(t)$, $n = 1, 2, \ldots$. We denote

$$\nu_{i0} = \min\{n : U_n = 0\} \text{ provided that } U_0 = i.$$

It follows from the ergodic theorem [6] that for a Markov chain with a finite set of states

$$P\{\nu_{i0} > n\} \leqslant c\rho^n \text{ for any } i = \overline{1,N} \tag{8}$$

for some $0 < \rho < 1$ and $c < \infty$. Taking into account estimates (7) and (8) we have

$$P(\tau_{00} > t) \leqslant c_0 e^{-q_0 t}$$

for some c_0 and q_0. So, according to the first part of the Theorem 3 this proves (3) and (4) for $q(t)$.

Example 2. Consider a two-phase queuing model. The first phase is a classical system $M/M/r/m$ with a Poisson input flow with rate λ, exponentially distributed service times with parameter μ, r servers, and m places for waiting of service. Customers served in the first phase arrive at the second phase. which is a system S with an infinite number of servers and $B(x)$ as a distribution function of service times. To employ our results, we need to describe the input flow for S. It is clear that the output flow from the first phase is MMP with the intensity

$$\lambda(t, \omega) = \mu \sum_{j=1}^{r-1} j\mathbb{I}(U(t) = j) + \mu r \mathbb{I}(r \leqslant U(t) \leqslant m)$$

where $U(t)$ is the number of customers in $M/M/r/m$.

We note that $U(t)$ is continuous-time Markov chain with the set of states $\{0, 1, 2, \ldots, r+m\}$. It is ergodic and we assume that it is stationary. So, $\lambda(t)$ is the stationary regenerative process and its points of regeneration are the moments when $U(t)$ gets into a fixed state, for example, zero. It follows from Corollary 1 that for the number of customers $q(t)$ in the second phase limit relations (3) and (4) take place. To employ Theorem 3 in the case of the countable control Markov chain $U(t)$ we need an auxiliary result.

Consider a Markov chain $\{X_n, n \geqslant 0\}$ with one class of communicating essential states $(0, 1, 2, \ldots)$. Denote by ν_{00} the number of steps to return to zero state. Let $\{\alpha_j\}_{j=-\infty}^{+\infty}$ be a sequence defining a probability measure P on the set $\{0, \pm 1, \pm 2, \ldots\}$ i.e. $P(j) = \alpha_j$ such that

$$\sum_{j=-\infty}^{+\infty} j\alpha_j = -\delta < 0, \quad \sum_{j=-\infty}^{+\infty} (j+\delta)^2 \alpha_j = \sigma^2 < \infty. \tag{9}$$

Lemma 2. *If for the probability distribution* $\{\alpha_j\}_{j=-\infty}^{+\infty}$ *condition* (9) *holds and*

$$P\{X_n - X_{n-1} > j | X_{n-1} = i\} \leqslant \sum_{i=j+1}^{\infty} \alpha_i \text{ for any } i = 0, 1, \ldots \text{ and } j = 0, \pm 1, \pm 2, \ldots \tag{10}$$

then for some $0 < \beta < 1$ *and any* $n \geqslant 0$

$$P(\nu_{00} > n) < (1-\beta)^n. \tag{11}$$

Proof. It follows from (10) that if some random variable ξ has the distribution $\{\alpha_j\}_{j=-\infty}^{+\infty}$, then the following stochastic inequality holds:

$$X_n - X_{n-1} \leqslant \xi.$$

Let $\{\xi_n\}_{n=1}^{\infty}$ be independent random variables with distribution $\{\alpha_j\}_{j=-\infty}^{+\infty}$ and $S_n = \xi_1 + \xi_2 + \ldots + \xi_n$. Then

$$P(\nu_{00} > n) = P(X_1 > 0, X_2 > 0, \ldots, X_n > 0 | X_0 = 0) \leqslant P(X_n > 0 | X_0 = 0) =$$

$$= P\left\{\sum_{k=1}^{n} (X_k - X_{k-1}) > 0 | X_0 = 0\right\} \leqslant P(S_n > 0). \tag{12}$$

According to the CLT we have

$$P(S_n > 0) \sim 1 - \Phi\left(\frac{\delta\sqrt{n}}{\sigma}\right),$$

for n sufficiently large. Here $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy$. Using the estimation [6]

$$1 - \Phi(x) < \frac{1}{x\sqrt{2\pi}}e^{-\frac{x^2}{2}},$$

we obtain the asymptotic inequality

$$P(S_n > 0) < \frac{1}{c\sqrt{2\pi n}}e^{-\frac{c^2 n}{2}} \tag{13}$$

as $n \to \infty$, where $c = \frac{\delta}{\sigma}$. It follows from (12) that for some $c_1 > 0$

$$P(\nu_{00} > n) < P(S_n > 0) \leqslant \frac{1}{c\sqrt{2\pi n}}e^{-\frac{c^2 n}{2}} \leqslant c_1\left(e^{-\frac{c^2}{2}}\right)^n$$

for n sufficiently large. So, (11) holds for $0 < \beta < 1$ and any $n \geqslant 0$.

Remark 1. Let the transition probabilities for X_n be given by the relations

$$p_{jj+1} = p_j,\ p_{jj-1} = 1 - p_j,\ p_{01} = 1 \text{ for } j > 0.$$

If for some $\varepsilon > 0$ and $j > j_0$

$$p_j \leqslant \frac{1}{2} - \varepsilon$$

then (11) holds. Here j_0 is some positive integer.

Denote by ν_{ij} the number of steps of the embedded Markov chain X_n to get into the state (j) provided that $X_0 = i$. Then

$$\nu_{j_0 j_0} = \nu_{j_0+1 j_0}\mathbb{I}\text{ (transition from state } j_0 \text{ to state } j_0+1)+$$

$$+\nu_{j_0-1 j_0}\mathbb{I}(\text{ transition from state } j_0 \text{ to state } j_0 - 1) + 1.$$

So,

$$P(\nu_{j_0 j_0} > n) = p_{j_0}P(\nu_{j_0+1 j_0} > n-1) + (1 - p_{j_0})P(\nu_{j_0-1 j_0} > n-1).$$

According to Lemma 2 there exists some $0 < \beta_1 < 1$ such that $P(\nu_{j_0+1 j_0} > n-1) \leqslant (1-\beta_1)^{n-1}$. In view of (8) there exists some $0 < \beta_2 < 1$ such that $P(\nu_{j_0-1 j_0} > n-1) \leqslant (1-\beta_2)^{n-1}$. So,

$$P(\nu_{j_0 j_0} > n) < c(1-\beta)^n,$$

for some constant c and $\beta = \min(\beta_1, \beta_2)$.

Example 3. Let $U(t)$ in (6) be a birth and death process with infinitesimal parameters $q_{ii+1} = a_i$, $q_{ii-1} = b_i$, $i \geqslant 0$, and $b_0 = 0$. If $\sum\limits_{i=1}^{\infty}\prod\limits_{j=1}^{i}\frac{a_{i-1}}{b_i} < \infty$ then $U(t)$ is ergodic and we assume that it is a stationary process. Then $\lambda(t,\omega)$, defined by (6) with $\lambda_j = b_j$, is also a stationary one. Moreover, it is a regenerative process. As points of its regeneration one may take moments θ_n when $U(t)$ gets into any fix state (i_0).

Let $\{t_n\}_{n=1}^{\infty}$ be moments of jumps $U(t)$. We consider the embedded Markov chain $U_n = U(t_n + 0)$. Everywhere below we denote by ν_{ij} the number of steps of the embedded Markov chain U_n to get into the state (j) provided that $U_0 = i$.

Here we assume that the following conditions are fulfilled.

1. For all $j > 0$ there exists $\gamma > 0$ such that $a_j + b_j \geqslant \gamma > 0$.
2. There exist $j_0 > 0$ and $\delta > 0$ such that $\frac{a_j}{a_j+b_j} \leqslant \frac{1}{2} - \delta$ for $j \geqslant j_0$.

Choosing $i_0 \geqslant j_0$ according to Remark 1 we get that

$$P(\nu_{i_0 i_0} > n) < (1-\beta)^n \text{ for some } 0 < \beta < 1.$$

The first condition provides $1 - G_{ij}(x) \leqslant e^{-\gamma x}$. In view of the first part of Theorem 3, convergences (3) and (4) hold.

Example 4. Consider a two-phase queuing model as in Example 2 but, in this case, we assume that $m = \infty$. So, output flow from the first phase is MMP with intensity

$$\lambda(t,\omega) = \mu \sum_{j=1}^{r-1} j\mathbb{I}(U(t) = j) + \mu r \mathbb{I}(U(t) \geqslant r)$$

where $U(t)$ is the number of the customers in $M/M/r/\infty$.

We notice that $U(t)$ is a continuous-time Markov chain with the set of states $\{0, 1, 2, \ldots\}$. Assume that $a < r\mu$; therefore $U(t)$ is ergodic. In addition, we suppose that $U(t)$ is a stationary process. Putting $b_j = \min(j, r)\mu$ and $a_j = a$, $j \geqslant 0$, and $i_0 = r$ in Example 3, we obtain that conditions 1 and 2 are fulfilled and convergences (3) and (4) take place.

5. Systems with semi-Markov modulated input flow

These flows form an important subclass of regenerative DSPP. In this case a random intensity is given by (6) where $\{U(t), t \in [0, +\infty)\}$ is a semi-Markov modulated process with values in $\{0, 1, 2, \ldots\}$ and $\{\lambda_k, \lambda_k < C, k \geqslant 0\}$. It is known (see, e.g., [7]) that the distribution of $U(t)$ is defined by two matrixes $\mathbb{P} = (p_{ij})$ and $\mathbb{G} = (G_{ij}(x))$. The first matrix consists of transition probabilities from state (i) to state (j), and the second one consists of distribution functions of the sojourn time in the state $i = 0, 1, 2 \ldots$ provided that the following state will be j. Let $\{t_n\}_{n=1}^{\infty}$ be moments of jumps $U(t)$ and $U_n = U(t_n + 0)$. Then $\mathbb{P}$ is a transition matrix for the Markov chain U_n.

We assume that the matrix $\mathbb{P}$ is ergodic, i.e., the Markov chain U_n is irreducible, aperiodic, and has a proper limit distribution. In this case, the process $U(t)$ has a stationary distribution, provided that

$$a_{ij} = \int_0^{\infty} (1 - G_{ij}(x))dx < \infty$$

for all i, j [7].

Further we assume that $U(t)$, and hence $\lambda(t)$ are stationary processes. As before, let $\{t_n\}_{n=1}^{\infty}$ be moments of jumps $U(t)$. We also note that $\lambda(t)$ is a regenerative process with moments of regeneration $\{\theta_i\}_{i=0}^{\infty}$ when $U(t)$ gets into a fixed state, for example, zero. We assume that

$$\theta_i = \inf_{n \geqslant 1} \{t_n > \theta_{i-1} : U_n = 0\}, \ \theta_0 = \inf \{t \geqslant 0 : U(t) = 0\}, \ i = 1, 2, \ldots,$$

$$\tau_i = \theta_i - \theta_{i-1}, \ \tau_0 = \theta_0, \ i = 1, 2, \ldots.$$

Our aim is to find conditions for convergences (3) and (4) for infinite channel system with semi-Markov modulated input flow. In view of Theorem 3, it is necessary to find the asymptotic behavior of the probability

$$P(\tau_0 > t) = \frac{1}{E\tau_1} \int_t^{\infty} P(\tau_1 > y)dy$$

as $t \to \infty$. Note that the period of regeneration τ_j, $j \geqslant 1$, consists of the time that the process $U(t)$ is situated in the zero state and the return time to the zero state after exit from it.

Theorem 4. *Let the following conditions be fulfilled.*

1) *There exists a distribution function $G(x)$ with a finite mean such that for x sufficiently large*

$$1 - G_{ij}(x) \leqslant 1 - G(x), \text{ for } i, j = 0, 1, \ldots. \tag{14}$$

2) *There are $\beta \in (0,1)$, constant $C < \infty$, such that*

$$P(\nu_{00} > n) \leqslant C(1-\beta)^n, \; n = 0, 1, \ldots.$$

Then for any $h \in (0,1)$, some constants C_1, C_2, and x sufficiently large, the following inequality holds:

$$P(\tau_1 > x) \leqslant C_1(1 - G(x^{1-h})) + C_2 e^{-\gamma x^h}. \tag{15}$$

Here $\gamma = -\ln(1-\beta)$.

Proof. For the Markov chain U_n let ν_{00} be the number of steps before return to zero state. Let, for $U(t)$, $\{\eta_j\}_{j=1}^{\nu_{00}}$ be the sojourn times in the relevant states. Then

$$\tau_{00} = \eta_1 + \eta_2 + \ldots + \eta_{\nu_{00}}.$$

For any $M < \infty$ we have

$$P(\tau_1 > x) \leqslant P(\sum_{j=0}^{\nu_{00}} \eta_j > x, \nu_{00} \leqslant M) + P(\sum_{j=0}^{\nu_{00}} \eta_j > x, \nu_{00} > M) \leqslant$$

$$\sum_{n=1}^{M} n \left(1 - G\left(\frac{x}{n}\right)\right) P(\nu_{00} = n) + P(\nu_{00} > M) \leqslant \left(1 - G\left(\frac{x}{M}\right)\right) E\nu_{00} + (1-\beta)^M.$$

Since $E\nu_{00} < \infty$, putting $M = [x^h]$ for $h \in (0,1)$ and $\gamma = -\ln(1-\beta)$, we obtain (15). Here $[x]$ is the integer part of x.

Corollary 2. *Let the conditions of Theorem 4 be satisfied. Then for a queuing system with input flow of intensity $\lambda(t)$ defined by (6), where $U(t)$ is a semi-Markov modulated process, we have*

1) *if $1 - G(x) \leqslant e^{-qx}$ for some $q > 0$, then convergences (3) and (4) hold*;
2) *if $1 - G(x) < cx^{-\delta}$ for $\delta > 1$, then (3) is fulfilled for $\left|\Delta - \frac{1}{2}\right| < \delta - \frac{3}{2}$, and (4) is fulfilled for $\delta > 2\Delta$. Here c is some positive constant.*

Proof. It follows from Theorem 4 that in the first case the asymptotic behavior of the correlation function $r(t)$ as $t \to \infty$ is exponential, so (3) and (4) are fulfilled.

In the second case, from (15) for $h \in (0,1)$ and any $\varepsilon > 0$ sufficiently small, we have

$$P(\tau_1 > x) \leqslant E\nu_{00} x^{-\delta(1-h)} + C_1 e^{-\gamma x^h} \leqslant C_2 x^{-(1-h)\delta}$$

as $x \to \infty$. Here C_1, C_2 are some constants. Therefore, we obtain that

$$|r(t)| \leqslant C_3 t^{1-(1-h)\delta} \tag{16}$$

as $t \to \infty$ and C_3 is a constant.

Hence, the first part of Theorem 3 is true in this case for any $0 < \Delta < 1$. Similarly, we establish the validity of the second part of Theorem 3 for $\delta > 2\Delta$.

Remark 2. Function $G(x)$ should be chosen sufficiently close to the family $\{G_{ij}(x)\}$. For example, if $\widetilde{G}(x) = \min_{i,j} G_{ij}(x)$ is a distribution function, then $G(x)$ should be taken equal to $\widetilde{G}(x)$.

Example 5. We assume that the semi-Markov process $U(t)$ has a finite set of states, and $\mathbb{P}$ an is ergodic matrix. Matrix $\mathbb{G} = (G_{ij}(x))$, as before, consists of distribution functions of the sojourn time of the state $i = 0, 1, 2, \ldots$ provided that the following state will be j.

We take $G(x) = \min_{i,j} G_{ij}(x)$ as the dominating function, which is the distribution function for a finite N.

So, according to Corollary 2, if $1 - G(x) \leqslant cx^{-\delta}$ for x sufficiently large then (3) holds for $\left|\Delta - \frac{1}{2}\right| < \delta - \frac{3}{2}$ and (4) holds for $\delta > 2\Delta$. However, if $1 - G(x) \leqslant e^{-qx}$ for some $q > 0$ and x sufficiently large, then Theorem 3 is fulfilled for any $0 < \Delta < 1$.

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REFERENCES

1. L. G. Afanasyeva, E. E. Bashtova, E. A. Chernavskaya, "Limit theorems for queuing system with an infinite number of servers," in: *XXXII International Seminar on Stability Problems for Stochastic Models* (2014), pp. 9–11.
2. N. Kaplan, "Limit theorems for a $GI/G/\infty$ queue," *Ann. Probab.*, **3**, No. 5, 780–789 (1975).
3. J. Grandell, "Doubly stochastic Poisson process," in: *Lecture Notes in Mathematics*, Vol. 529, Springer (1976), pp. 1–276.
4. L. G. Afanasyeva and E. V. Bulinskaya , *Random Processes in Queuing Theory and Inventory Management*, Izd. Moskovskogo Universiteta, Moscow (1990).
5. B. A. Sevastyanov, *Branching Processes*, Nauka, Moscow (1971).
6. W. Feller, *An Introduction to Probability Theory and Its Applications*, Wiley, New York (1970).
7. V. S. Koroljuk, S. M. Brodie, and A. F. Turbin, "Semi-Markov processes and their applications," *Results of Science and Tehn., Ser. Theor. Probab. Math. Stat. Theor. Cybern.*, **11**, 47–97 (1974).
8. H. Thorisson, "The coupling of regenerative processes," *Adv. Appl. Probab.*, **15**, 531–561 (1983).
9. L. G. Afanasyeva and E. E.Bashtova, "Coupling method for asymptotic analysis of queues with regenerative input and unreliable server," *Queueing Syst.*, **76**, No. 2, 125–147 (2014).
10. L. G. Afanasyeva, E. E. Bashtova, and E. V. Bulinskaya, "Limit Theorems for Semi-Markov Queues and Their Applications," *Commun. Stat. — Simul. Comput.,* **41**, 1–22 (2012).
11. W. Whitt, *Stochastic Process Limits: an Introduction to Stochastic-process Limits and Their Application to Queues*, Springer, New York (2001).
12. K. Weining, K. Ramanan. "Asymptotic Approximations for Stationary Distributions of Many-Server Queues with Abandonment," *Ann. Appl. Probab.*, **22**, No. 2, 477–521 (2012).
13. W. Whitt, "Efficiency-Driven Heavy-Traffic Approximations for Many-Server Queues with Abandonments," *Manag. Sci.*, **50**, No 10, 1449–1461 (2004).
14. B. DAuria, "Stochastic decomposition of the queue in a random environment," *Oper. Res. Lett.*, **35**, No, 6, 805–12 (2007).
15. A. Massey and W. Whitt, "Networks of Infinite-Server Queues with Non-stationary Poisson Input," *Queueing Syst.*, **13**, No. 1–3, 183–250 (1993).
16. L. G. Afanasyeva and I. V. Rudenko , "Queuing systems $GI|G|\infty$ and their applications to the analysis of traffic patterns," *Theor. Probab. Appl.*, **57**, No. 3, 427–452 (2012).
17. L. Lipsky, D. Derek, and G. Swapna, *New frontiers in applied probability*, (2011).

18. E. B. Yarovaya, “Models of branching walks and their applications in the theory of reliability,” *Automat. Remote Control*, No. 7, 29–46 (2010).

19. S. Asmussen, *Applied Probability and Queues*, Wiley, Chichester (1987).