

# QUANTUM GROUPS: FROM THE KULISH–RESHETIKHIN DISCOVERY TO CLASSIFICATION

B. Kadets,\* E. Karolinsky,\* I. Pop,<sup>†</sup> and A. Stolin<sup>‡</sup>

UDC 517.9

*The aim of this paper is to provide an overview of results about classification of quantum groups which were obtained by the authors. Bibliography: 17 titles.*

**Dedicated to P. P. Kulish on the occasion of his 70th birthday**

## 1. INTRODUCTION

The first example of a quantum group was found by Kulish and Reshetikhin in [13]. They discovered what was later named  $U_q(\mathfrak{sl}_2)$  in relation to the study of the inverse quantum scattering method. Later, Drinfeld [3] and Jimbo [9] independently developed a general notion of quantum group. Today there are many different approaches to what a quantum group is, and the term has no clear meaning. Informally speaking, a quantum group is a deformation of a universal enveloping algebra of some Lie algebra  $\mathfrak{g}$ . Of course, the precise meaning should be given to the term deformation. We use the following definition.

**Definition 1.1.** A quantum group is a topologically free cocommutative mod  $\hbar$  Hopf algebra over  $\mathbb{C}[[\hbar]]$  such that  $H/\hbar H$  is a universal enveloping algebra of some Lie algebra  $\mathfrak{g}$  over  $\mathbb{C}$ .

It is well known that many problems about Lie groups become simpler when they are written in the language of Lie algebras. In general, the existence of almost one-to-one correspondence between Lie groups and Lie algebras is one of the central parts of Lie theory. Therefore, it is desirable to obtain a notion of quantum algebra that will help to simplify problems about quantum groups gradually. The first natural attempt was to look at the linear part of the comultiplication of a quantum group  $H$ . Indeed, one can define a co-Poisson structure  $\delta : U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$  by the formula

$$\delta(x) = \frac{\Delta(a) - \Delta^{21}(a)}{\hbar} \bmod \hbar,$$

where  $x \equiv a \bmod \hbar$ . Furthermore, from a co-Poisson structure on  $U(\mathfrak{g})$  one gets a Lie bialgebra structure on  $\mathfrak{g}$ , and the co-Poisson structure is uniquely determined by this Lie bialgebra structure. The process of recovering the (nonunique) quantum group structure from the Lie bialgebra structure is known as quantization.

The following problem naturally arises.

**Conjecture 1.2** (Drinfeld’s quantization conjecture). Any Lie bialgebra can be quantized.

The conjecture was solved by Etingof and Kazhdan in [5, 6].

Kazhdan and Etingof not only proved Drinfeld’s quantization conjecture but found a correct notion of quantum algebra. This was very important because it was not difficult to see that there might be many different quantizations of a given Lie bialgebra over  $\mathbb{C}$ . They constructed

---

\*Kharkov National University, Kharkov, Ukraine, e-mail: borja.kadets@gmail.com, ekarolinsky@univer.kharkov.ua.

<sup>†</sup>Chalmers University of Technology, Göteborg, Sweden, e-mail: iulia.pop@hb.se.

<sup>‡</sup>Gothenburg University, Göteborg, Sweden, e-mail: astolin@chalmers.se.

a canonical co-Poisson structure on  $U(\mathfrak{g}) \otimes \mathbb{C}[[\hbar]]$ . This structure is much finer than the co-Poisson structure discussed above. The Lie groups – Lie algebras correspondence has an analogy in the quantum world.

**Theorem 1.3.** *Let  $\text{Qgroup}$  be the category of quantum groups in the sense of Definition 1.1. Let  $\text{LieBialg}$  be the category of topologically free Lie bialgebras over  $\mathbb{C}[[\hbar]]$  with  $\delta \equiv 0 \pmod{\hbar}$ . Then there exists a dequantization functor  $\text{deQuant} : \text{Qgroup} \rightarrow \text{LieBialg}$  that is an equivalence of categories.*

In their solution of Drinfeld’s quantization conjecture, Etingof and Kazhdan constructed a functor  $\text{Quant} : \text{LieBialg} \rightarrow \text{Qgroup}$ , which informally can be called *universal quantization formula* or *quantum Baker–Campbell–Hausdorff formula*. They proved that if one starts with a Lie bialgebra  $L[[\hbar]]$  and first applies the functor  $\text{Quant}$  to it and then applies  $\text{deQuant}$ , the resulting Lie bialgebra will be isomorphic to  $L[[\hbar]]$ . The same is true if one starts with a quantum group  $H$ :  $\text{Quant}(\text{deQuant}(H))$  will be isomorphic to  $H$ .

One of applications of the Lie groups – Lie algebras correspondence is the classification of semisimple Lie groups because the classification of semisimple Lie algebras is a much easier problem. In the same way, one can use Theorem 1.3 as an approach to classification of quantum groups over semisimple Lie algebras. This was done in the papers [10, 11]. The rest of the paper is devoted to an exposition of the main results of these papers.

## 2. FIRST STEPS OF THE CLASSIFICATION

Let  $\mathfrak{g}$  be a simple Lie algebra over  $\mathbb{C}$ . We have seen that the classification of quantum groups over  $\mathfrak{g}$  is equivalent to the classification of Lie bialgebra structures on  $\mathfrak{g}[[\hbar]] := \mathfrak{g} \otimes \mathbb{C}[[\hbar]]$ . It is easy to see that any Lie bialgebra structure on  $\mathfrak{g}[[\hbar]]$  gives rise to a Lie bialgebra structure on  $\mathfrak{g}((\hbar)) := \mathfrak{g} \otimes \mathbb{C}((\hbar))$ , and any Lie bialgebra structure on  $\mathfrak{g}((\hbar))$  becomes a Lie bialgebra structure on  $\mathfrak{g}[[\hbar]]$  after a multiplication by an appropriate power of  $\hbar$ . Therefore, it is enough to classify Lie bialgebra structures on  $\mathfrak{g}((\hbar))$ .

Let us first look at the classification of Lie bialgebra structures on semisimple Lie algebras over an algebraically closed field  $\mathbb{F}$  of characteristic zero. This classification was obtained by Belavin and Drinfeld [1]. We now give a brief outline of their results. Let  $\delta$  be a Lie bialgebra structure on  $\mathfrak{g}$ . First, one notices that the “compatibility condition” for  $\delta$  is equivalent to the fact that  $\delta$  is a cocycle. From the triviality of cohomology of simple Lie algebras we see that there exists  $r \in \mathfrak{g} \otimes \mathfrak{g}$  such that  $\delta = dr$ . The condition that  $\delta$  is a Lie bialgebra structure can be rewritten in terms of  $r$ . It turns out that after an appropriate scaling,  $r$  should satisfy the classical Yang–Baxter equation. There are two quite different cases,  $r$  skewsymmetric or nonskewsymmetric. In the first case, there is no hope to obtain a meaningful classification. However, there is a lot of structure associated to a skewsymmetric  $r$ -matrix; these objects are intimately related to quasi-Frobenius Lie algebras [1]. In the second case, Belavin and Drinfeld found explicit formulas for  $r$ -matrices up to conjugation.

**Theorem 2.1.** *Let  $\mathfrak{g}$  be a simple Lie algebra over an algebraically closed field of characteristic zero. Then any Lie bialgebra structure on  $\mathfrak{g}$  is coboundary. Let  $r$  be a corresponding  $r$ -matrix. If  $r$  is not skewsymmetric, then*

$$r = r_0 + \sum_{\alpha > 0} e_{-\alpha} \otimes e_{\alpha} + \sum_{\alpha \in \text{Span}(\Gamma_1)^+} \sum_{k \in \mathbb{N}} e_{-\alpha} \wedge e_{\tau^k(\alpha)}$$

for some root decomposition. Here  $\Gamma_1$  and  $\Gamma_2$  are subsets of the set of simple roots,  $\tau : \Gamma_1 \rightarrow \Gamma_2$  is an isometric bijection, and for every  $\alpha \in \Gamma_1$  there exists  $k \in \mathbb{N}$  such that  $\tau^k(\alpha) \in \Gamma_2 \setminus \Gamma_1$ . The triple  $(\Gamma_1, \Gamma_2, \tau)$  is called admissible. The tensor  $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$  must satisfy the following two conditions:

- (1)  $r_0 + r_0^{21} = \sum t_k \otimes t_k$ , where  $t_k$  is an orthonormal basis of  $\mathfrak{h}$ ;
- (2)  $(\tau(\alpha) \otimes \text{id} + \text{id} \otimes \alpha)r_0 = 0$  for any  $\alpha \in \Gamma_1$ .

It is worth noticing that there is an equivalent way to distinguish skewsymmetric and nonskewsymmetric  $r$ -matrices. In the first case, the Drinfeld double  $D(\mathfrak{g})$  is isomorphic to  $\mathfrak{g} \otimes \mathbb{F}[\varepsilon]$ ,  $\varepsilon^2 = 0$ ; in the second case,  $D(\mathfrak{g}) \simeq \mathfrak{g} \oplus \mathfrak{g}$ , see [17].

We want to obtain a version of the Belavin–Drinfeld classification over the nonclosed field  $\mathbb{C}((\hbar))$ . Let again  $\mathfrak{g}$  be a simple Lie algebra over  $\mathbb{C}$ . First notice that we have a natural notion of equivalence for Lie bialgebras on  $\mathfrak{g}((\hbar))$ :  $\delta_1 \sim \delta_2$  if and only if there exist  $\lambda \in \mathbb{C}((\hbar))$  and  $X \in G(\mathbb{C}((\hbar)))$  such that  $\delta_1 = \lambda \text{Ad}_X \delta_2$ . Here  $G$  is an algebraic group associated to  $\mathfrak{g}$ .

Any Lie bialgebra structure on  $\mathfrak{g}((\hbar))$  can be lifted to  $\mathfrak{g} \otimes \overline{\mathbb{C}((\hbar))}$ . Over the algebraically closed field  $\overline{\mathbb{C}((\hbar))}$ , we have the Belavin–Drinfeld classification. Therefore, any Lie bialgebra structure on  $\mathfrak{g}((\hbar))$  is given by an  $r$ -matrix of the form  $\lambda \text{Ad}_X r$ , where  $r$  is an  $r$ -matrix from the Belavin–Drinfeld list or a skewsymmetric  $r$ -matrix. One can prove that for a nonskew matrix, up to equivalence,  $\lambda$  is either 1 or  $\sqrt{\hbar}$ . Therefore, for any nonskew matrix from the Belavin–Drinfeld list there are two sets  $H_{BD}^1(r_{BD})$  and  $\overline{H}_{BD}^1(r_{BD})$  of equivalence classes of  $r$ -matrices.  $H_{BD}^1(r_{BD})$  parametrizes equivalence classes of  $r$ -matrices of the form  $\text{Ad}_X r_{BD}$  which define a Lie bialgebra structure on  $\mathfrak{g}((\hbar))$ , and, respectively,  $\overline{H}_{BD}^1(r_{BD})$  parametrizes equivalence classes of matrices of the form  $\sqrt{\hbar} \text{Ad}_X r_{BD}$ . We call  $\overline{H}_{BD}^1(r_{BD})$  and  $H_{BD}^1(r_{BD})$  the sets of, respectively, twisted and nontwisted Belavin–Drinfeld cohomologies. Analogously, for a skewsymmetric  $r$ -matrix  $r$  we define the Frobenius cohomology set  $H_F^1(r)$ .

There is an alternative way to see the difference between twisted and nontwisted Lie bialgebra structures. Let us look at the structure of the Drinfeld double. It easily follows from methods developed in [15] that there are three possible cases:  $D(\mathfrak{g}((\hbar)))$  can be isomorphic to  $\mathfrak{g}((\hbar)) \oplus \mathfrak{g}((\hbar))$ ,  $\mathfrak{g}((\hbar))[\sqrt{\hbar}]$ , or to  $\mathfrak{g}((\hbar))[\varepsilon]$ , where  $\varepsilon^2 = 0$ . These possibilities precisely correspond to the nontwisted, twisted, and skew cases, respectively.

We have shown that all Lie bialgebra structures on  $\mathfrak{g}$  fall into one of the three types: nontwisted, twisted, or skew. In what follows, we examine each case in more detail.

### 3. NONTWISTED CASE

We have defined  $H_{BD}^1(r_{BD})$  as the set of equivalence classes of Lie bialgebra structures. However, there is an equivalent definition which appeals only to the inner structure of  $\mathfrak{g}((\hbar))$ . In what follows,  $G$  is an algebraic group that corresponds to  $\mathfrak{g}$ .

**Definition 3.1.** The *centralizer*  $C(r)$  of an  $r$ -matrix  $r$  is the set of all  $X \in G(\overline{\mathbb{C}((\hbar))})$  such that  $\text{Ad}_X r = r$ .

**Definition 3.2.**  $X \in G(\overline{\mathbb{C}((\hbar))})$  is called a *nontwisted Belavin–Drinfeld cocycle* for  $r_{BD}$  if  $X^{-1} \sigma(X) \in C(r_{BD})$  for any  $\sigma \in \text{Gal}(\overline{\mathbb{C}((\hbar))}/\mathbb{C}((\hbar)))$ . The set of nontwisted cocycles will be denoted by  $Z(r_{BD}) = Z(G, r_{BD})$ .

**Definition 3.3.** Two cocycles  $X_1, X_2 \in Z(r_{BD})$  are called *equivalent* if there exist  $Q \in G(\mathbb{C}((\hbar)))$  and  $C \in C(r_{BD})$  such that  $X_1 = QX_2C$ .

**Definition 3.4.** The set of equivalence classes of nontwisted cocycles is denoted by  $H_{BD}^1(r_{BD}) = H_{BD}^1(G, r_{BD})$  and called the *nontwisted Belavin–Drinfeld cohomology*.

We were able to compute  $H_{BD}^1$  for the algebras of  $A - D$  series. First let us make a small remark about the  $A_n$  case. In this case,  $\mathfrak{g}((\hbar))$  is naturally acted upon by the group  $GL(n)$ , and we can compute the cohomology with respect to conjugation by  $GL(n)$  or  $SL(n)$ . To distinguish between these cases we write  $H_{BD}^1(GL(n), r_{BD})$  and  $H_{BD}^1(SL(n), r_{BD})$ .

If  $(\Gamma_1, \Gamma_2, \tau)$  is an admissible triple, then the set  $\alpha, \tau(\alpha), \dots, \tau^k(\alpha)$ , where  $\alpha \in \Gamma_1 \setminus \Gamma_2$  and  $\tau^k(\alpha) \in \Gamma_2 \setminus \Gamma_1$ , will be called a string of  $\tau$ . The following table describes  $H_{BD}^1$  for algebras of type  $A - D$ . The cohomology is called trivial if  $|H_{BD}^1(r_{BD})| = 1$ .

Algebra	Triple type	$H_{BD}^1$ for an arbitrary field	$H_{BD}^1$ for $\mathbb{C}((\hbar))$
$A_n$		trivial ( $GL(n)$ case)	
$B_n$		trivial	
$C_n$		trivial	
$D_n$	there exists a string of $\tau$ that contains $\alpha_{n-1}$ and $\alpha_n$	$F^*/(F^*)^2$	two elements
	$\alpha_{n-1}$ and $\alpha_n$ do not belong to the same string of $\tau$	trivial	

Table 1

**Remark 3.5.** In this paper,  $\alpha_n$  and  $\alpha_{n-1}$  are the branchendpoints in the Dynkin diagram for  $D_n$ .

**Remark 3.6.** One can similarly define the Belavin–Drinfeld cohomologies over an arbitrary field  $F$  as a tool to understand Lie bialgebra structures on  $\mathfrak{g}(F)$ .

The result for  $H_{BD}^1(SL(n), r_{BD})$  is more interesting. Let  $\alpha_{i_1}, \dots, \alpha_{i_k}$  be a string of  $\tau$ ,  $\tau(\alpha_{i_p}) = \alpha_{i_{p+1}}$ . If  $\tau(\alpha_{i_p})$  is not defined, then anyway we define the corresponding string which consists of one element  $\{\alpha_{i_p}\}$  only.

For any string  $S = \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$  of  $\tau$ , we define the weight of  $S$  by  $w_S = \sum_p i_p$ . Moreover, for any Belavin–Drinfeld triple we also formally consider the string  $\{\alpha_n\}$  with weight  $n$ .

Let  $N$  be the greatest common divisor of the weights of all strings.

**Theorem 3.7.** *The number of elements of  $H_{BD}^1(SL(n), r)$  is  $N$ . Each cohomology class contains a diagonal matrix  $D = A_1 A_2$ , where  $A_2 \in C(GL(n), r)$  and  $A_1 \in \text{diag}(n, \mathbb{C}((\hbar)))$ . Two such diagonal matrices  $D_1 = A_1 A_2$  and  $D_2 = B_1 B_2$  are contained in the same class of  $H_{BD}^1(SL(n), r)$  if and only if  $\det(A_1) = \det(B_1)$  in  $\mathbb{C}((\hbar))^*/(\mathbb{C}((\hbar))^*)^N$ .*

#### 4. TWISTED CASE

As in the nontwisted case, there is a way to define  $\overline{H}_{BD}^1$  without mentioning Lie bialgebra structures.

**Theorem 4.1.**  *$a\text{Ad}_X r_{BD}$  defines a Lie bialgebra structure on  $\mathfrak{g}(\mathbb{C}((\hbar)))$  if and only if  $X$  is a nontwisted cocycle for the field  $\mathbb{C}((\hbar))[\sqrt{\hbar}]$  and  $\text{Ad}_{X^{-1}\sigma_0(X)} r_{BD} = r_{BD}^{21}$ . Here  $\sigma_0$  is the nontrivial element of the group  $\text{Gal}(\mathbb{C}((\hbar))[\sqrt{\hbar}]/\mathbb{C}((\hbar)))$ .*

To deal with the condition  $\text{Ad}_{X^{-1}\sigma_0(X)} r_{BD} = r_{BD}^{21}$ , we classified all the triples  $(\Gamma_1, \Gamma_2, \tau)$  such that  $r_{BD}^{21}$  and  $r_{BD}$  are conjugate. In each case, we found a suitable  $S \in \mathbb{C}((\hbar))$  such that  $r_{BD}^{21} = \text{Ad}_S r_{BD}$ . Then we can define Belavin–Drinfeld cocycles and cohomologies similarly to the nontwisted case. In all cases,  $S^2 = \pm 1$ .

**Definition 4.2.**  $X \in G(\overline{\mathbb{C}(\hbar)})$  is called a *Belavin–Drinfeld twisted cocycle* if  $X^{-1}\sigma(X) \in C(r_{BD})$  and  $SX^{-1}\sigma_0(X) \in C(r_{BD})$  for any  $\sigma \in \text{Gal}(\overline{\mathbb{C}(\hbar)}/\mathbb{C}(\hbar)[\sqrt{\hbar}])$ . The set of Belavin–Drinfeld twisted cocycles is denoted by  $\overline{Z}(r_{BD}) = \overline{Z}(G, r_{BD})$ .

**Definition 4.3.** Two twisted cocycles  $X_1, X_2$  are called *equivalent* if there exist  $Q \in G(\mathbb{C}(\hbar))$  and  $C \in C(r_{BD})$  such that  $X_1 = QX_2C$ . The set of equivalence classes of twisted cocycles is called the *twisted Belavin–Drinfeld cohomology* and denoted by  $\overline{H}_{BD}^1(r_{BD}) = \overline{H}_{BD}^1(G, r_{BD})$ .

Algebra	Triple type	$\overline{H}_{BD}^1$ for $\mathbb{C}(\hbar)$		
$A_n$	$s\tau = \tau^{-1}s$ , where $s$ is the nontrivial automorphism of the Dynkin diagram	one element		
	other	empty		
$B_n$	Drinfeld-Jimbo	one element		
	not DJ	empty		
$C_n$	Drinfeld-Jimbo	one element		
	not DJ	empty		
$D_n$	even $n$	Drinfeld-Jimbo	one element	
		not DJ	empty	
	odd $n$	$\Gamma_1 = \{\alpha_{n-1}\}$ $\tau(\alpha_{n-1}) = \alpha_n$ ; $\Gamma_1 = \{\alpha_n\}$ $\tau(\alpha_n) = \alpha_{n-1}$ ;	two elements	
		$\Gamma_1 = (\alpha_{n-1}, \alpha_k), k \neq n$ $\tau(\alpha_{n-1}) = \alpha_k, \tau(\alpha(k)) = \alpha_n$ ; $\Gamma_1 = (\alpha_n, \alpha_k), k \neq n-1$ $\tau(\alpha_n = \alpha_k), \tau(\alpha_k) = \alpha_{n-1}$		
		Drinfeld-Jimbo		one element
		not DJ		empty

Table 2

Here the cohomology for  $\mathfrak{sl}_n$  is considered with respect to the group  $GL(n)$ . For the results for  $A_n$  over an arbitrary field, see [16].

## 5. SKEWSYMMETRIC CASE

Following the pattern of [1], it can be easily proved that the classification of Lie bialgebra structures related to skew (triangular)  $r$ -matrices on  $\mathfrak{g}(\hbar)$  is equivalent to the classification of quasi-Frobenius Lie subalgebras of  $\mathfrak{g}(\hbar)$ . This can be used to prove that if  $r$  is skewsymmetric, then  $r$  has to be defined over  $\mathbb{C}(\hbar)$ . However, different  $r$ -matrices defined over  $\mathbb{C}(\hbar)$  can be conjugate over  $\overline{\mathbb{C}(\hbar)}$ . We can define the Frobenius cohomology similarly to the Belavin–Drinfeld cohomology. We call two  $r$ -matrices equivalent if there exist  $a \in \mathbb{C}(\hbar)$  and  $X \in G(\mathbb{C}(\hbar))$  such that  $r_1 = a\text{Ad}_X r_2$ . If  $r$  defines a Lie bialgebra structure on  $\mathfrak{g}(\hbar)$ , then we define the Frobenius cohomology set  $H_F^1(r)$  to be the set of equivalence classes of  $r$ -matrices that are conjugate to  $r$  over  $\overline{\mathbb{C}(\hbar)}$ . We do not have a classification of skew  $r$ -matrices even over an algebraically closed field, but this cohomology can be computed in a way similar to the Belavin–Drinfeld case.

**Definition 5.1.** The *centralizer*  $C(r)$  of an  $r$ -matrix  $r$  is the set of all  $X \in G(\overline{\mathbb{C}(\hbar)})$  such that  $\text{Ad}_X r = r$ .

**Definition 5.2.**  $X \in G(\overline{\mathbb{C}(\hbar)})$  is called a *nontwisted Frobenius cocycle* for  $r$  if  $X^{-1}\sigma(X) \in C(r)$  for any  $\sigma \in \text{Gal}(\overline{\mathbb{C}(\hbar)}/\mathbb{C}(\hbar))$ . The set of nontwisted cocycles will be denoted by  $Z_F(r) = Z_F(G, r)$ .

**Definition 5.3.** Two cocycles  $X_1, X_2 \in Z_F(r)$  are called *equivalent* if there exist  $Q \in G(\mathbb{C}(\hbar))$  and  $C \in C(r)$  such that  $X_1 = QX_2C$ .

**Definition 5.4.** The set of equivalence classes of Frobenius cocycles is denoted by  $H_F^1(r) = H_F^1(G, r)$  and called the *Frobenius cohomology*.

**Example 5.5.** Let  $r_J$  be the Jordan  $r$ -matrix, i.e.,  $r_J = E \wedge H$ . Then  $H_F^1(r_J)$  is trivial. Here  $\{E, F, H\}$  is the standard basis in  $\mathfrak{sl}_2$ .

## 6. HISTORICAL REMARKS

Quantum groups (as in Definition 1.1) were defined by Drinfeld in his talk at the International Congress of Mathematicians in Berkeley, 1986. Relations between quantum groups and quantum algebras (quantization and dequantization functors, quantum Baker–Campbell–Hausdorff formula) were obtained by Etingof and Kazhdan in a series of papers [5, 6].

The first example of a quantum group of nontwisted type is due to Kulish and Reshetikhin [13]. Generalizations for all simple Lie algebras were obtained by Drinfeld and Jimbo in [3, 9], where they found quantum groups which quantize Lie bialgebra structures on  $\mathfrak{g}$  defined by  $\Gamma_1 = \Gamma_2 = \emptyset$ .

Further classes of Lie bialgebra structures on  $\mathfrak{g}$ , related to certain triples  $(\Gamma_1, \Gamma_2, \tau)$ , were quantized by Kulish and Mudrov in [12].

Finally, Etingof, Schiffman, and Schedler quantized all Lie bialgebra structures defined by all admissible triples  $(\Gamma_1, \Gamma_2, \tau)$  [7].

There are no explicit formulas for quantum groups related to the twisted Belavin–Drinfeld cohomologies.

Construction of quantum groups of skewsymmetric type appeared in the work of Drinfeld [4] by means of a certain twisting element  $F$ . The first explicit formula for  $F$  is due to Coll, Gerstenhaber, and Giaquinto [2]. This formula was used by Kulish and Stolin to explicitly quantize a certain nonstandard Lie bialgebra structure on the polynomial Lie algebra  $\mathfrak{sl}_2[u]$ .

This paper is dedicated to Petr P. Kulish on the occasion of his 70-years jubilee. The authors are thankful for valuable remarks to G. Rozenblum who joins the congratulations.

## REFERENCES

1. A. Belavin and V. Drinfeld, “Triangle equations and simple Lie algebras,” *Soviet Sci. Rev. Sect. C. Math. Phys. Rev.*, **4**, 93–165 (1984).
2. V. Coll, M. Gerstenhaber, and A. Giaquinto, “An explicit deformation formula with non-commuting derivations,” *Israel Math. Conf. Proceedings*, **1**, 396–403 (1989).
3. V. Drinfeld, “Hopf algebras and the quantum Yang–Baxter equation,” *Dokl. Akad. Nauk SSSR*, **283**, 1060–1064 (1985).
4. V. Drinfeld, “Constant quasiclassical solitons of the Yang–Baxter quantum equation,” *Soviet Math. Doklady*, **28**, 667–671 (1983).
5. P. Etingof and D. Kazhdan, “Quantization of Lie bialgebras. I,” *Sel. Math. (NS)*, **2**, 1–41 (1996).
6. P. Etingof and D. Kazhdan, “Quantization of Lie bialgebras. II,” *Sel. Math. (NS)*, **4**, 213–232 (1998).

7. P. Etingof, T. Schedler, and O. Schiffmann, “Explicit quantization of dynamical  $r$ -matrices for finite dimensional semisimple Lie algebras,” *J. Amer. Math. Soc.*, **13**, 595–609 (2000).
8. P. Etingof and O. Schiffmann, *Lectures on Quantum Groups*, Intern. Press, Cambridge (1988).
9. M. Jimbo, “A  $q$ -difference analogue of  $U\mathfrak{g}$  and the Yang–Baxter equation,” *Lett. Math. Phys.*, **10**, 63–69 (1985).
10. B. Kadets, E. Karolinsky, I. Pop, and A. Stolin, “Classification of quantum groups and Belavin–Drinfeld cohomologies,” [ArXiv:1303.4046 \[math.QA\]](#).
11. B. Kadets, E. Karolinsky, I. Pop, and A. Stolin, “Classification of quantum groups and Belavin–Drinfeld cohomologies for orthogonal and symplectic Lie algebras,” [ArXiv:1502.00403 \[math.QA\]](#).
12. P. P. Kulish and A. I. Mudrov, “Universal  $R$ -matrix for esoteric quantum groups,” *Lett. Math. Phys.*, **47**, 139–148 (1999).
13. P. P. Kulish and N. Yu. Reshetikhin, “Quantum linear problem for the sine-Gordon equation and higher representations,” *J. Sov. Math.*, **23**, 2435–2441 (1983).
14. P. P. Kulish and A. A. Stolin, “Deformed Yangians and integrable models. Quantum groups and integrable systems. II,” *Czechoslovak J. Phys.*, **47**, 1207–1212 (1997).
15. F. Montaner, A. Stolin, and E. Zelmanov, “Classification of Lie bialgebras over current algebras,” *Selecta Math. (N.S.)*, **16**, 935–962 (2010).
16. I. Pop and A. Stolin, “Classification of quantum groups and Lie bialgebra structures on  $\mathfrak{sl}(n, \mathbb{F})$ . Relations with Brauer group,” [ArXiv:1402.3083 \[math.QA\]](#).
17. A. Stolin, “Some remarks on Lie bialgebra structures on simple complex Lie algebras,” *Comm. Algebra*, **27**, 4289–4302 (1999).