DOI 10.1007/s10958-016-2725-1
Journal of Mathematical Sciences, Vol. 213, No. 4, March, 2016

ASYMPTOTIC MODELS OF ANISOTROPIC HETEROGENEOUS ELASTIC WALLS OF BLOOD VESSELS

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UDC 517.958:539.3(5):531.3-324

Using the dimension reduction procedure for a three-dimensional elasticity system, we derive a two-dimensional model for elastic laminate walls of a blood vessel. In the case of a sufficiently small wall thickness, we derive a system of limit equations coupled with the Navier–Stokes equations through the stress and velocity, i.e., dynamic and kinematic conditions on the interior surface of the wall. We deduce explicit formulas for the effective rigidity tensor of the wall in two natural cases. We show that if the blood flow remains laminar, then the cross-section of the orthotropic homogeneous blood vessel becomes circular. Bibliography: 30 *titles. Illustrations*: 2 *figures.*

1 Introduction

1.1. Formulation of the problem. Blood vessels form one of the most complicated and important systems (the circulatory system) in a human body which is exposed to various risks and is poorly amenable to medical treatments. Mathematical modelling of blood transport in arteries, veins, capillaries, and other blood vessels is a classical problem which is still actual nowadays (cf. [1]–[3] and [4, Section 8]). Although the existing models are usually based on the anisotropic and composite structure of blood vessel walls (cf. Figure 1 and [4, 5]), the analysis in this direction is far from being completed yet. In this way, our paper makes a next step in derivation of adequate governing relations that carefully take into account the laminated structure of an elastic blood vessel wall and the complicated composite structure of each laminate wall layer as well. For this purpose we consider a flow of a viscous incompressible fluid (blood) in

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Translated from *Problemy Matematicheskogo Analiza* **83**, December 2015, pp. 93-109.

a cylindrical vessel having an arbitrary cross-section. The vessel wall can consist of several layers of anisotropic materials. Our goal is twofold: first, to derive a model where a three-dimensional but thin anisotropic wall of the vessel is replaced with a boundary surface and, second, to obtain an explicit relation between the Hooke tensors for three- and two-dimensional models. We obtain such a model under the assumption that the wall thickness is small in comparison with the vessel diameter, whereas the diameter is small compared with the length of the part of the vessel under consideration. In this part of the vessel, the blood flow is laminar because the hydrostatic pressure prevails over the hydrodynamic forces. This allows us to conclude in Section 4 that the circular cross-section of the blood vessel is optimal in a certain sense. Moreover, the fact that the flow is laminar and the elastic wall material is strong and tough results in a small wall displacements. Hence the dimensional reduction procedure can be applied to the elastic vessel wall.

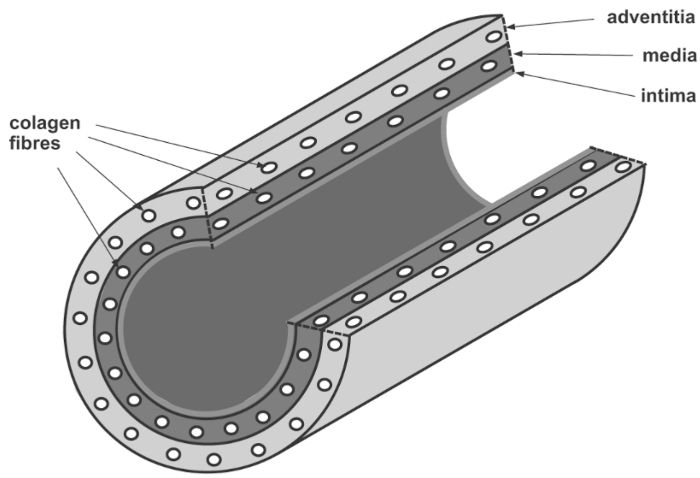

Figure 1. The blood vessel wall consisting of three layers reinforced by collagen fibres.

The dimension reduction procedure for the three-dimensional Navier–Stokes equations in a blood vessel was developed in [6], where the two-dimensional wall model was considered. Our results, especially explicit formulas in Section 4, provide concrete values for the elastic moduli used in [6] in the orthotropic rigidity tensor of the vessel wall. Thus, in this paper, the main attention is paid to the formal asymptotic analysis resulting in these explicit formulas while we dot not justify the asymptotics by several reasons.

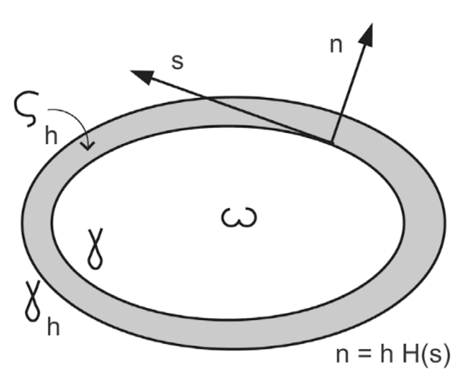

Figure 2. The cross-section of the blood vessel.

First, the dimension reduction for a thin elastic cylindrical shell under a fixed external loading, i.e., with prescribed hydrodynamical forces, follows a standard scheme (cf., for example, [7]–[10]). In particular, the paper [11] contains a detailed proof of the error estimate in a similar situation.

Second, the evaluation of effective elastic properties of blood vessels is the most urgent problem in simulation of circulatory system (cf. [12]).

We give a mathematical formulation of the problem. Let ω be a two-dimensional simply connected bounded domain enveloped by a smooth contour γ. In a neighborhood $\mathscr{V}$ of γ, we introduce the natural curvilinear orthogonal coordinates (n, s), where n is the oriented distance to γ ($n > 0$ outside ω and $n < 0$ inside ω) and s is the arc length along γ, measured counter-clockwise. Let H be a smooth positive function on γ, and let h be a small positive parameter. Setting $\gamma_h = \{y \in \mathscr{V},\ n = hH(s)\}$, we denote by ς_h the domain between γ and γ_h (cf. Figure 2). Then the lumen of the vessel is given by $\Omega = \omega \times \mathbb{R}$ and the vessel wall is $\Sigma_h = \varsigma_h \times \mathbb{R}$. An appropriate rescaling makes the parameters and coordinates dimensionless.

The flow in the vessel is described by the velocity vector $\mathbf{v} = (v_1, v_2, v_3)$ and the pressure p satisfying the Navier–Stokes equations

$$\begin{aligned} &\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nu\Delta\mathbf{v} = -\nabla p, \quad \text{in } \Omega \\ &\nabla\cdot\mathbf{v} = 0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where ρ_b is the fluid density and ν is the kinematic viscosity related to the dynamic viscosity μ by $\nu = \mu/\rho_b$. The stress state of the linear elastic wall is described by the displacement vector $\mathbf{u} = (u_1, u_2, u_3)$ and the stress tensor $\sigma = \{\sigma_{jk}\}_{j,k=1}^3$ satisfying the nonstationary elasticity equations

$$\frac{\partial\sigma_{j1}}{\partial x_1} + \frac{\partial\sigma_{j2}}{\partial x_2} + \frac{\partial\sigma_{j3}}{\partial x_3} = \rho\frac{\partial^2 u_j}{\partial t^2} \quad \text{in } \Sigma_h, \quad \text{j=1,2,3,} \tag{1.2}$$

and the Hooke law

$$\sigma_{jk} = \sum_{p,q=1}^3 A_{jk}^{pq}\varepsilon_{pq}, \quad j,k = 1,2,3, \quad \varepsilon_{pq} = \frac{1}{2}\Big(\frac{\partial u_p}{\partial x_q} + \frac{\partial u_q}{\partial x_p}\Big), \tag{1.3}$$

where ρ is the mass density, $\varepsilon = \{\varepsilon_{jk}\}_{j,k=1}^3$ is the strain tensor, and $\mathbf{A} = \{A_{jk}^{pq}\}$ is the rigidity tensor (also called the *Hooke tensor*) consisting of the moduli of elasticity of the wall material and possessing the standard symmetry and positivity properties:

$$A_{jk}^{pq} = A_{pq}^{jk} = A_{pq}^{kj}, \qquad \sum_{j,k,p,q=1}^3 A_{jk}^{pq}\xi_{jk}\xi_{pq} \geqslant C_A \sum_{j,k=1}^3 |\xi_{jk}|^2,$$

where $C_A > 0$ and $\{\xi_{jk}\}$ is an arbitrary symmetric 3×3-matrix. Within the Eulerian framework (this is our simplifying assumption), the actual position of the interior surface $\Gamma = \gamma \times \mathbb{R}$ at the moment t is given by $\{x + u(x,t) : x \in \Gamma\}$; it corresponds to stretching the elastic wall caused by the pulsatory blood flow.

The exterior surface $\Gamma_h = \gamma_h \times \mathbb{R}$ is assumed to be traction free[1)], i.e.,

$$\sigma_{j1}n_1 + \sigma_{j2}n_2 = 0 \quad \text{on } \Gamma_h. \quad \text{j=1,2,3,} \tag{1.4}$$

[1)] The Robin boundary condition can be also used for describing an interaction between the surrounding tissue and the blood vessel (cf., for example, [13] and [14]).

where $\mathbf{n} = (n_1, n_2, 0)$ is the outward unit normal to Γ_h. It is natural to impose two conditions on the interior surface Γ. The first one is that the fluid velocity coincides with the velocity of the elastic wall, i.e., the kinematic no-slip boundary condition holds

$$\mathbf{v} = \partial_t \mathbf{u} \quad \text{on } \Gamma, \tag{1.5}$$

whereas the second (dynamic) condition says that the hydrodynamic force is equal to the normal stress vector (traction comes with minus because the normal $\mathbf{n}$ is interior for Σ_h):

$$\sigma_\Gamma := \boldsymbol{\sigma} \cdot \mathbf{n} = \rho_b \mathbf{F}, \tag{1.6}$$

where $\mathbf{F} = (F_n, F_z, F_s)$,

$$\begin{aligned} F_n &= -p + \nu \frac{\partial v_n}{\partial n}, \\ F_s &= \frac{\nu}{2}\Big(\frac{\partial v_n}{\partial s} + \frac{\partial v_s}{\partial n} - \varkappa u_s\Big), \\ F_z &= \frac{\nu}{2}\Big(\frac{\partial v_n}{\partial z} + \frac{\partial v_z}{\partial n}\Big), \end{aligned} \tag{1.7}$$

where v_n and v_s are the velocity components in the direction of the normal $\mathbf{n}$ and the tangent $\mathbf{s}$ respectively, whereas v_z is the longitudinal velocity component ($z = x_3$). Finally, $\varkappa(s)$ is the curvature of γ at the point s.

We assume that

$$\rho = \frac{1}{h}\rho\Big(\frac{n}{h}, s, z\Big), \quad \mathbf{A} = \frac{1}{h}\mathbf{A}\Big(\frac{n}{h}, s, z\Big)$$

satisfy one of the following conditions:

(I) (a heterogeneous wall material): $\rho(\zeta, s, z)$ and $\mathbf{A}(\zeta, s, z)$ are smooth on $\overline{\Sigma}_1$, where $\Sigma_1 = \{(\zeta, s, z) \,:\, s \in \gamma,\ \zeta \in (0, H(s)), z \in \mathbb{R}\}$,

(II) (a laminate wall with layers of piecewise constant thickness): $H(s) = 1$, whereas ρ and $\mathbf{A}$ are defined as follows. Let $h_1, \dots, h_N$ be given numbers such that $h_1, \dots, h_N > 0$, $h_1 + \ldots + h_N = h$, $a_0 = 0, a_j = a_{j-1} + h_j$, $j = 1, \dots, N$. Then $\rho(\zeta, s, z) = \rho^j(s, z)$, $\mathbf{A}(\zeta, s, z) = \mathbf{A}^j(s, z)$, $\zeta \in (a_{j-1}/h, a_j/h)$, where ρ^j and $\mathbf{A}^j$ are independent of ζ.

Our goal is to derive a two-dimensional model of a blood vessel wall under Assumption (I) which simplifies the demonstration to some extent. However, the walls of veins and arteries involve composite laminate elastic structures, and so we give explicit formulas under Assumption (II) attributed mainly to peripheral veins (cf. Subsection 4.1). In arteries and voluminous veins, bundles of collagen fibres must be taken into account as well (cf. Subsection 4.2). Note that the dimension reduction procedure intrinsically admits passing to various limits and a straightforward approach is to approximate composites with piecewise constant elastic moduli by those having smooth heterogeneous properties and then, in the final integral formula for effective moduli (cf. Subsection 3.3), to return to the piecewise constant case.

1.2. Results. The dimension reduction plays an important role in mathematical modelling of engineering problems, where certain elements have small size in some directions. The theory of rods, plates, shells, elastic multi-structures etc. are examples worth mentioning. There are many papers on this topic that describe approximate models and justify them mathematically to a different extent of rigor by using various methods and approaches. Note that there are many classical engineering theories for laminated plates and shells (cf., for example, [15]).

We apply the rigorous dimension reduction procedure which was developed for problems in elasticity in [7, 8, 10, 12, 16] and for general elliptic problems in [17]. The main difficulty stems from the fact that the anisotropic wall has laminated structure. Our approach is based on several important ideas (cf. Section 2):

1) application of the matrix notation for equations in the elasticity theory referred to as the Voigt–Mandel notation in mechanics,

2) rearrangement of components of stress and strain vectors, which reflects different order asymptotic behavior of "normal" and "tangent" components of the corresponding tensors and is closely related to the notion of surface enthalpy [18].

The crucial point of our asymptotic approach is to construct an operator $\mathbf{U} \to \sigma_\Gamma$ of the Dirichlet–to–Neumann type, where $\mathbf{U}$ is a given displacement on the boundary Γ and σ_Γ is the corresponding normal stress vector on Γ. This relation is obtained in Section 3 and the leading term of σ_Γ on Γ is expressed through a hyperbolic operator on Γ applied to $\mathbf{U}$. By this fact, the equilibrium equation (1.6) becomes a hyperbolic system for $\mathbf{U}$ with the right-hand side $-h^{-1}\rho_b\mathbf{F}$ (cf. (3.34)). Combining it with the Navier–Stokes system (1.1) and the kinematic condition (1.5), we obtain a system of constitutive relations describing an interaction between the blood flow in the vessel and the elastic wall. Similar models were considered (cf. [4, Chapter 8] and [6] and the references therein), but only in the case of vessels with circular cross-section and isotropic homogeneous walls.

In Section 4, we analyze the model. In particular, we discuss connections between the elastic coefficient in our model and the elastic coefficient of the vessel wall.

Various laminate composite structures of blood vessel walls are well-known (cf. [4, Chapter 8]) and, as outlined above, we apply the dimension reduction procedure to approximate a thin anisotropic elastic wall by an anisotropic shell in order to derive an explicit formula for the limit rigidity matrix (3.3). In contrast to usual mathematical models of vessels, we do not assume *a priori* that the cross-section is circular (cf. Subsections 4.4 and 4.5). This allows us to consider the wall strains caused by such damages of blood vessels as irregular calcification (hyalinosis, arterial calcinosis), oblong atherosclerotic deposits (atherosclerotic plaque), and/or various surgical exposures.

2 Elastic Walls

The immediate objective of our asymptotic analysis of the elasticity problem (1.2), (1.3) with the boundary conditions (1.4) and

$$u_j = U_j \quad \text{on } \Gamma, \quad \text{j=1,2,3}, \tag{2.1}$$

is to compute the normal stress vector σ_Γ on the boundary Γ. Here, $\mathbf{U} = (U_1, U_2, U_3)$ is a given displacement vector on Γ.

We use the following notation for points inside Σ_h: $x = (x_1, x_2, x_3) = (y, z)$, where $y = (y_1, y_2) = (x_1, x_2)$ and $z = x_3$.

2.1. Elastic fields in the curvilinear coordinates. We introduce the orthogonal system of curvilinear coordinates (n, s, z) in $\mathscr{V}$, where n and s are defined in Section 1. In particular, the contour γ is given by $(x_1, x_2) = \zeta(s)$, $0 \leqslant s \leqslant |\gamma|$, where $|\gamma|$ is the length of γ, which, by rescalling, is assumed to be equal to 1. Let (n_1, n_2) be the unit outward normal vector to the boundary γ of ω. Then $n_1 = \zeta_2'(s)$, $n_2 = -\zeta_1'(s)$, and

$$(x_1, x_2) = (\zeta_1(s), \zeta_2(s)) + n(\zeta_2'(s), -\zeta_1'(s)), \quad x_3 = z \tag{2.2}$$

in the neighborhood $\mathscr{V}$. Since this system of coordinates is orthogonal, we can use the notation and general formulas in [19, Appendix C] to write all the elasticity relations in this local coordinate system. In particular, the corresponding orthonormal basis is $\mathbf{n} = (-\zeta_2'(s), \zeta_1'(s), 0)$, $\mathbf{s} = (\zeta_1'(s), \zeta_2'(s), 0)$, $\mathbf{z} = (0,0,1)$ and the scale factors are given by $H_n = H_z = 1$, $H_s = 1+n\varkappa(s)$, where $\varkappa(s) = \zeta_2''(s)\zeta_1'(s) - \zeta_1''(s)\zeta_2'(s)$ is the curvature of γ. The Jacobian of the transformation (2.2) is denoted by J, and $J = H_n H_s H_z = 1+n\varkappa(s)$. The components of the displacement vector in this coordinate system are expressed as $u_n = n_1u_1 + n_2u_2$ and $u_s = -n_2u_1 + n_1u_2$, $u_z = u_3$. The components of the strain tensor are given by

$$\begin{aligned}
&\varepsilon_{nn} = \frac{\partial u_n}{\partial n}, \quad \varepsilon_{ss} = \frac{1}{J}\Big(\frac{\partial u_s}{\partial s} + \varkappa u_n\Big), \ \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \\
&\varepsilon_{ns} = \varepsilon_{sn} = \frac{1}{2}\Big(\frac{\partial u_s}{\partial n} + \frac{1}{J}\Big(\frac{\partial u_n}{\partial s} - \varkappa u_s\Big)\Big), \\
&\varepsilon_{sz} = \varepsilon_{zs} = \frac{1}{2}\Big(\frac{1}{J}\frac{\partial u_z}{\partial s} + \frac{\partial u_s}{\partial z}\Big), \quad \varepsilon_{zn} = \varepsilon_{nz} = \frac{1}{2}\Big(\frac{\partial u_z}{\partial n} + \frac{\partial u_n}{\partial z}\Big).
\end{aligned} \tag{2.3}$$

We also need the derivatives of the basis vectors:

$$\begin{aligned}
&\frac{\partial \mathbf{n}}{\partial n} = \frac{\partial \mathbf{n}}{\partial z} = \frac{\partial \mathbf{s}}{\partial n} = \frac{\partial \mathbf{s}}{\partial z} = \frac{\partial \mathbf{z}}{\partial n} = \frac{\partial \mathbf{z}}{\partial s} = \frac{\partial \mathbf{z}}{\partial z} = 0, \\
&\frac{\partial \mathbf{n}}{\partial s} = \varkappa(s)\mathbf{s}, \quad \frac{\partial \mathbf{s}}{\partial s} = -\varkappa(s)\mathbf{n}.
\end{aligned}$$

Using these relations, we obtain the elasticity equations in Σ_h

$$\begin{aligned}
&\frac{\partial \sigma_{nn}}{\partial n} + \frac{1}{J}\varkappa(\sigma_{nn} - \sigma_{ss}) + \frac{1}{J}\frac{\partial \sigma_{sn}}{\partial s} + \frac{\partial \sigma_{zn}}{\partial z} = \rho\partial_t^2 u_n, \\
&\frac{\partial \sigma_{sn}}{\partial n} + 2\frac{1}{J}\varkappa\sigma_{sn} + \frac{1}{J}\frac{\partial \sigma_{ss}}{\partial s} + \frac{\sigma_{sz}}{\partial z} = \rho\partial_t^2 u_s, \\
&\frac{\partial \sigma_{zn}}{\partial n} + \frac{1}{J}\varkappa\sigma_{zn} + \frac{1}{J}\frac{\partial \sigma_{sn}}{\partial s} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\partial_t^2 u_z
\end{aligned} \tag{2.4}$$

(cf. [19, Appendix C]).

2.2. The matrix notation. We use the matrix, rather than tensor, notation. We denote by $\mathscr{U} = (u_1, u_2, u_3)^T$ a column vector with components u_1, u_2, and u_3. Using the Voigt–Mandel notation (cf., for example, [16, 20, 21]), we introduce the strain and stress columns

$$\begin{aligned}
&\boldsymbol{\varepsilon}(\mathscr{U}) = (\varepsilon_{11}, \sqrt{2}\varepsilon_{12}, \sqrt{2}\varepsilon_{13}, \varepsilon_{22}, \varepsilon_{33}, \sqrt{2}\varepsilon_{32})^T, \\
&\boldsymbol{\sigma}(\mathscr{U}) = (\sigma_{11}, \sqrt{2}\sigma_{12}, \sqrt{2}\sigma_{13}, \sigma_{22}, \sigma_{33}, \sqrt{2}\sigma_{32})^T.
\end{aligned} \tag{2.5}$$

The factor $\sqrt{2}$ is used for equalizing the Euclidian norm of columns and the norm of the corresponding tensors and the superscript T denotes the transpose of the corresponding vector/matrix. Moreover, the Hooke law in (1.3) converts into

$$\boldsymbol{\sigma}(\mathscr{U}) = A\boldsymbol{\varepsilon}(\mathscr{U}), \tag{2.6}$$

where A is a symmetric positive definite 6×6-matrix (the rigidity (Hooke) matrix) whose entries are related to the entries of the rigidity tensor $\mathbf{A} = \{A_{ij}^{pq}\}$ by

$$A_{11} = A_{11}^{11}, \quad A_{12} = \sqrt{2}A_{11}^{12}, \quad A_{13} = \sqrt{2}A_{11}^{13}, \quad A_{14} = A_{11}^{22}, \quad A_{15} = A_{11}^{33}, \quad A_{16} = \sqrt{2}A_{11}^{23},$$
$$A_{21} = \sqrt{2}A_{12}^{11}, \quad A_{22} = 2A_{12}^{12}, \quad A_{23} = 2A_{12}^{13}, \quad A_{24} = \sqrt{2}A_{12}^{22}, \dots$$

Let $\varphi \in [0, 2\pi)$. Consider the orthogonal transformation

$$x \to \widehat{x} = \theta x, \quad \theta = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.7}$$

which is the rotation about the z-axis by the angle φ. Then the displacement, strain, and stress column vectors are transformed as

$$\widehat{\mathscr{U}} = \theta\mathscr{U}, \quad \widehat{\boldsymbol{\varepsilon}} = \Theta^T\boldsymbol{\varepsilon}, \quad \widehat{\boldsymbol{\sigma}} = \Theta^T\boldsymbol{\sigma}, \tag{2.8}$$

where the 6×6-matrix Θ is orthogonal and given by

$$\Theta = \begin{pmatrix} \cos^2\varphi & \sqrt{2}\sin\varphi\cos\varphi & 0 & \sin^2\varphi & 0 & 0 \\ -\sqrt{2}\sin\varphi\cos\varphi & \cos^2\varphi - \sin^2\varphi & 0 & \sqrt{2}\sin\varphi\cos\varphi & 0 & 0 \\ 0 & 0 & \cos\varphi & 0 & 0 & \sin\varphi \\ \sin^2\varphi & -\sqrt{2}\sin\varphi\cos\varphi & 0 & \cos^2\varphi & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\sin\varphi & 0 & 0 & \cos\varphi \end{pmatrix}.$$

This formulas is easily verified and can be found in [16, Chapter 2]. Note that the orthogonality property of Θ in (2.8) is valid due to the presence of the factor $\sqrt{2}$ in (2.5).

Comparing (2.8) and (2.6), we conclude that the change of variables (2.7) leads to the following transformation of the rigidity matrix: $A \mapsto \mathscr{A} = \Theta^T A\,\Theta$. Using the notation (2.5), we write the last formula in (1.3) in the matrix form

$$\boldsymbol{\varepsilon}(\mathscr{U}) = D(\nabla_x)\mathscr{U},$$

where $\nabla_x = \operatorname{grad}$ and $D(\nabla_x)$ is a 6×3-matrix of first order differential operators,

$$D(\xi) = \begin{pmatrix} \xi_1 & \frac{1}{\sqrt{2}}\xi_2 & \frac{1}{\sqrt{2}}\xi_3 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}\xi_1 & 0 & \xi_2 & 0 & \frac{1}{\sqrt{2}}\xi_3 \\ 0 & 0 & \frac{1}{\sqrt{2}}\xi_1 & 0 & \xi_3 & \frac{1}{\sqrt{2}}\xi_2 \end{pmatrix}^T.$$

2.3. The surface rearrangement for stresses and strains. As shown, for example, in [18], it is convenient to rearrange components in the stress and strain vectors. First, let us introduce the strain and stress columns in the orthogonal curvilinear coordinates (n, s, z):

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{nn}, \sqrt{2}\varepsilon_{ns}, \sqrt{2}\varepsilon_{nz}, \varepsilon_{ss}, \varepsilon_{zz}, \sqrt{2}\varepsilon_{zs})^T,$$
$$\boldsymbol{\sigma}(\mathbf{u}) = (\sigma_{nn}, \sqrt{2}\sigma_{ns}, \sqrt{2}\sigma_{nz}, \sigma_{ss}, \sigma_{zz}, \sqrt{2}\sigma_{zs})^T,$$

where $\mathbf{u}$ is the column vector $(u_n, u_s, u_z)^T$. Then the Hooke law takes the form

$$\boldsymbol{\sigma}(\mathbf{u}) = \mathscr{A}\boldsymbol{\varepsilon}(\mathbf{u}), \tag{2.9}$$

where $\mathscr{A} = \Theta(\varphi)^T A \Theta(\varphi)$. Here, φ is the angle between y_1-axis and the normal $\mathbf{n}$ depending on s, but independent of n and z. We introduce two columns

$$\begin{aligned} \boldsymbol{\eta}(\mathbf{u}) &= (\sigma_{nn}, \sqrt{2}\sigma_{ns}, \sqrt{2}\sigma_{nz}, \varepsilon_{ss}, \varepsilon_{zz}, \sqrt{2}\varepsilon_{zs})^T, \\ \boldsymbol{\xi}(\mathbf{u}) &= (-\varepsilon_{nn}, -\sqrt{2}\varepsilon_{ns}, -\sqrt{2}\varepsilon_{nz}, \sigma_{ss}, \sigma_{zz}, \sqrt{2}\sigma_{zs})^T. \end{aligned} \tag{2.10}$$

The important property of this rearrangement is that all the components in $\boldsymbol{\eta}(\mathbf{u})$ are "observable" on the surface Γ. This means that the stress column $\sigma^\dagger(\mathbf{u}) = (\sigma_{nn}, \sqrt{2}\sigma_{ns}, \sqrt{2}\sigma_{nz})^T$ implies the traction on Γ given in the elasticity problem data, and the strain column

$$\varepsilon^\sharp(\mathbf{u}) = (\varepsilon_{ss}, \varepsilon_{zz}, \sqrt{2}\varepsilon_{zs})^T \tag{2.11}$$

can be evaluated from components of the displacement vector on Γ and their derivatives with respect to s and z, i.e., along the surface Γ only (cf. (2.3)). The columns

$$\begin{aligned} \varepsilon^\dagger(\mathbf{u}) &= (\varepsilon_{nn}, \sqrt{2}\varepsilon_{ns}, \sqrt{2}\varepsilon_{nz})^T, \\ \sigma^\sharp(\mathbf{u}) &= (\sigma_{ss}, \sigma_{zz}, \sqrt{2}\sigma_{zs})^T \end{aligned} \tag{2.12}$$

gathered into a column in (2.10), do not possess the above properties and can be regarded as "unobservable." Indeed, to compute the components in (2.12), one has to differentiate the displacements in n (cf. (2.3)). Therefore, one needs to know those displacements inside the body that are unobservable.

We represent the rigidity matrix $\mathscr{A}$ blockwise

$$\mathscr{A} = \begin{pmatrix} A^{\dagger\dagger} & A^{\dagger\sharp} \\ A^{\sharp\dagger} & A^{\sharp\sharp} \end{pmatrix}, \tag{2.13}$$

where all the blocks are 3×3-matrices, the matrices $A^{\dagger\dagger}$ and $A^{\sharp\sharp}$ are positive definite, and $A^{\dagger\sharp} = (A^{\sharp\dagger})^T$. We write (2.9) in the form

$$\begin{aligned} \sigma^\dagger(\mathbf{u}) &= A^{\dagger\dagger}\varepsilon^\dagger(\mathbf{u}) + A^{\dagger\sharp}\varepsilon^\sharp(\mathbf{u}), \\ \sigma^\sharp(\mathbf{u}) &= A^{\sharp\dagger}\varepsilon^\dagger(\mathbf{u}) + A^{\sharp\sharp}\varepsilon^\sharp(\mathbf{u}). \end{aligned}$$

Then

$$\begin{aligned} \sigma^\sharp(\mathbf{u}) &= (A^{\sharp\sharp} - A^{\sharp\dagger}(A^{\dagger\dagger})^{-1}A^{\dagger\sharp})\varepsilon^\sharp(\mathbf{u}) + A^{\sharp\dagger}A_{\dagger\dagger}^{-1}\sigma^\dagger(\mathbf{u}), \\ -\varepsilon^\dagger(\mathbf{u}) &= (A^{\dagger\dagger})^{-1}A^{\dagger\sharp}\varepsilon^\sharp(\mathbf{u}) - (A^{\dagger\dagger})^{-1}\sigma^\dagger(\mathbf{u}). \end{aligned}$$

Thus, we get the following relation connecting the $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ columns:

$$\boldsymbol{\xi}(\mathbf{u}) = \mathbf{Q}\boldsymbol{\eta}(\mathbf{u}), \quad \mathbf{Q} = \begin{pmatrix} Q^{\dagger\dagger} & Q^{\dagger\sharp} \\ Q^{\sharp\dagger} & Q^{\sharp\sharp} \end{pmatrix},$$

$Q^{\sharp\sharp} = A^{\sharp\sharp} - A^{\sharp\dagger}(A^{\dagger\dagger})^{-1}A^{\dagger\sharp} > 0$, $Q^{\dagger\dagger} = -(A^{\dagger\dagger})^{-1} < 0$, $Q^{\sharp\dagger} = A^{\sharp\dagger}(A^{\dagger\dagger})^{-1}$, $Q^{\dagger\sharp} = (A^{\dagger\dagger})^{-1}A^{\dagger\sharp}$. The positivity of $Q^{\sharp\sharp}$ follows from the relations $0 < \mathbf{a}^T\mathscr{A}\mathbf{a} = (a^\sharp)^T Q^{\sharp\sharp} a^\sharp$, $\mathbf{a} = (-(A^{\dagger\dagger})^{-1}A^{\dagger\sharp}a^\sharp, a^\sharp)^T$ for any $a^\sharp \in \mathbb{R}^3 \setminus \{0\}$. It is clear that Q is symmetric and invertible, but not positive definite.

Remark 2.1. According to [18], the quantity $\frac{1}{2}\boldsymbol{\xi}(\mathbf{u})^T\boldsymbol{\eta}(\mathbf{u})$ is the density of the surface enthalpy. This particular Gibbs functional naturally appears in the asymptotic analysis of thin layers and surface structures such as elastic coatings, phase interfaces, propagating cracks etc.

3 The Dimension Reduction Procedure

The main goal of this section is to show that the leading term of σ_Γ obtained from the solution to the problem (1.2), (1.3) and satisfying the boundary conditions (1.4) and $\mathbf{u} = \mathbf{U}$ on Γ has the form

$$\sigma_\Gamma = -hD^\sharp(\varkappa(s), -\partial_s, -\partial_z)^T\, \overline{Q}^{\sharp\sharp}(s,z) D^\sharp(\varkappa(s), \partial_s, \partial_z)\mathbf{U}(s,z) - h\overline{\rho}(s,z)\partial_t^2\mathbf{U}(s,z), \tag{3.1}$$

where $\varkappa$ is defined after formula (1.7),

$$D^\sharp(\varkappa, \partial_s, \partial_z) = \begin{pmatrix} \varkappa & \partial_s & 0 \\ 0 & 0 & \partial_z \\ 0 & \frac{1}{\sqrt{2}}\partial_z & \frac{1}{\sqrt{2}}\partial_s \end{pmatrix} \tag{3.2}$$

$$\overline{Q}^{\sharp\sharp}(s,z) = \begin{pmatrix} \overline{Q}_{11}^{\sharp\sharp} & \overline{Q}_{12}^{\sharp\sharp} & \overline{Q}_{13}^{\sharp\sharp} \\ \overline{Q}_{21}^{\sharp\sharp} & \overline{Q}_{22}^{\sharp\sharp} & \overline{Q}_{23}^{\sharp\sharp} \\ \overline{Q}_{31}^{\sharp\sharp} & \overline{Q}_{32}^{\sharp\sharp} & \overline{Q}_{33}^{\sharp\sharp} \end{pmatrix}(s,z) = \int_0^{H(s)} Q^{\sharp\sharp}(\zeta, s, z)\, d\zeta, \tag{3.3}$$

$$\overline{\rho}(s,z) = \int_0^{H(s)} \rho(\zeta, s, z)\, d\zeta. \tag{3.4}$$

The matrix $Q^{\sharp\sharp}$ is the Schur complement of the block $A^{\dagger\dagger}$ of the matrix $\mathscr{A}$, defined in (2.13), i.e.,

$$Q^{\sharp\sharp} = A^{\sharp\sharp} - A^{\sharp\dagger}(A^{\dagger\dagger})^{-1}A^{\dagger\sharp}. \tag{3.5}$$

3.1. The asymptotic ansatz and leading term. We suppose that ρ and $\mathbf{A}$ satisfy one of conditions (I) or (II) from Section 1. Therefore, Equation (2.9) takes the form

$$\boldsymbol{\sigma}(\mathbf{u}; n, s, z) = \mathscr{A}(\zeta, s, z)\boldsymbol{\varepsilon}(\mathbf{u}; n, s, z), \tag{3.6}$$

where $\zeta = h^{-1}n$ is regarded as the fast variable or the stretched transversal coordinate.

We look for an asymptotic solution to the problem (1.2), (1.3) satisfying the boundary conditions (1.4) and (2.1) in the form

$$\mathbf{u}^h(n,s,z) = \mathbf{u}^0(s,z) + h\mathbf{u}'(\zeta,s,z) + h^2\mathbf{u}''(\zeta,s,z) + \ldots \tag{3.7}$$

The superscript h on the left-hand side of (3.7) emphasizes the dependence of the solution on the small parameter h. On the right-hand side of (3.7), $\mathbf{u}^0$ stands for the leading term which is independent of the fast variable, as we will explain below. In Section 4, we find the correction terms $\mathbf{u}'$ and $\mathbf{u}''$ and derive a limit system of differential equations for $\mathbf{u}^0 = \mathbf{U}$. All functions may depend also on the parameter t, but we will not indicate this dependence explicitly.

Substituting $\partial_n = h^{-1}\partial_\zeta$ into (2.3), we find

$$\boldsymbol{\varepsilon}(\mathbf{u}^h) = h^{-1}D(\partial_\zeta, 0, 0)\mathbf{u}^0 + \ldots \tag{3.8}$$

Here and in the sequel, the dots stand for higher-order terms which are not important for the current step of asymptotic procedure. Similarly, the elasticity equations (2.4) take the form

$$h^{-1}D(\partial_\zeta, 0, 0)^T\boldsymbol{\sigma}(\mathbf{u}^h) + \ldots = \ldots \tag{3.9}$$

Moreover, since the gradient operator ∇_x in the curvilinear coordinates goes to $(\partial_n, J^{-1}\partial_s, \partial_z)$, the normal $\mathbf{n}^h$ on the exterior boundary $\Gamma_h = \gamma_h \times \mathbb{R}$ has the form

$$\mathbf{n}^h(n,s) = (1 + J(n,s)^{-2}h^2|\partial_s H(s)|^2)^{-1/2}(1, -hJ(n,s)^{-1}\partial_s H(s), 0)^T. \tag{3.10}$$

Hence $n_n^h = 1 + O(h^2)$, $n_s^h = O(h)$, and $n_z^h = 0$ which converts the boundary condition (1.4) to

$$D(1,0,0)^T \boldsymbol{\sigma}(\mathbf{u}^h) + \ldots = 0. \tag{3.11}$$

From (3.9), (3.11), (3.8), (3.6), and (2.1) we get the mixed boundary value problem for the system of ordinary equations in ζ with the parameters $(s,z) \in \Gamma$

$$\begin{aligned} &- D(\partial_\zeta, 0, 0)^T \mathscr{A}(\zeta, s, z) D(\partial_\zeta, 0, 0)\mathbf{u}^0(\zeta, s, z) = 0, \quad \zeta \in \Upsilon(s), \\ &D(1,0,0)^T \mathscr{A}(H(s), s, z) D(\partial_\zeta, 0, 0)\mathbf{u}^0(H(s), s, z) = 0, \\ &\mathbf{u}^0(0, s, z) = \mathbf{U}(s, z). \end{aligned} \tag{3.12}$$

Since the matrix $\mathscr{A}$ is symmetric and positive definite and the rank of $D(1,0,0)$ is equal to 3, the 3×3-matrix

$$\mathbf{a} = D(1,0,0)^T \mathscr{A} D(1,0,0) \tag{3.13}$$

is also symmetric and positive definite. In this notation, the differential operator in the first line of (3.12) takes the form $-\partial_\zeta \mathbf{a}(\zeta, s, z)\partial_\zeta$ and the operator in the second line is as follows: $\mathbf{a}(\zeta, s, z)\partial_\zeta$. Hence the problem (3.12) has a unique solution independent of ζ:

$$\mathbf{u}^0(\zeta, s, z) = \mathbf{U}(s, z). \tag{3.14}$$

3.2. The first correction term. Since $\mathbf{u}^0$ is independent of ζ, we get

$$\boldsymbol{\varepsilon}(\mathbf{u}^h; n, s, z) = \boldsymbol{\varepsilon}^0(s, z) + D(\partial_\zeta, 0, 0)\mathbf{u}'(\zeta, s, z) + \ldots$$

where

$$\boldsymbol{\varepsilon}^0 = \Big(0, \frac{1}{\sqrt{2}}(\partial_s u_n^0 - \varkappa u_s^0), \frac{1}{\sqrt{2}}\partial_z u_n^0, \partial_s u_s^0 + \varkappa u_n^0, \partial_z u_z^0, \frac{1}{\sqrt{2}}(\partial_s u_z^0 + \partial_z u_s^0)\Big)^T. \tag{3.15}$$

Collecting coefficients of order h^{-1} in the elasticity equations, we arrive at the system of ordinary differential equations

$$-D(\partial_\zeta, 0, 0)^T \mathbf{A}(\zeta, s, z) D(\partial_\zeta, 0, 0)\mathbf{u}'(\zeta, s, z) = D(\partial_\zeta, 0, 0)^T \mathbf{A}(\zeta, s, z)\boldsymbol{\varepsilon}^0(s, z), \quad \zeta \in \Upsilon(s). \tag{3.16}$$

The boundary conditions (1.4) on the exterior boundary Γ_h imply

$$D(1,0,0)^T \mathbf{A}(H(s), s, z) D(\partial_\zeta, 0, 0)\mathbf{u}'(H(s), s, z) = -D(1,0,0)^T \mathbf{A}(H(s), s, z)\boldsymbol{\varepsilon}^0(s, z). \tag{3.17}$$

Furthermore, we derive the second boundary condition

$$\mathbf{u}'(0, s, z) = 0 \tag{3.18}$$

because the right-hand side of (2.1) contains no term of order h. Since the matrix differential operator on the left-hand side of (3.17) can be written as

$$D(1,0,0)^T \mathbf{A}(\zeta, s, z) D(1,0,0)\partial_\zeta = \mathbf{a}(\zeta, s, z)\partial_\zeta,$$

we can solve (3.16), (3.17) and use the matrix (3.13) to have

$$\partial_\zeta \mathbf{u}'(\zeta, s, z) = -\mathbf{a}(\zeta, s, z)^{-1} D(1,0,0)^T \mathbf{A}(\zeta, s, z)\boldsymbol{\varepsilon}^0(s, z). \tag{3.19}$$

Taking into account (3.18), we obtain

$$\mathbf{u}'(\zeta, s, z) = -\int_0^\zeta \mathbf{a}(\tau, s, z)^{-1} D(1,0,0)^T \mathbf{A}(\tau, s, z)\boldsymbol{\varepsilon}^0(s, z) d\tau.$$

We can calculate the trace of the leading term of the normal stresses on Γ:

$$\begin{aligned} &D(1,0,0)^T \boldsymbol{\sigma}(\mathbf{u}^h; 0, s, z) = D(1,0,0)^T \mathbf{A}(0, s, z)(\boldsymbol{\varepsilon}^0(s, z) \\ &+ D(1,0,0)\partial_s \mathbf{u}'(0, s, z) + \ldots) = D(1,0,0)^T \mathbf{A}(0, s, z)\boldsymbol{\varepsilon}^0(s, z) \\ &- D(1,0,0)^T \mathbf{A}(0, s, z) D(1,0,0) \mathbf{a}(0, s, z)^{-1} D(1,0,0)^T \mathbf{A}(0, s, z)\boldsymbol{\varepsilon}^0(s, z) + \ldots = 0 + \ldots , \end{aligned}$$

where we used the equality (3.13) to show that the leading term vanishes. In other words, the first couple of asymptotic terms in the ansatz (3.7) brings zero traction on the interior surface contacting blood. In Section 4, we show that the traction generated by the third term $h^2\mathbf{u}''$ becomes nontrivial and is given by the matrix differential operator applied to the vector (3.13). In the asymptotic analysis, it is convenient to endow formally the rigidity matrix $\mathbf{A}$ with the order h^{-1}, which means that the elastic wall is thin, but hard. Certainly, this is true for both arteries and veins.

3.3. The second correction term. As in Subsection 3.2, we conclude that the term $\mathbf{u}''$ in (3.7) satisfies the same mixed boundary value problem for the system of ordinary differential equations in ζ but with new right-hand sides $\mathbf{f}''$ and $\mathbf{g}''$:

$$\begin{aligned} &- D(\partial_\zeta, 0, 0)^T \mathbf{A}(\zeta, s, z) D(\partial_\zeta, 0, 0)\mathbf{u}''(\zeta, s, z) = \mathbf{f}''(\zeta, s, z),\ \zeta \in \Upsilon(s), \\ &D(1,0,0)^T \mathbf{A}(H(s), s, z) D(\partial_\zeta, 0, 0)\mathbf{u}''(H(s), s, z) = \mathbf{g}''(s, z), \\ &\mathbf{u}''(0, s, z) = 0. \end{aligned} \tag{3.20}$$

To find $\mathbf{f}''$ and $\mathbf{g}''$, we take into account the lower-order terms in the strain columns $\boldsymbol{\varepsilon}(\mathbf{u}^0)$ and $\boldsymbol{\varepsilon}(\mathbf{u}')$. First, we obtain

$$\boldsymbol{\varepsilon}(\mathbf{u}^0; n, s, z) = \boldsymbol{\varepsilon}^0(s, z) + h\boldsymbol{\varepsilon}^1(\zeta, s, z) + \ldots, \tag{3.21}$$

where $\boldsymbol{\varepsilon}^0$ is given by (3.15). According to (2.3), we set

$$\boldsymbol{\varepsilon}^1 = -\zeta\varkappa\Big(0, \frac{1}{\sqrt{2}}(\partial_s u_n - \varkappa u_s), \frac{1}{\sqrt{2}}\partial_s u_z, \partial_s u_s + \varkappa u_n, 0, 0\Big)^T, \tag{3.22}$$

where the factor $-\zeta\varkappa(s)$ comes from the decomposition

$$J(n, s)^{-1} = (1 + n\varkappa(s))^{-1} = 1 - h\zeta\varkappa(s) + O(h^2).$$

For $\boldsymbol{\varepsilon}(\mathbf{u}')$ we have

$$\boldsymbol{\varepsilon}(\mathbf{u}'; n, s, z) = h^{-1} D(\partial_\zeta, 0, 0)\mathbf{u}'(\zeta, s, z) + \boldsymbol{\varepsilon}'(\zeta, s, z) + \ldots , \tag{3.23}$$

where, as in (3.15),

$$\boldsymbol{\varepsilon}' = \Big(0, \frac{1}{\sqrt{2}}(\partial_s u_n' - \varkappa u_s'), \frac{1}{\sqrt{2}}\partial_z u_n', \partial_s u_s' + \varkappa u_n', \partial_z u_z', \frac{1}{\sqrt{2}}(\partial_s u_z' + \partial_z u_s')\Big)^T. \tag{3.24}$$

Formulas (3.21) and (3.23) allow us to compute the traction on Γ

$$\begin{aligned} &D(1,0,0)^T \boldsymbol{\sigma}(\mathbf{u}^h; 0, s, z) = h\big(D(1,0,0)^T \mathbf{A}(0,s,z) D(\partial_\zeta, 0, 0)\mathbf{u}''(0,s,z) \\ &+ D(1,0,0)^T \mathbf{A}(0,s,z)(\boldsymbol{\varepsilon}^1(0,s,z) + \boldsymbol{\varepsilon}'(0,s,z))\big) + \dots \\ &= h\big(D(1,0,0)^T \mathbf{A}(0,s,z) D(\partial_\zeta,0,0)\mathbf{u}''(0,s,z) + D(1,0,0)^T \mathbf{A}(0,s,z)\boldsymbol{\varepsilon}'(0,s,z)\big) + \dots, \end{aligned} \tag{3.25}$$

where we used that $\boldsymbol{\varepsilon}^1(\zeta, s, z) = 0$ for $\zeta = 0$ due to the factor ζ in (3.22). Solving (3.20), we get

$$D(1,0,0)^T \mathbf{A}(0,s,z) D(\partial_\zeta, 0, 0)\mathbf{u}''(0,s,z) = \int_0^{H(s)} \mathbf{f}''(\zeta,s,z)d\zeta + \mathbf{g}''(s,z).$$

Therefore, for calculating the next term for the traction on Γ it suffices to determine the right-hand sides $\mathbf{f}''$ and $\mathbf{g}''$.

To compute $\mathbf{f}''$, we need the terms (3.22) and (3.24) and asymptotic expansion of the matrix differential operator on the left-hand side of the equilibrium equations (2.4) which is

$$-h^{-1}D(\partial_\zeta,0,0)^T - h^0(D(0,\partial_s,\partial_z)^T + \varkappa(s)\mathbb{K}) + \dots,$$

where

$$\mathbb{K} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$\begin{aligned} \mathbf{f}''(\zeta,s,z) &= D(\partial_\zeta,0,0)^T \mathbf{A}(\zeta,s,z)(\boldsymbol{\varepsilon}^1(\zeta,s,z) + \boldsymbol{\varepsilon}'(\zeta,s,z)) + (D(0,\partial_s,\partial_z)^T \\ &\quad + \varkappa(s)\mathbb{K})\mathbf{A}(\zeta,s,z)\big(\boldsymbol{\varepsilon}^0(s,z) + D(\partial_\zeta,0,0)\mathbf{u}'(\zeta,s,z)\big) + \rho(\zeta,s,z)\partial_t^2\mathbf{u}^0(s,z), \end{aligned} \tag{3.26}$$

where the right-hand side of (2.4) was taken into account.

To calculate $\mathbf{g}''$, we recall that $D(\mathbf{n}^h(s,z)) = D(1,0,0) - hD(0,\partial_s H(s), 0) + \dots$ in view of (3.10). Therefore,

$$\begin{aligned} \mathbf{g}''(s,z) &= -D(1,0,0)^T \mathbf{A}(H(s),s,z)(\boldsymbol{\varepsilon}^1(H(s),s,z) + \boldsymbol{\varepsilon}'(H(s),s,z)) \\ &\quad + D(0,\partial_s H(s),0)^T \mathbf{A}(H(s),s,z)(\boldsymbol{\varepsilon}^0(s,z) + D(\partial_\zeta,0,0)\mathbf{u}'(H(s),s,z)). \end{aligned} \tag{3.27}$$

From (3.26) and (3.27) it follows that

$$\begin{aligned} \int_0^{H(s)} \mathbf{f}''(\zeta,s,z)d\zeta + \mathbf{g}''(s,z) &= \Bigg(\int_0^{H(s)} D(\partial_\zeta,0,0)^T \mathbf{A}(\zeta,s,z)(\boldsymbol{\varepsilon}^1(\zeta,s,z) + \boldsymbol{\varepsilon}'(\zeta,s,z))d\zeta \\ &- D(1,0,0)^T \mathbf{A}(H(s),s,z)(\boldsymbol{\varepsilon}^1(H(s),s,z) + \boldsymbol{\varepsilon}'(H(s),s,z))\Bigg) \end{aligned}$$

$$+\left(\int_0^{H(s)} (D(0,\partial_s,\partial_z)^T + \varkappa(s)\mathbb{K})\mathbf{A}(\zeta,s,z)(\boldsymbol{\varepsilon}^0(s,z) + D(\partial_\zeta,0,0)\mathbf{u}'(\zeta,s,z))d\zeta\right.$$

$$\left. + D(0,\partial_s H(s),0)^T\mathbf{A}(H(s),s,z)(\boldsymbol{\varepsilon}^0(s,z) + D(\partial_\zeta,0,0)\mathbf{u}'(H(s),s,z))\right)$$

$$-\int_0^{H(s)} \rho(\zeta,s,z)d\zeta\partial_t^2\mathbf{u}^0(s,z) =: I_1 + I_2 - I_3.$$

Integrating by parts, we find $I_1 = -D(1,0,0)^T\mathbf{A}(0,s,z)\boldsymbol{\varepsilon}'(0,s,z)$, which cancels the last term in (3.25). Furthermore, one can directly check that

$$I_2 = (D(0,\partial_s,\partial_z)^T + \varkappa(s)\mathbb{K})\int_0^{H(s)} \mathbf{A}(\zeta,s,z)(\boldsymbol{\varepsilon}^0(s,z) + D(\partial_\zeta,0,0)\mathbf{u}'(\zeta,s,z))d\zeta.$$

Using (3.19), we get

$$I_2 = (D(0,\partial_s,\partial_z)^T + \varkappa(s)\mathbb{K})\mathbf{M}(s,z)\boldsymbol{\varepsilon}^0(s,z), \tag{3.28}$$

where

$$\mathbf{M}(s,z) = \int_0^{H(s)} \big(\mathbf{A}(\zeta,s,z) - \mathbf{A}(\zeta,s,z)D(1,0,0)\mathbf{a}(\zeta,s,z)^{-1}D(1,0,0)^T\mathbf{A}(\zeta,s,z)\big)d\zeta \tag{3.29}$$

is a symmetric 6×6 matrix.

Lemma 3.1. *The matrix* $\mathbf{M}$ *has the form*

$$\mathbf{M}(s,z) = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \overline{Q}^{\sharp\sharp}(s,z) \end{pmatrix}, \tag{3.30}$$

where $\mathbb{O}$ *is the null* 3×3*-matrix and* $\overline{Q}^{\sharp\sharp}(s,z)$ *is given by* (3.3) *and* (3.5).

Proof. By (2.13), (3.12), and (3.32), the integrand in (3.29) is expressed as

$$\begin{pmatrix} A^{\dagger\dagger} & A^{\dagger\sharp} \\ A^{\sharp\dagger} & A^{\sharp\sharp} \end{pmatrix} - \begin{pmatrix} A^{\dagger\dagger} & A^{\dagger\sharp} \\ A^{\sharp\dagger} & A^{\sharp\sharp} \end{pmatrix}\begin{pmatrix} \mathbb{E} \\ \mathbb{O} \end{pmatrix}\left[\begin{pmatrix} \mathbb{E} & \mathbb{O} \end{pmatrix}\begin{pmatrix} A^{\dagger\dagger} & A^{\dagger\sharp} \\ A^{\sharp\dagger} & A^{\sharp\sharp} \end{pmatrix}\begin{pmatrix} \mathbb{E} \\ \mathbb{O} \end{pmatrix}\right]^{-1}$$

$$\times \begin{pmatrix} \mathbb{E} & \mathbb{O} \end{pmatrix}\begin{pmatrix} A^{\dagger\dagger} & A^{\dagger\sharp} \\ A^{\sharp\dagger} & A^{\sharp\sharp} \end{pmatrix} = \begin{pmatrix} A^{\dagger\dagger} & A^{\dagger\sharp} \\ A^{\sharp\dagger} & A^{\sharp\sharp} \end{pmatrix} - \begin{pmatrix} \mathbb{A}^{\dagger\dagger} \\ \mathbb{A}^{\sharp\dagger} \end{pmatrix}(A^{\dagger\dagger})^{-1}\begin{pmatrix} A^{\dagger\dagger} & A^{\dagger\sharp} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & A^{\sharp\sharp} - A^{\sharp\dagger}(A^{\dagger\dagger})^{-1}A^{\dagger\sharp} \end{pmatrix} = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & Q^{\sharp\sharp} \end{pmatrix},$$

where $Q^{\sharp\sharp}$ is defined by (3.5) and

$$\mathbb{E} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.31}$$

The above relation together with (3.3) yields (3.30). $\square$

Let $D^\sharp(\varkappa(s), \partial_s, \partial_z)$ be defined by (3.2). Lemma 3.1 together with (3.14) shows that (3.28) can be written as

$$I_2 = -D^\sharp(\varkappa(s), -\partial_s, -\partial_z)^T \overline{Q}^{\sharp\sharp}(s,z)\boldsymbol{\varepsilon}^\sharp(\mathbf{U}; s, z), \tag{3.32}$$

where $\boldsymbol{\varepsilon}^\sharp$ is given by (2.11). Using the evident equality $J(0,s) = 1$, we get

$$\boldsymbol{\varepsilon}^\sharp(\mathbf{U}) = \Big(\partial_s u_s + \varkappa u_n, \partial_z u_z, \frac{1}{\sqrt{2}}(\partial_s u_z + \partial_z u_s)\Big)^T = D^\sharp(\varkappa, \partial_s, \partial_z)\mathbf{U}.$$

Finally, since $I_3 = \overline{\rho}(s,z)\partial_t^2 \mathbf{U}(s,z)$, where $\overline{\rho}$ is defined by (3.4), we arrive at the following expression for the normal stress vector on Γ:

$$\begin{aligned}\sigma_\Gamma = D(1,0,0)^T \boldsymbol{\sigma}(\mathbf{u}^h; 0, s, z) = -h\overline{\rho}(s,z)\partial_t^2 \mathbf{U}(s,z) \\ - hD^\sharp(\varkappa(s), -\partial_s, -\partial_z)^T\, \overline{Q}^{\sharp\sharp}(s,z) D^\sharp(\varkappa(s), \partial_s, \partial_z)\mathbf{U}(s,z) + \ldots,\end{aligned} \tag{3.33}$$

which together with (3.14) leads to (3.1).

Remark 3.1. Writing the Hooke law (2.6) in the form $\boldsymbol{\varepsilon}(u) = \mathbf{B}\boldsymbol{\sigma}(u)$, where $\mathbf{B} = \mathbf{A}^{-1}$ is the compliance matrix, we observe that $Q^{\sharp\sharp} = B^{\sharp\sharp}$.

3.4. The model of the vessel wall. The flow in the cylindrical vessel Ω is described by the Navier–Stokes equations (1.1), where $\mathbf{v}$ is the velocity vector, p is the pressure, and ν is the kinematic viscosity. On the vessel walls, it is assumed that the wall velocity coincides with the fluid velocity described by the relation (1.5) and the hydrodynamic force is equal to the normal stress vector on the boundary. By (3.33), the latter means

$$D^\sharp(\varkappa(s), -\partial_s, -\partial_z)^T \overline{Q}^{\sharp\sharp}(s,z) D^\sharp(\varkappa(s), \partial_s, \partial_z)\mathbf{U}(s,z) + \overline{\rho}(s,z)\partial_t^2 \mathbf{U}(s,z) = -h^{-1}\rho_b \mathbf{F}(s,z) \tag{3.34}$$

on $\partial\Omega$, where $\rho_b \mathbf{F}$ is the hydrodynamic force with components given by (1.7), ρ_b is the blood density, and $D^\sharp$ is defined by (3.2).

4 Analysis of the Model

4.1. The additive property of the rigidity matrix. Let $\Sigma_h = \varsigma_h \times \mathbb{R}$ be a laminated wall consisting of K layers of thickness h_k. In each layer, the rigidity matrix $A_{(k)}$ is assumed to be constant. The relations (3.4) and (3.3) take the form

$$\overline{\rho}(s,z) = \sum_{j=1}^{N} \frac{h_j}{h}\rho^j(s,z),$$

$$\overline{Q}^{\sharp\sharp} = \sum_{k=1}^{K} \frac{h_k}{h} Q^{\sharp\sharp}_{(k)},$$

where h_k/h is the normalized thickness of the kth layer and $Q^{\sharp\sharp}_{(k)}$ is the block of the matrix $Q_{(k)}$ constructed by using the matrix $A_{(k)}$ according to (3.5).

4.2. The rigidity matrix for an arterial wall. The laminate structure of walls depends on the type of blood vessels. The most studied vessels are arteries (cf. [1, 3, 4, 5, 22]) whose walls consist of three layers: intima, media, and adventitia. The internal layer, which is just a very thin film, does not affect elastic properties of the wall. However, the media and adventitia layers are composites formed by bundles of collagen fibers of a homogeneous material consisting of muscle cells. In each layer, the bundles are usually modeled by two families of fibres that are wound around the cylinder under the angles $\pm\varphi_m$ and $\pm\varphi_a$ to the z-axis respectively. Here, $\varphi_m, \varphi_a \in (0, \pi/2)$. As a result, we obtain composite materials reinforced by periodic families of rigid rods.

There are several approaches for determining elastic properties of laminated composite walls of arteries. For example, a nonlinear rheological stress/strain relation is proposed in [5] for the entire arterial wall, as well as in the case of dissection of the media and adventitia. For estimating elastic properties of vessels by solving inverse problems we refer to [23] and [24]. However, for these rheological relations no two-dimensional model still exists. In this paper, we use a technique based on the linear homogenization theory, which allows us to compute explicitly the matrix (3.3) with the boundary condition (3.1).

Applying the asymptotic homogenization procedure developed in [9, 25], we obtain the following representation of the rigidity matrix:

$$\overline{Q}^{\sharp\sharp} = Q^{\sharp\sharp}_{(c)} + Q^{\sharp\sharp}_{(m)} + Q^{\sharp\sharp}_{(a)}, \tag{4.1}$$

where

$$Q^{\sharp\sharp}_{(m)} = E_m \sum_{\pm} \Theta^\sharp(\pm\varphi_m)\mathbb{E}\Theta^\sharp(\pm\varphi_m)^T,$$
$$Q^{\sharp\sharp}_{(a)} = E_a \sum_{\pm} \Theta^\sharp(\pm\varphi_a)\mathbb{E}\Theta^\sharp(\pm\varphi_a)^T.$$

Here,

$$Q_{(c)} = \begin{pmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & 2\mu \end{pmatrix}$$

is the rigidity matrix of an isotropic filler with the Lamé constants λ and μ, which are small with respect to the Young moduli E_m and E_a of the collagen fibers from the media and adventitia layers, i.e., $\mu, \lambda \ll E_m, E_c$. Furthermore, the matrix $\mathbb{E}$ is given by (3.31) and

$$\Theta^\sharp(\varphi) = \begin{pmatrix} \cos^2\varphi & \sin^2\varphi & -\sqrt{2}\sin\varphi\cos\varphi \\ \sin^2\varphi & \cos^2\varphi & \sqrt{2}\sin\varphi\cos\varphi \\ \sqrt{2}\sin\varphi\cos\varphi & -\sqrt{2}\sin\varphi\cos\varphi & \cos^2\varphi - \sin^2\varphi \end{pmatrix}$$

(cf. [16, Chapter 2] for details). In particular,

$$Q^{\sharp\sharp}_{(a)} = 2E_a \begin{pmatrix} \sin^4\varphi_a & \sin^2\varphi_a\cos^2\varphi_a & 0 \\ \sin^2\varphi_a\cos^2\varphi_a & \cos^4\varphi_a & 0 \\ 0 & 0 & 2\sin^2\varphi_a\cos^2\varphi_a \end{pmatrix}. \tag{4.2}$$

A similar formula for $Q^{\sharp\sharp}_{(m)}$ can be obtained from (4.2) with φ_a replaced by φ_m. Thus, the composite material of the artery wall after averaging is orthotropic with the main orthotropy axes directed along the z- and s- axes.

The matrices $Q^{\sharp\sharp}_{(m)}$ and $Q^{\sharp\sharp}_{(a)}$ are not positive definite. For example, $(\cos^2\varphi_a, -\sin^2\varphi, 0)^T$ belongs to the kernel of the matrix (4.2). However, $Q^{\sharp\sharp}_{(m)} + Q^{\sharp\sharp}_{(a)}$ and, consequently, the matrix (4.1) is positive definite provided that $\varphi_m \neq \varphi_a$. The last condition is satisfied by arteries (cf. [1, 3, 4, 5, 22]). Thus, the main elastic cyclic load is taken by collagen fibers and their location determines the orthotropic properties of the wall.

On the other hand, each layer is anisotropic and has different resistence properties in different directions depending on the angles φ_a and φ_m respectively. Under separation of media and adventitia layers, such loads, caused by blood pulsation, lead to oscillations of large amplitude, which can be a reason of dissonance in media and adventitia layers. This observation can explain the well-known fact in medical practice that the artery dissection (separation of layers) can lead to aneurysm and even may stimulate crushing of vessel walls.

4.3. The stability estimate and Green formula for the limit problem. The goal of this subsection is to present the Green formula and obtain a stability estimate for the problem (1.1), (3.34), (1.5). For the sake of simplicity, we consider the Navier–Stokes system without convective acceleration, i.e., instead of (1.1), we consider the Stokes system

$$\partial_t \mathbf{v} - \nu\Delta\mathbf{v} = -\nabla p \quad \text{in } \Omega.$$

We consider a pair $(\mathbf{V}, \mathbf{W})$, where $\mathbf{V}$ is a solenoidal vector-valued function on $\Omega\times[0,T]$ and $\mathbf{W}$ is a vector-valued function on $\Gamma\times[0,T]$, T is a positive number. We assume that $\mathbf{V}|_\Gamma = \partial_t\mathbf{W}$ and note that the pair $(\mathbf{v}, \mathbf{U})$ in (1.1) and (3.34) also possesses these properties. Multiplying the equation by a solenoidal vector field $\mathbf{V} = (V_1, V_2, V_3)$ and using the Green formula for the Stokes system (cf. [26, Chapter 3, Section 2]), we get

$$\int\limits_\Omega (\partial_t\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p)\cdot\mathbf{V}dx = \frac{\nu}{2}\int\limits_\Omega \Big(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k}\Big)\Big(\frac{\partial V_k}{\partial x_i} + \frac{\partial V_i}{\partial x_k}\Big)dx$$
$$- \int\limits_\Gamma \mathscr{T}_{ik}(\mathbf{v})V_i n_k dS_\Gamma + \int\limits_\Omega \partial_t\mathbf{v}\cdot\mathbf{V}dx, \tag{4.3}$$

where summation over repeating indexes is assumed, the dot denotes the inner product of two vectors, and

$$\mathscr{T}_{ik}(\mathbf{v}) = -\delta_i^k p + \nu\Big(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k}\Big).$$

Here, δ_i^k is the Kronecker delta. By (1.6) and (3.34),

$$\int\limits_\Gamma \mathscr{T}_{ik}(\mathbf{v})V_i n_k dS_\Gamma = -\frac{h}{\rho_b}\Big(a(\mathbf{U},\mathbf{V}) + \int\limits_\Gamma \overline{\rho}(s,z)\partial_t^2\mathbf{U}\cdot\mathbf{V}dx\Big)$$

with

$$a(\mathbf{U},\mathbf{V}) = \int\limits_\Gamma \overline{Q}^{\sharp\sharp}(s,z)D^\sharp(\varkappa(s),\partial_s,\partial_z)\mathbf{U}(s,z)\cdot D^\sharp(\varkappa(s),\partial_s,\partial_z)\mathbf{V}dS_\Gamma.$$

Using this identity, we write (4.3) in the form

$$\int_\Omega (\partial_t \mathbf{v} - \nu\Delta\mathbf{v} + \nabla p)\cdot \mathbf{V}dx = \frac{\nu}{2}\int_\Omega \Big(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k}\Big)\Big(\frac{\partial V_k}{\partial x_i} + \frac{\partial V_i}{\partial x_k}\Big)dx$$

$$+ \frac{h}{\rho_b}\left(a(\mathbf{U}, \partial_t \mathbf{W}) + \int_\Gamma \overline{\rho}(s,z)\partial_t^2 \mathbf{U}\cdot\partial_t \mathbf{W} dS_\Gamma\right) + \int_\Omega \partial_t \mathbf{v}\cdot\mathbf{V}dx. \tag{4.4}$$

Setting $(\mathbf{V}, \mathbf{W}) = (\mathbf{v}, \mathbf{U})$ and integrating over $[0, T]$, we find

$$\frac{\nu}{2}\int_0^T\int_\Omega \Big(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k}\Big)\Big(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k}\Big)dx + \frac{1}{2}\int_\Omega |\mathbf{v}|^2 dx|_{t=T} + \frac{h}{2\rho_b}\left(a(\mathbf{U},\mathbf{U})|\right.$$

$$\left. + \int_\Gamma \overline{\rho}(s,z)|\partial_t\mathbf{U}|^2 dS_\Gamma\right)\Big|_{t=T} = \frac{1}{2}\int_\Omega |\mathbf{v}|^2 dx|_{t=0} + \frac{h}{2\rho_b}\left(a(\mathbf{U},\mathbf{U}) + \int_\Gamma \overline{\rho}(s,z)|\partial_t \mathbf{U}|^2 dS_\Gamma\right)\Big|_{t=0}, \tag{4.5}$$

where we recalled that the right-hand side in (4.4) vanishes in view of (1.1). Since the problem (1.1), (3.34), (1.5) is supplied with initial conditions for $\mathbf{v}$ and $\mathbf{U}$, the equality (4.5) implies the stability estimate for the norm of the solution in terms of the norms of the initial conditions. The relation (4.4) can be used to define a weak solution to the problem (1.1), (3.34), (1.5).

4.4. On the shape of the vessel cross-section. The relations (3.1) and (3.3) allow us to find the geometry of the wall, which is based on two observations. First, according to [6, 22, 27, 28, 29], the vessel walls are mainly subject to homogeneous hydrostatical pressure at low velocities, i.e., the leading term for F in (1.6) and (3.34) is of the form

$$F_n(s,z,t) = -p^0(z,t) + \dots,$$
$$F_s(s,z,t) = 0 + \dots,$$
$$F_z(s,z,t) = 0 + \dots.$$

Let us assume that both derivatives of $p^0(z,t)$ with respect to z and t are small, i.e., the hydrostatic pressure changes slowly in time and along the vessel. Then the system (3.34) takes the form

$$\begin{aligned} &\varkappa(\overline{Q}_{11}^{\sharp\sharp}(\varkappa u_n^0 + \partial_s u_s^0) + \overline{Q}_{13}^{\sharp\sharp} 2^{-1/2}\partial_s u_z^0) = h^{-1}p^0, \\ &-\partial_s(\overline{Q}_{11}^{\sharp\sharp}(\varkappa u_n^0 + \partial_s u_s^0) + \overline{Q}_{13}^{\sharp\sharp} 2^{-1/2}\partial_s u_z^0) = 0, \\ &-2^{1/2}\partial_s(\overline{Q}_{31}^{\sharp\sharp}(\varkappa u_n^0 + \partial_s u_s^0) + \overline{Q}_{33}^{\sharp\sharp} 2^{-1/2}\partial_s u_z^0) = 0. \end{aligned} \tag{4.6}$$

Comparing the first and second equations, we see that $\varkappa$ is independent of s and, consequently, $\varkappa = \varkappa_0 = \text{const}$, which implies that the cross-section is a disc. From the first and third equations in (4.6) we get

$$\overline{Q}_{11}^{\sharp\sharp}(\varkappa_0 u_n^0 + \partial_s u_s^0) + \overline{Q}_{13}^{\sharp\sharp} 2^{-1/2}\partial_s u_z^0 = h^{-1}\varkappa_0^{-1}p^0,$$

$$\overline{Q}_{31}^{\sharp\sharp}(\varkappa_0 u_n^0 + \partial_s u_s^0) + \overline{Q}_{33}^{\sharp\sharp} 2^{-1/2}\partial_s u_z^0 = c_0.$$

Therefore,

$$u_n^0 = \frac{1}{q^\sharp}\Big((\overline{Q}_{31}^{\sharp\sharp}\varkappa_0 + \overline{Q}_{33}^{\sharp\sharp}2^{-1/2})\frac{p_0}{h\varkappa_0} - (\overline{Q}_{11}^{\sharp\sharp}\varkappa_0 + \overline{Q}_{13}^{\sharp\sharp}2^{-1/2})c_0\Big),$$

$$\partial_s u_z^0 = \frac{1}{q^\sharp}(\overline{Q}_{11}^{\sharp\sharp}\varkappa_0 c_0 - \overline{Q}_{31}^{\sharp\sharp}p_0),$$

where $q^\sharp = \overline{Q}_{11}^{\sharp\sharp}\varkappa_0(\overline{Q}_{31}^{\sharp\sharp}\varkappa_0 + \overline{Q}_{33}^{\sharp\sharp}2^{-1/2}) - \overline{Q}_{31}^{\sharp\sharp}\varkappa_0(\overline{Q}_{11}^{\sharp\sharp}\varkappa_0 + \overline{Q}_{13}^{\sharp\sharp}2^{-1/2})$.

4.5. The vessel cross-section revisited. It is reasonable to assume that the vessel wall is subject to residual stresses, which can be explained as follows. The blood circulation system is formed in the embryonic state of organism and then develops through the growth of collagen fibres and muscle cells which is different for them. This fact may cause residual stresses in elastic walls which can be described by the additional terms

$$D^\sharp(\varkappa(s), -\partial_s, -\partial_z)(g_s(s), g_z(s), 0)^T = (\varkappa(s)g_s(s) - \partial_s g_z(s), 0, 0)^T \tag{4.7}$$

on the right-hand side of (3.34), which do not destroy the structure of the system (4.6) and lead to the same conclusion about the circular shape of the cross-section of vessel as in Subsection 4.2.

The situation is different if external forces are taken into account. These forces can be caused, for example, by an asymmetric position of a surgical suture. In this case, the right-hand side of (3.34) has the additional stress term $\tau(s) = (\tau_n(s), \tau_s(s), \tau_z(s))$ and (4.6) takes the form

$$\begin{aligned}
&\varkappa(\overline{Q}_{11}^{\sharp\sharp}(\varkappa u_n^0 + \partial_s u_s^0) + \overline{Q}_{13}^{\sharp\sharp}2^{-1/2}\partial_s u_z^0) = p^0 + \tau_n,\\
&-\partial_s(\overline{Q}_{11}^{\sharp\sharp}(\varkappa u_n^0 + \partial_s u_s^0) + \overline{Q}_{13}^{\sharp\sharp}2^{-1/2}\partial_s u_z^0) = \tau_s,\\
&-2^{1/2}\partial_s(\overline{Q}_{31}^{\sharp\sharp}(\varkappa u_n^0 + \partial_s u_s^0) + \overline{Q}_{33}^{\sharp\sharp}2^{-1/2}\partial_s u_z^0) = \tau_z.
\end{aligned} \tag{4.8}$$

Integrating the second equation, we get

$$\overline{Q}_{11}^{\sharp\sharp}(\varkappa u_n^0 + \partial_s u_s^0) + \overline{Q}_{13}^{\sharp\sharp}2^{-1/2}\partial_s u_z^0 = -\int_0^s \tau_s(\rho)d\rho + c_1. \tag{4.9}$$

Let the right-hand side of (4.9) be a continuous 1-periodic function, i.e., $\int_0^1 \tau_s(\rho)d\rho = 0$. From (4.8) and (4.9) it follows that $\varkappa\Big(-\int_0^s \tau_s(\rho)d\rho + C_1\Big) = p^0 + \tau_n$. Since $\varkappa$ is the curvature of a closed curve with the unit length, we have

$$\int_0^1 \varkappa(s)ds = 2\pi \tag{4.10}$$

provided that the origin is located inside the curve. Hence for C_1 we obtain the equation

$$2\pi = \int_0^1 (p^0 + \tau_n(s))\Big(-\int_0^s \tau_s(\rho)d\rho + C_1\Big)^{-1} ds.$$

Then we find the curve from the curvature.

Let $\varkappa = \varkappa(s)$ be given. It is convenient to assume that $\varkappa$ is given for all s and is periodic with period 1. We assume that the relation (4.10) is valid and reconstruct the curve $\zeta(s)$. We have the following explicit formulas for ζ (cf., for example, [30, Chapter 3, Section 5]):

$$\zeta_1(s) = \int_0^s \sin\alpha(\rho)d\rho + x_0, \quad \zeta_2(s) = \int_0^s \sin\alpha(\rho)d\rho + y_0, \tag{4.11}$$

where $\alpha(s) = \int_0^s \varkappa(\rho)d\rho$ and x_0, y_0 are constants. By the periodicity of $\varkappa$ and (4.10), we get $\alpha(s+1) = 2\pi + \alpha(s)$. Hence the sufficient conditions for the curve (4.11) to be closed are as follows:

$$\int_0^1 \sin\alpha(\rho)d\rho = 0, \quad \int_0^1 \cos\alpha(\rho)d\rho = 0. \tag{4.12}$$

If $\varkappa$ is positive, then we change variable $y = \alpha(\rho)$, $dy = \varkappa(\rho)d\rho$, in (4.12), which yields

$$\int_0^{2\pi} \frac{\sin y\, dy}{\varkappa(s(y))} = 0, \quad \int_0^{2\pi} \frac{\cos y\, dy}{\varkappa(s(y))} = 0.$$

The violation of (4.12), which means that the wall shape is unstable and supports the same artery pathologies as in the case of dissection described in Subsection 4.4. The deviation of the cross-section shape from the circular one causes the deterioration of the blood permeability: it is known that, among all cross-sections of the set perimeter, it is the circular cross section that provides the largest stream of a fluid for the Poiseuille flow. However, the local distortion of the artery shape represents a risk of secondary significance for the vascular system because the basic threat follows from a decrease in the cross-section area by means of the formation and accumulation of atherosclerotic damages.

Acknowledgments

We thank D. S. Kolesnikov and O. A. Nazarova for consultations on medical questions.

S. Nazarov was supported by the Russian Foundation for Basic Research (project No. 15-01-02175) and Linköping University (Sweden). V. Kozlov acknowledges the support of the Swedish Research Council (VR).

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Submitted on August 3, 2015