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ORTHOGONAL BASIS FOR WAVELET FLOWS

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We present an orthogonal basis for discrete wavelets in the case of comb structure of the spline-wavelet decomposition and estimate the time of computation of this decomposition by a concurrent computing system with computer communication surrounding taken into account. Bibliography: 9 *titles.*

Wavelet decompositions are widely used, which strongly stimulates their further development. Unlike classical wavelet decompositions [1, 2], the approach proposed in [3] does not require to construct a wavelet basis. On the other hand, the approach of [3] provides asymptotically optimal (with respect to an N-diameter of standard compact sets) spline-wavelet approximations [4]. In the computer realization of decompositions, the knowledge of a basis allows us to diminish the computation time essentially. We note that the wavelet bases have not been considered earlier for the above-mentioned spline-wavelets. It turns out that, in this case, to diminish the computation time, it suffices to know a basis of discrete wavelets.

In this paper, we obtain an orthogonal (in the Euclidean space) basis of discrete wavelets in the case of a spline-wavelet decomposition of the comb structure [5]. We estimate the computation time necessary to realize this decomposition by a concurrent computing system with computer communication surrounding taken into account.

The paper consists of eight sections. In the first four sections, we construct the first order spline-wavelet decomposition. In Section 5, we construct the orthogonal basis for the space of discrete wavelets, whereas Section 6 deals with continuous basis wavelets. Section 7 is devoted to the time of computation of the main flow by successive and concurrent computing systems with computer communication surrounding taken into account. Section 8 contains a similar study for numerical realization of the wavelet flow.

1 Preliminaries

1.1. The space of first order splines. For any natural m we introduce the notation: $J_m \stackrel{\text{def}}{=} \{0, 1, \ldots, m\}$ and $J'_m \stackrel{\text{def}}{=} \{-1, 0, 1, \ldots, m\}$. Let N be a natural number. On $[a, b]$, we consider a grid

$$X : \quad a = x_0 < x_1 < \ldots < x_{N-2} < x_{N-1} < x_N = b, \tag{1.1}$$

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and denote $S_j \stackrel{\text{def}}{=} (x_j, x_{j+1}) \cup (x_{j+1}, x_{j+2})$, $j \in J_{N-2}$,

$$G \stackrel{\text{def}}{=} \bigcup_{i \in J_{N-1}} (x_i, x_{i+1}), \quad S_{-1} \stackrel{\text{def}}{=} (x_0, x_1), \quad S_{N-1} \stackrel{\text{def}}{=} (x_{N-1}, x_N).$$

Let $A \stackrel{\text{def}}{=} \{\mathbf{a}_i\}_{i \in J'_{N-1}}$ be a complete chain of two-dimensional vectors, i.e., $\det(\mathbf{a}_{j-1}, \mathbf{a}_j) \neq 0$, $j \in J_{N-1}$ (cf. [5]).

We consider a two-component vector-valued function $\varphi(t)$ with continuous components on $[a, b]$ which are linearly independent on any interval $(a', b') \subset G$.

We define the functions $\omega_j(t)$, $t \in G$, $j \in J'_{N-1}$, by the approximate relations

$$\sum_{j \in J'_{N-1}} \mathbf{a}_j \, \omega_j(t) = \varphi(t) \quad \forall t \in G, \quad \omega_j(t) \equiv 0 \quad \forall t \in G \backslash S_j, \ j \in J'_{N-1}.$$

Thus, the functions $\omega_j(t)$, $j \in J'_{N-1}$, are defined on the set G by

$$\omega_{-1}(t) = \begin{cases} \dfrac{\det(\varphi(t), \mathbf{a}_0)}{\det(\mathbf{a}_{-1}, \mathbf{a}_0)}, & t \in (x_0, x_1), \\ 0, & t \in G \backslash S_{-1}, \end{cases} \tag{1.2}$$

$$\omega_j(t) = \begin{cases} \dfrac{\det(\mathbf{a}_{j-1}, \varphi(t))}{\det(\mathbf{a}_{j-1}, \mathbf{a}_j)}, & t \in (x_j, x_{j+1}), \\ \dfrac{\det(\varphi(t), \mathbf{a}_{j+1})}{\det(\mathbf{a}_j, \mathbf{a}_{j+1})}, & t \in (x_{j+1}, x_{j+2}), \\ 0, & t \in G \backslash S_j, \end{cases} \quad j \in J_{N-2}, \tag{1.3}$$

$$\omega_{N-1}(t) = \begin{cases} \dfrac{\det(\mathbf{a}_{N-2}, \varphi(t))}{\det(\mathbf{a}_{N-2}, \mathbf{a}_{N-1})}, & t \in (x_{N-1}, x_N), \\ 0, & t \in G \backslash S_{N-1}. \end{cases} \tag{1.4}$$

Below, we consider the space of first order splines $\mathbb{S}_N = \mathbb{S}_N(X, A, \varphi) \stackrel{\text{def}}{=} Cl_p \mathscr{L}\{\omega_j\}_{j \in J'_{N-1}}$, where $\mathscr{L}\{\ldots\}$ is the span of the functions in the curly brackets and Cl_p is the closure in the topology of pointwise convergence. By assumptions on $\varphi(t)$, the functions $\omega_j(t)$, $j \in J'_{N-1}$, are linearly independent on G and, consequently, form a basis for the space $\mathbb{S}_N$. Consequently, we have $\dim \mathbb{S}_N = N + 1$.

Remark 1.1. As is known [6], the functions $\omega_j(t)$, $j \in J'_{N-1}$, can be extended by continuity to the points of X if and only if $\mathbf{a}_i = \varphi(x_{i+1})$ for all $i \in J'_{N-1}$.

1.2. Continuous first order splines. We consider the case

$$\varphi(t) = \begin{pmatrix} 1, & t \end{pmatrix}^T \quad \mathbf{a}_j = \mathbf{a}_j^* \stackrel{\text{def}}{=} \varphi(x_{j+1}). \tag{1.5}$$

The functions obtained from (1.2)–(1.4) are denoted by $\omega_j^*(t)$ so that

$$\omega_{-1}^*(t) = \begin{cases} \dfrac{x_1 - t}{x_1 - x_0}, & t \in (x_0, x_1), \\ 0, & t \in G \backslash S_{-1}, \end{cases}$$

$$\omega_j^*(t) = \begin{cases} \dfrac{t - x_j}{x_{j+1} - x_j}, & t \in (x_j, x_{j+1}), \\ \dfrac{x_{j+2} - t}{x_{j+2} - x_{j+1}}, & t \in (x_{j+1}, x_{j+2}), \quad j \in J_{N-2}, \\ 0, & t \in G \backslash S_j, \end{cases}$$

$$\omega_{N-1}^*(t) = \begin{cases} \dfrac{t - x_{N-1}}{x_N - x_{N-1}}, & t \in (x_{N-1}, x_N), \\ 0, & t \in G \backslash S_{N-1}. \end{cases}$$

Sometimes, we need the uniform grid $\overline{X}_h \overset{\text{def}}{=} \{x_j \mid x_j = a + jh, j = 0, 1, 2, \ldots, N\}$, where $N \geqslant 5$ is a natural number. In the case of the uniform grid $\overline{X}_h$, the objects under considerations are marked with overline $^-$. For the uniform grid $\overline{X}_h$ we have $\overline{\omega}_j(t) = \omega(t/h - j)\chi_{[a,b]}$, $j = -1, 0, 1, \ldots, N-1$, where

$$\omega(x) \overset{\text{def}}{=} \begin{cases} x, & x \in (0,1), \\ 2 - x, & x \in (1,2), \\ 0, & x \notin [0,1], \end{cases}$$

and $\chi_{[a,b]}$ is the characteristic function of $[a, b]$.

2 Two-Interval Comb Structure

Let s and r be natural numbers, and let $s < r < \lfloor N/2 \rfloor$. We remove the points

$$x_{2s+1}, x_{2s+3}, x_{2s+5}, \ldots, x_{2r-1} \tag{2.1}$$

from the grid (1.1) and consider the enlarged grid

$$\widetilde{X} : a = \widetilde{x}_0 < \widetilde{x}_1 < \widetilde{x}_2 < \ldots < \widetilde{x}_{\widetilde{N}-1} < \widetilde{x}_{\widetilde{N}} = b,$$

where $\widetilde{N} = N - r + s$,

$$\widetilde{x}_i = x_i, \quad 0 \leqslant i \leqslant 2s,$$

$$\widetilde{x}_{i'} = x_{2i'-2s}, \quad 2s + 1 \leqslant i' \leqslant s + r,$$

$$\widetilde{x}_{i''} = x_{r-s+i''}, \quad s + r + 1 \leqslant i'' \leqslant N - r + s.$$

We set

$$\widetilde{S}_j \overset{\text{def}}{=} (\widetilde{x}_j, \widetilde{x}_{j+1}) \cup (\widetilde{x}_{j+1}, \widetilde{x}_{j+2}), \quad j \in J_{\widetilde{N}-2},$$

$$\widetilde{G} \overset{\text{def}}{=} \bigcup_{i \in J_{\widetilde{N}-1}} (\widetilde{x}_i, \widetilde{x}_{i+1}), \quad \widetilde{S}_{-1} \overset{\text{def}}{=} (\widetilde{x}_0, \widetilde{x}_1), \quad \widetilde{S}_{\widetilde{N}-1} \overset{\text{def}}{=} (\widetilde{x}_{\widetilde{N}-1}, \widetilde{x}_{\widetilde{N}})$$

and

$$I_*^{h} \overset{\text{def}}{=} \{-1, 0, \dots, 2s-2\},$$
$$I_*^{m} \overset{\text{def}}{=} \{2s-1, \dots, s+r-1\},$$
$$I_*^{t} \overset{\text{def}}{=} \{s+r, s+r+1, \dots, \widetilde{N}-1\}.$$

It is obvious that $I_*^h \cup I_*^m \cup I_*^t = J\,'_{\widetilde{N}-1}$. We consider the chain of vectors $\widetilde{A} \overset{\text{def}}{=} \{\widetilde{\mathbf{a}}_{-1}, \widetilde{\mathbf{a}}_0, \dots, \widetilde{\mathbf{a}}_{\widetilde{N}-1}\}$, such that the following assumption holds.

(A) A chain vectors $\widetilde{A} \overset{\text{def}}{=} \{\widetilde{\mathbf{a}}_j\}$ is complete, and

$$\widetilde{\mathbf{a}}_j = \mathbf{a}_j, \quad j \in I_*^h,$$
$$\widetilde{\mathbf{a}}_j = \mathbf{a}_{2j-2s+1}, \quad j \in I_*^m,$$
$$\widetilde{\mathbf{a}}_j = \mathbf{a}_{j+r-s}, \quad j \in I_*^t.$$

The set $\{X, A, \widetilde{X}, \widetilde{A}\}$ is called the *two-interval comb structure* [5].

We consider the system of functions $\{\widetilde{\omega}_j\}_{j\in J'_{\widetilde{N}-1}}$ obtained from the relations

$$\sum_{j\in J'_{\widetilde{N}-1}} \widetilde{\mathbf{a}}_j\, \widetilde{\omega}_j(t) = \varphi(t) \quad \forall t \in \widetilde{G},$$

$$\widetilde{\omega}_j(t) \equiv 0 \quad \forall t \in \widetilde{G}\backslash \widetilde{S}_j,\ j \in J\,'_{\widetilde{N}-1},$$

so that

$$\widetilde{\omega}_{-1}(t) = \begin{cases} \dfrac{\det\,(\varphi(t), \widetilde{\mathbf{a}}_0)}{\det\,(\widetilde{\mathbf{a}}_{-1}, \widetilde{\mathbf{a}}_0)}, & t \in (\widetilde{x}_0, \widetilde{x}_1), \\ 0, & t \in \widetilde{G}\backslash \widetilde{S}_{-1}, \end{cases}$$

$$\widetilde{\omega}_j(t) = \begin{cases} \dfrac{\det\,(\widetilde{\mathbf{a}}_{j-1}, \varphi(t))}{\det\,(\widetilde{\mathbf{a}}_{j-1}, \widetilde{\mathbf{a}}_j)}, & t \in (\widetilde{x}_j, \widetilde{x}_{j+1}), \\ \dfrac{\det\,(\varphi(t), \widetilde{\mathbf{a}}_{j+1})}{\det\,(\widetilde{\mathbf{a}}_j, \widetilde{\mathbf{a}}_{j+1})}, & t \in (\widetilde{x}_{j+1}, \widetilde{x}_{j+2}), \\ 0, & t \in \widetilde{G}\backslash \widetilde{S}_j, \end{cases} \qquad j \in J_{\widetilde{N}-2},$$

$$\widetilde{\omega}_{\widetilde{N}-1}(t) = \begin{cases} \dfrac{\det\,(\widetilde{\mathbf{a}}_{\widetilde{N}-2}, \varphi(t))}{\det\,(\widetilde{\mathbf{a}}_{\widetilde{N}-2}, \widetilde{\mathbf{a}}_{\widetilde{N}-1})}, & t \in (\widetilde{x}_{\widetilde{N}-1}, \widetilde{x}_{\widetilde{N}}), \\ 0, & t \in \widetilde{G}\backslash \widetilde{S}_{\widetilde{N}-1}. \end{cases}$$

Remark 2.1. We assume that all functions under consideration are restricted to the set G.

We introduce the space $\widetilde{\mathbb{S}} \overset{\text{def}}{=} Cl_p \mathscr{L}\{\widetilde{\omega}_j\}_{j\in J\,'_{\widetilde{N}-1}}$. It is obvious that $\dim \widetilde{\mathbb{S}} = \widetilde{N}+1$.

We will need the following assertions (cf. Theorems 1 and 2 in [5]).

1. For $t \in G$

$$\widetilde{\omega}_j(t) \equiv \omega_j(t), \quad j \in I_*^h,$$

$$\widetilde{\omega}_j(t) \equiv \omega_{r-s+j}(t), \quad j \in I_*^t.$$

2. If $0 \leqslant i \leqslant r-s-1$, then

$$\widetilde{\omega}_{2s+i}(t) \equiv \frac{\det(\mathbf{a}_{2s+2i-1}, \varphi(t))}{\det(\mathbf{a}_{2s+2i-1}, \mathbf{a}_{2s+2i+1})}, \quad t \in (\widetilde{x}_{2s+i}, \widetilde{x}_{2s+i+1}).$$

If $-1 \leqslant i \leqslant r-s-2$, then

$$\widetilde{\omega}_{2s+i}(t) \equiv \frac{\det(\varphi(t), \mathbf{a}_{2s+2i+3})}{\det(\mathbf{a}_{2s+2i+1}, \mathbf{a}_{2s+2i+3})}, \quad t \in (\widetilde{x}_{2s+i+1}, \widetilde{x}_{2s+i+2}).$$

3. Under Assumption (A), the inclusion $\widetilde{\mathbb{S}} \subset \mathbb{S}$ holds.

3 The Embedding Matrix

3.1. The space C_X and matrix of embedding. In what follows, we use the linear space C_X introduced earlier by the author (cf., for example, [3]).

Let (c,d) be an interval of the real axis. Denote by $C\langle c,d\rangle$ the linear space of functions that are continuous on (c,d) and have finite limits at the endpoints of (c,d). We introduce the linear space of functions as the direct product of the spaces $C\langle x_i, x_{i+1}\rangle$, $i = 0,1,\ldots,N-1$, namely,

$$C_X \stackrel{\text{def}}{=} \bigotimes_{i=0}^{N-1} C\langle x_i, x_{i+1}\rangle.$$

It is clear that $\omega_j \in C_X$, $j \in J'_{N-1}$, and $\widetilde{\mathbb{S}} \subset \mathbb{S} \subset C_X$.

Let $\{g_i\}_{i\in J'_{N-1}}$ be a realization of the system of linear functionals over C_X, biorthogonal to the system of functions $\{\omega_j\}_{j\in J'_{N-1}}$, $\langle g_i, \omega_j\rangle = \delta_{i,j}$, such that $\operatorname{supp} g_j \subset [x_j, x_j+\varepsilon)$, $j \in J_{N-1}$, $\operatorname{supp} g_{-1} \subset [x_0, x_0+\varepsilon)$, where $\varepsilon > 0$ is an arbitrary positive number (the existence of such realizations was established in [3]). We consider the matrix $\mathfrak{P}$ with entries

$$\mathfrak{p}_{i,j} \stackrel{\text{def}}{=} \langle g_j, \widetilde{\omega}_i\rangle, \quad i \in J'_{\widetilde{N}-1}, \quad j \in J'_{N-1}. \tag{3.1}$$

We denote $\operatorname{supp}\,{}_{+}\widetilde{\omega}_i = [\widetilde{x}_i, \widetilde{x}_{i+2})$ and consider three groups of the values of j:

$$I_H \stackrel{\text{def}}{=} \{-1,0,1,\ldots,2s-2\},$$
$$I_T \stackrel{\text{def}}{=} \{2r, 2r+1,\ldots,N-1\},$$
$$I_M \stackrel{\text{def}}{=} \{2s-1, 2s,\ldots,2r-1\}.$$

It is obvious that $I_H \cup I_M \cup I_T = J'_{N-1}$ and $I_*^h = I_H$. We need the following formulas proved in [5, Theorem 4]:

$$\mathfrak{p}_{ij} = \delta_{i,j} \quad \forall i \in J'_{\widetilde{N}-1},\ j \in \{-1,0,1,\ldots,2s-1\}, \tag{3.2}$$

$$\mathfrak{p}_{ij} = \delta_{j,i-s+r} \quad \forall i \in J'_{\widetilde{N}-1},\ j \in I_T; \tag{3.3}$$

whereas the remaining elements are computed for $q \in \{s, s+1,\ldots,r-1\}$ by the formulas

$$\mathfrak{p}_{i,2q} = \mathfrak{p}_{i,2q+1} = 0 \quad \forall i \in J'_{\widetilde{N}-1}\backslash\{s+q-1, s+q\}, \tag{3.4}$$

$$\mathfrak{p}_{s+q-1,2q+1} = 0, \quad \mathfrak{p}_{s+q,2q+1} = 1, \tag{3.5}$$

$$\mathfrak{p}_{s+q-1,2q} = \frac{\det(\mathbf{a}_{2q}, \mathbf{a}_{2q+1})}{\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q+1})}, \quad \mathfrak{p}_{s+q,2q} = \frac{\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q})}{\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q+1})}. \tag{3.6}$$

According to formula (3.2), we have

$$\begin{aligned}&\mathfrak{p}_{ij} = \delta_{i,j} \quad \forall \in {J'}_{\widetilde{N}-1},\ j \in \ \{-1,0,\dots,2s-1\}\\&\Longrightarrow [\mathfrak{P}^T\mathbf{a}]_j = a_j \quad \forall j \in \{-1,0,\dots,2s-1\}.\end{aligned} \tag{3.7}$$

From (3.3) we find

$$\begin{aligned}&\mathfrak{p}_{ij} = \delta_{j,i-s+r} \quad \forall i \in {J'}_{\widetilde{N}-1},\ j \in I_T = \{2r, 2r+1,\dots,N-1\}\\&\Longrightarrow [\mathfrak{P}^T\mathbf{a}]_j = a_{j+s-r} \quad \forall j \in \{2r, 2r+1,\dots,N-1\}.\end{aligned} \tag{3.8}$$

By (3.4) and (3.5), for $q \in \{s, s+1,\dots,r-1\}$ we have

$$\begin{aligned}&\mathfrak{p}_{i,2q+1} = 0 \ \ \forall i \in {J'}_{\widetilde{N}-1}\backslash\{s+q-1, s+q\}, \quad \mathfrak{p}_{s+q-1,2q+1} = 0,\ \mathfrak{p}_{s+q,2q+1} = 1\\&\Longrightarrow [\mathfrak{P}^T\mathbf{a}]_{2q+1} = a_{s+q} \ \ \forall q \in \{s, s+1,\dots,r-1\}.\end{aligned} \tag{3.9}$$

By (3.4) and (3.6), for $q \in \{s, s+1,\dots,r-1\}$ we have

$$\begin{aligned}&\mathfrak{p}_{i,2q} = 0 \quad \forall i \in {J'}_{\widetilde{N}-1}\backslash\{s+q-1, s+q\},\\&\mathfrak{p}_{s+q-1,2q} = \frac{\det(\mathbf{a}_{2q}, \mathbf{a}_{2q+1})}{\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q+1})},\\&\mathfrak{p}_{s+q,2q} = \frac{\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q})}{\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q+1})}\end{aligned}$$

so that $[\mathfrak{P}^T\mathbf{a}]_{2q} = \mathfrak{p}_{s+q-1,2q}\, a_{s+q-1} + \mathfrak{p}_{s+q,2q}\, a_{s+q}\ q \in \{s, s+1,\dots,r-1\}$. Consequently,

$$[\mathfrak{P}^T\mathbf{a}]_{2q} = \frac{\det(\mathbf{a}_{2q}, \mathbf{a}_{2q+1})}{\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q+1})}\, a_{s+q-1} + \frac{\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q})}{\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q+1})}\, a_{s+q} \quad \forall q \in \{s, s+1,\dots,r-1\}. \tag{3.10}$$

3.2. The embedding matrix of first order splines. The matrix $\mathfrak{P}$ and its entries are marked with asterisk $*$ if we deal with first order splines considered in Subsection 1.2.

Theorem 3.1. *The entries of* $\mathfrak{P}^*$ *are computed by the formulas*

$$\begin{aligned}&\mathfrak{p}^*_{ij} = \delta_{i,j} \quad \forall i \in {J'}_{\widetilde{N}-1},\ j \in \{-1,0,1,\dots,2s-1\},\\&\mathfrak{p}^*_{ij} = \delta_{j,i-s+r}, \quad \forall i \in {J'}_{\widetilde{N}-1},\ j \in I_T,\end{aligned}$$

whereas the remaining entries are computed for $q \in \{s,\dots,r-1\}$ *by the formulas*

$$\begin{aligned}&\mathfrak{p}^*_{i,2q} = \mathfrak{p}_{i,2q+1} = 0 \quad \forall i \in {J'}_{\widetilde{N}-1}\backslash\{s+q-1, s+q\},\\&\mathfrak{p}^*_{s+q-1,2q+1} = 0, \quad \mathfrak{p}^*_{s+q,2q+1} = 1,\\&\mathfrak{p}^*_{s+q-1,2q} = \frac{x_{2q+2} - x_{2q+1}}{x_{2q+2} - x_{2q}}, \quad \mathfrak{p}^*_{s+q,2q} = \frac{x_{2q+1} - x_{2q}}{x_{2q+2} - x_{2q}}.\end{aligned} \tag{3.11}$$

The proof is easily obtained from the relations (3.7)–(3.10) and (1.5).

In the case of a uniform grid $\overline{X}_h$, formulas (3.11) become simpler: $\overline{\mathfrak{p}}_{s+q-1,2q} = \overline{\mathfrak{p}}_{s+q,2q} = 1/2$.

4 The Extension Matrix

4.1. The extension matrix for the space of first order splines. We consider a system of functionals $\{\widetilde{g}_i\}_{i \in J'_{\widetilde{N}-1}}$, biorthogonal to $\{\widetilde{\omega}_j\}_{j \in J'_{\widetilde{N}-1}}$, i.e., $\langle \widetilde{g}_i, \widetilde{\omega}_j \rangle = \delta_{i,j}$, and such that

$$\operatorname{supp} \widetilde{g}_i \subset [\widetilde{x}_i, \widetilde{x}_i + \varepsilon) \quad \forall \varepsilon > 0, \quad i \in J_{\widetilde{N}-1}, \quad \operatorname{supp} \widetilde{g}_{-1} \subset (\widetilde{x}_0, \widetilde{x}_0 + \varepsilon).$$

Using the relation $\langle \widetilde{g}_i, \varphi \rangle = \widetilde{\mathbf{a}}_i$, we compute the $\widetilde{N}+1 \times N+1$-matrix $\mathfrak{Q}$ with entries $\mathfrak{q}_{i,j} \stackrel{\text{def}}{=} \langle \widetilde{g}_i, \omega_j \rangle$, $i \in J'_{\widetilde{N}-1}$, $j \in J'_{N-1}$. We use the following formulas proved in [5, Theorem 6].

1. For all $i \in J'_{\widetilde{N}-1}$

$$\mathfrak{q}_{i,j} = \begin{cases} \delta_{i,j}, & j \in I_H, \\ \delta_{i,j+s-r}, & j \in I_T. \end{cases}$$

2. In addition,

$$\begin{aligned} &\mathfrak{q}_{2s-1,j} = \delta_{2s-1,j} \quad \forall j \in J'_{N-1}, \\ &\mathfrak{q}_{s+q,j} = 0 \quad \forall j \in J'_{N-1} \backslash \{2q-1, 2q\}, \quad s \leqslant q \leqslant r-1. \end{aligned} \tag{4.1}$$

3. Finally,

$$\mathfrak{q}_{s+q,2q} = \frac{\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q+1})}{\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q})} \neq 0, \tag{4.2}$$

$$\mathfrak{q}_{s+q,2q-1} = \frac{\det(\mathbf{a}_{2q+1}, \mathbf{a}_{2q})}{\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q})} \neq 0, \quad s \leqslant q \leqslant r-1. \tag{4.3}$$

We note that the fact that the expressions (4.2) and (4.3) are different from zero follows from Assumption (A).

4.2. The extension matrix for first order splines. In the case of first order splines, the following assertion holds.

Theorem 4.1. *The entries of the matrix $\mathfrak{Q}^*$ are computed by the following formulas.*

1. *For all $i \in J'_{\widetilde{N}-1}$*

$$\mathfrak{q}^*_{i,j} = \delta_{i,j}, \quad j \in I_H, \tag{4.4}$$

$$\mathfrak{q}^*_{i,j} = \delta_{i,j+s-r}, \quad j \in I_T. \tag{4.5}$$

2. *In addition,*

$$\mathfrak{q}^*_{2s-1,j} = \delta_{2s-1,j} \quad \forall j \in J'_{N-1}, \tag{4.6}$$

$$\mathfrak{q}^*_{s+q,j} = 0 \quad \forall j \in J'_{N-1} \backslash \{2q-1, 2q\}, \quad s \leqslant q \leqslant r-1. \tag{4.7}$$

3. *Finally,*

$$\mathfrak{q}^*_{s+q,2q} = \frac{x_{2q+2} - x_{2q}}{x_{2q+1} - x_{2q}}, \tag{4.8}$$

$$\mathfrak{q}^*_{s+q,2q-1} = \frac{x_{2q+1} - x_{2q+2}}{x_{2q+1} - x_{2q}}, \quad s \leqslant q \leqslant r-1. \tag{4.9}$$

Proof. Formulas (4.4)–(4.7) immediately follow from (4.1)–(4.3), whereas formulas (4.8) and (4.9) are obtained from (4.2) and (4.3) with (1.5) taken into account. □

In the case of a uniform grid $\overline{X}_h$, formulas (4.8), (4.9) are simplified:

$$\overline{\mathfrak{q}}_{s+q,2q} = 2, \quad \overline{\mathfrak{q}}_{s+q,2q-1} = -1.$$

5 Decomposition of Wavelet Flows

5.1. Projection operator. We consider the projection operator P from the space $\mathbb{S}$ to the subspace $\widetilde{\mathbb{S}}$ given by the formula

$$Pu \overset{\text{def}}{=} \sum_{j\in J'_{\widetilde{N}-1}} \langle \widetilde{g}_j, u\rangle\, \widetilde{\omega}_j \quad \forall\, u \in \mathbb{S} \tag{5.1}$$

and introduce the operator $Q = \mathscr{I} - P$, where $\mathscr{I}$ is the identity operator in $\mathbb{S}$. As a result, we obtain the direct decomposition [6]

$$\mathbb{S} = \widetilde{\mathbb{S}} \dotplus \mathbb{W}, \tag{5.2}$$

called the *first order spline-wavelet decomposition of the space* $\mathbb{S}$, where $\widetilde{\mathbb{S}}$ is said to be the *main space* and $\mathbb{W} \overset{\text{def}}{=} Q\mathbb{S}$ is referred to as the *wavelet space*.

Let $u \in \mathbb{S}$. Using (5.1) and (5.2), we find $u = \widetilde{u} + w$, where

$$u = \sum_{j\in J'_{N-1}} c_j\omega_j, \quad \widetilde{u} = \sum_{i\in J'_{\widetilde{N}-1}} a_i\widetilde{\omega}_i, \quad w = \sum_{j\in J'_{N-1}} b_j\omega_j, \tag{5.3}$$

$a_i \overset{\text{def}}{=} \langle \widetilde{g}_i, u\rangle$, $b_j, c_j \in \mathbb{R}^1$. Introducing the main $\mathbf{a}$, wavelet $\mathbf{b}$, and original $\mathbf{c}$ flows by the formulas

$$\mathbf{a} \overset{\text{def}}{=} (a_{-1}, \ldots, a_{\widetilde{N}-1})^T, \quad \mathbf{b} \overset{\text{def}}{=} (b_{-1}, \ldots, b_{N-1})^T, \tag{5.4}$$

$$\mathbf{c} \overset{\text{def}}{=} (c_{-1}, \ldots, c_{N-1})^T, \tag{5.5}$$

we write the decomposition formula [6]

$$\mathbf{b} = \mathbf{c} - \mathfrak{P}^T\mathfrak{Q}\mathbf{c}, \tag{5.6}$$

$$\mathbf{a} = \mathfrak{Q}\mathbf{c}. \tag{5.7}$$

5.2. Orthogonal basis for the space of first order wavelet flows. We identify the space $\mathscr{C}$ of flows with the Euclidean space $\mathbb{R}^{N+1}$ equipped with the standard inner product

$$(\mathbf{x}, \mathbf{y}) \overset{\text{def}}{=} \sum_{i=1}^{N+1} x_i y_i \quad \forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^{N+1},$$

$\mathbf{x} = (x_1, x_2, \ldots, x_{N+1})$, $\mathbf{y} = (y_1, y_2, \ldots, y_{N+1})$. Let $\mathscr{A}$ be the $\widetilde{N} + 1$-dimensional space of the main flows $\mathbf{a}$, and let $\mathscr{B}$ be the space of all possible wavelet flows $\mathbf{b}$. Each of the systems $\{\omega_j\}_{j\in J'_{N-1}}$ and $\{\widetilde{\omega}_j\}_{j\in J'_{\widetilde{N}-1}}$ consists of linearly independent elements, whereas the flows $\mathbf{c}$, $\mathbf{a}$, $\mathbf{b}$

are connected with the representation (5.2) by formulas (5.3)–(5.5), we see that the spaces $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are linearly isomorphic to the spaces $\widetilde{\mathbb{S}}$, $\mathbb{W}$, and $\mathbb{S}$ respectively:

$$\mathscr{A} \sim \widetilde{\mathbb{S}}, \quad \mathscr{B} \sim \mathbb{W}, \quad \mathscr{C} \sim \mathbb{S}, \tag{5.8}$$

so that $\mathscr{C} = \mathscr{A} \dotplus \mathscr{B}$. It is easy to prove [7] that

$$\mathscr{B} = \ker \mathfrak{Q}. \tag{5.9}$$

Theorem 5.1. *For the spline-wavelet decomposition under consideration the space $\mathscr{B}$ of first order wavelet flows can be represented as*

$$\mathscr{B} = \Big\{ \mathbf{b} \;\Big|\; \mathbf{b} = \sum_{q=s+1}^{r} \alpha_q \mathbf{b}_q \quad \forall \alpha_i \in \mathbb{R}^1,\ i = s+1, s+2, \dots, r \Big\}, \tag{5.10}$$

where for $q = s+1, \dots, r-1$ the vector $\mathbf{b}_q$ is defined by the symbolic determinant

$$\mathbf{b}_q \stackrel{\text{def}}{=} \det \begin{pmatrix} \mathbf{e}_{2q-1} & \mathbf{e}_{2q} \\ \det(\mathbf{a}_{2q+1}, \mathbf{a}_{2q}) & \det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q+1}) \end{pmatrix}, \tag{5.11}$$

and for $q = r$

$$\mathbf{b}_r = \mathbf{e}_{2r-1}. \tag{5.12}$$

Proof. Taking into account the structure of the matrix $\mathfrak{Q}$, we have

$$\ker \mathfrak{Q} = \Big\{ \mathbf{b} \;\Big|\; \mathbf{b} = \sum_{q=s+1}^{r-1} \alpha_q \big(\mathfrak{q}_{s+q,2q} \mathbf{e}_{2q-1} - \mathfrak{q}_{s+q,2q-1} \mathbf{e}_{2q} \big) + \alpha_r \mathbf{e}_{2r-1} \quad \forall \alpha_q \in \mathbb{R}^1,\ q = s+1, \dots, r-1 \Big\}.$$

Substituting the relations (4.2), (4.3) and taking into account the arbitrariness of α_i (which allows us to ignore the appearing denominators), we find

$$\ker \mathfrak{Q} = \Big\{ \mathbf{b} \;\Big|\; \mathbf{b} = \sum_{q=s+1}^{r-1} \alpha_q \big(\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q+1})) \mathbf{e}_{2q-1} - \det(\mathbf{a}_{2q+1}, \mathbf{a}_{2q}) \mathbf{e}_{2q} \big) + \alpha_r \mathbf{e}_{2r-1} \quad \forall \alpha_q \in \mathbb{R}^1,\ q = s+1, \dots, r \Big\}.$$

Taking into account (5.11), we obtain the relation (5.10). Formula (5.12) is obvious. □

Definition 5.1. A system of vectors $\{\mathbf{v}_i\}_{i=1,2,\dots,M}$ of the space $\mathbb{R}^K$ is called a *system of zero multiplicity* if

$$[\mathbf{v}_i]_s [\mathbf{v}_j]_s = 0 \quad \forall s \in \{1, 2, \dots, K\}\ \forall i, j \in \{1, 2, \dots, M\}, i \neq j.$$

It is easy to see that a system of vectors of zero multiplicity in $\mathbb{R}^K$ consists of mutually orthogonal vectors.

Theorem 5.2. *The system $\{\mathbf{b}_l\}_{l=s+1\dots,r}$ is a system of zero multiplicity. This system is an orthogonal basis for the space $\mathscr{B}$ of wavelet flows.*

Theorem 5.2 follows from Theorem 5.1.

We consider a system of vectors $\{\widetilde{\mathbf{b}}_l\}_{l=s+1\dots,r}$ of the form

$$\begin{aligned}&\widetilde{\mathbf{b}}_q \overset{\text{def}}{=} \mathbf{e}_{2q-1} + \frac{\det(\mathbf{a}_{2q}, \mathbf{a}_{2q+1})}{\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q+1})}\mathbf{e}_{2q} \quad q \in \{s+1, \dots, r-1\},\\ &\widetilde{\mathbf{b}}_r = \mathbf{e}_{2r-1}.\end{aligned} \tag{5.13}$$

Corollary 5.1. *The system* $\{\widetilde{\mathbf{b}}_l\}_{l=s+1\dots,r}$ *is a system of zero multiplicity and is an orthogonal basis for the space* $\mathscr{B}$ *of wavelet flows.*

5.3. Orthogonal basis in the case of first order wavelet flows. In the condition (1.5), the objects under consideration are marked with asterisk $*$; the corresponding flows are referred to as *first order flows.*

Theorem 5.3. *Under the condition* (1.5)*, the space* $\mathscr{B}^*$ *of wavelet flows has the form*

$$\mathscr{B}^* = \Big\{\mathbf{b} \;\Big|\; \mathbf{b} = \sum_{q=s+1}^{r} \alpha_q \mathbf{b}_q^* \quad \forall \alpha_i \in \mathbb{R}^1,\ i = s+1, s+2, \dots, r\Big\}, \tag{5.14}$$

where the vectors $\mathbf{b}_q^*$ *are defined by*

$$\mathbf{b}_q^* \overset{\text{def}}{=} (x_{2q+2} - x_{2q})\mathbf{e}_{2q-1} + (x_{2q+2} - x_{2q+1})\mathbf{e}_{2q}, \quad q = s+1, \dots, r-1, \tag{5.15}$$

$$\mathbf{b}_r^* = \mathbf{e}_{2r-1}, \quad q = r. \tag{5.16}$$

Proof. Formulas (5.14)–(5.16) are obtained by inserting (1.5) into (5.11). □

From (5.13), (5.15), and (5.16) it follows that the system $\{\widetilde{\mathbf{b}}_q^*\}$ of vectors

$$\begin{aligned}&\widetilde{\mathbf{b}}_q^* \overset{\text{def}}{=} \mathbf{e}_{2q-1} + \frac{x_{2q+2} - x_{2q+1}}{x_{2q+2} - x_{2q}}\mathbf{e}_{2q} \quad \forall q \in \{s+1\dots, r-1\},\\ &\widetilde{\mathbf{b}}_r^* \overset{\text{def}}{=} \mathbf{e}_{2r-1}\end{aligned}$$

is also an orthonormal basis for the first order wavelet flows. In the case of a uniform grid $\overline{X}_h$, for a basis for the space of wavelet flows one can take the vectors

$$\begin{aligned}&\overline{\mathbf{b}}_q \overset{\text{def}}{=} \mathbf{e}_{2q-1} + \mathbf{e}_{2q}/2, \quad q \in \{s+1\dots, r-1\},\\ &\overline{\mathbf{b}}_r = \mathbf{e}_{2r-1}.\end{aligned}$$

6 Basis Wavelets

6.1. Representations of basis wavelets. By the linear isomorphism (5.8), the image of a basis for the space $\mathscr{B}$ is a basis for the space $\mathbb{W}$. We denote by ψ the bijection generating this isomorphism $\mathscr{B} \mapsto \mathbb{W}$ and find the basis wavelets $w_q \overset{\text{def}}{=} \psi(\mathbf{b}_q)$.

Theorem 6.1. *The basis wavelets* $w_q(t)$ *have the form*

$$w_q(t) = \det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q+1})\omega_{2q-1}(t) + \det(\mathbf{a}_{2q}, \mathbf{a}_{2q+1})\omega_{2q}(t), \quad q = s+1, \dots, r-1, \tag{6.1}$$

$$w_r(t) = \omega_{2r-1}(t). \tag{6.2}$$

Proof. Using the third equality in (5.3), we find

$$w_q(t) = \sum_{j \in J'_{N-1}} [\mathbf{b}_q]_j \omega_j(t). \tag{6.3}$$

By (5.11) with $q = s+1, \ldots, r-1$, only two components of the vector $\mathbf{b}_q$ are different from zero:

$$\begin{aligned} [\mathbf{b}_q]_{2q-1} &= \det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q+1}), \\ [\mathbf{b}_q]_{2q} &= -\det(\mathbf{a}_{2q+1}, \mathbf{a}_{2q}). \end{aligned} \tag{6.4}$$

Substituting (6.4) into (6.3), we find (6.1). In the case $q = r$, we obtain (6.2) from (5.12). □

Theorem 6.2. *The basis wavelets $w_q(t)$, $q = s+1, \ldots, r-1$, can be written as*

$$w_q(t) = \begin{cases} \det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q+1}) \dfrac{\det(\mathbf{a}_{2q-2}, \varphi(t))}{\det(\mathbf{a}_{2q-2}, \mathbf{a}_{2q-1})}, & t \in (x_{2q-1}, x_{2q}), \\ \det(\varphi(t), \mathbf{a}_{2q+1}), & t \in (x_{2q}, x_{2q+2}), \\ 0, & t \notin (x_{2q-1}, x_{2q+2}). \end{cases} \tag{6.5}$$

Proof. By (1.3),

$$\omega_{2q-1}(t) = \begin{cases} \dfrac{\det(\mathbf{a}_{2q-2}, \varphi(t))}{\det(\mathbf{a}_{2q-2}, \mathbf{a}_{2q-1})}, & t \in (x_{2q-1}, x_{2q}), \\ \dfrac{\det(\varphi(t), \mathbf{a}_{2q})}{\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q})}, & t \in (x_{2q}, x_{2q+1}), \\ 0, & t \notin (x_{2q-1}, x_{2q+1}), \end{cases} \tag{6.6}$$

$$\omega_{2q}(t) = \begin{cases} \dfrac{\det(\mathbf{a}_{2q-1}, \varphi(t))}{\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q})}, & t \in (x_{2q}, x_{2q+1}), \\ \dfrac{\det(\varphi(t), \mathbf{a}_{2q+1})}{\det(\mathbf{a}_{2q}, \mathbf{a}_{2q+1})}, & t \in (x_{2q+1}, x_{2q+2}), \\ 0, & t \notin (x_{2q}, x_{2q+2}), \end{cases} \tag{6.7}$$

We consider the relation (6.1) with $q = s+1, \ldots, r-1$ for each of the intervals (x_{2q-1}, x_{2q}), (x_{2q}, x_{2q+1}), and (x_{2q+1}, x_{2q+2}).

1. Let $t \in (x_{2q-1}, x_{2q})$. Then (6.1), (6.6), and (6.7) imply

$$w_q(t) = \det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q+1}) \frac{\det(\mathbf{a}_{2q-2}, \varphi(t))}{\det(\mathbf{a}_{2q-2}, \mathbf{a}_{2q-1})} \quad \forall t \in (x_{2q-1}, x_{2q}). \tag{6.8}$$

2. For $t \in (x_{2q}, x_{2q+1})$ from (6.1), (6.6), and (6.7) we find

$$\begin{aligned} w_q(t) = {} & [\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q})]^{-1} [\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q+1}) \det(\varphi(t), \mathbf{a}_{2q}) \\ & + \det(\mathbf{a}_{2q}, \mathbf{a}_{2q+1}) \det(\mathbf{a}_{2q-1}, \varphi(t))]. \end{aligned} \tag{6.9}$$

For two-dimensional vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, and $\mathbf{u}$ we have

$$\det(\mathbf{x}, \mathbf{z}) \det(\mathbf{u}, \mathbf{y}) + \det(\mathbf{y}, \mathbf{z}) \det(\mathbf{x}, \mathbf{u}) = \det(\mathbf{u}, \mathbf{z}) \det(\mathbf{x}, \mathbf{y}). \tag{6.10}$$

We use (6.10) in (6.9) by setting $\mathbf{x} = \mathbf{a}_{2q-1}$, $\mathbf{y} = \mathbf{a}_{2q}$, $\mathbf{z} = \mathbf{a}_{2q+1}$, and $\mathbf{u} = \varphi(t)$. Then, in the second square brackets in (6.9), we have the product $\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q}) \det(\varphi(t), \mathbf{a}_{2q+1})$ instead of the sum. Therefore, (6.9) implies

$$w_q(t) = \det(\varphi(t), \mathbf{a}_{2q+1}) \quad \forall t \in (x_{2q}, x_{2q+1}). \tag{6.11}$$

3. Let $t \in (x_{2q+1}, x_{2q+2})$. Then (6.1), (6.6), and (6.7) imply

$$w_q(t) = \det(\varphi(t), \mathbf{a}_{2q+1}) \quad \forall t \in (x_{2q+1}, x_{2q+2}). \tag{6.12}$$

Taking into account (6.8), (6.11), and (6.12), we obtain (6.5). □

Remark 6.1. It is easy to verify that if $\mathbf{a}_i = \varphi(x_{i+1})$, $i \in J'_{N-1}$, then the basis wavelets $w_q(t)$ are continuous.

6.2. Basis wavelets of the first order. We consider continuous first order splines (cf. Subsection 1.2) on a grid X. The objects under consideration are marked with asterisk $*$.

Theorem 6.3. *The basis wavelets $w_q^*(t)$ have the form*

$$w_q^*(t) = (x_{2q+2} - x_{2q})\omega_{2q-1}^*(t) - (x_{2q+1} - x_{2q+2})\omega_{2q}^*(t), \quad q = s+1, \dots, r-1, \tag{6.13}$$

$$w_r^*(t) = \omega_{2r-1}^*(t). \tag{6.14}$$

Indeed, substituting (1.5) into (6.1), we find (6.13). The relation (6.14) follows from (6.2).

In the case of a uniform grid $\overline{X}_h$, the basis wavelets $\overline{w}_q(t)$ can be represented as

$$\overline{w}_q(t) = 2\overline{\omega}_{2q-1}(t) + \overline{\omega}_{2q}(t), \quad q = s+1, \dots, r-1,$$
$$\overline{w}_r(t) = \overline{\omega}_{2r-1}(t).$$

6.3. Connection with the notion of interference. The following relations were proved in [8]: if $q \in \{s+1, s+2, \dots, r-1\}$, then the components of the wavelet flow are expressed as

$$b_{2q} = \varkappa_q b_{2q-1}, \tag{6.15}$$

where the constant $\varkappa_q$ is independent of the original flow $\mathbf{c}$ and is defined by

$$\varkappa_q \stackrel{\text{def}}{=} \frac{\det(\mathbf{a}_{2q}, \mathbf{a}_{2q+1})}{\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q+1})}, \quad q \in \{s+1, s+2, \dots, r-1\}. \tag{6.16}$$

The linear dependence between the components of the numerical flow is called an *interference* and proportionalilty of neighboring components with coefficient independent of the original flow is called a *standing wave* [8].

The relations (6.15) and (6.16) show that the generation of first order wavelets on the two-interval comb structure $\{X, A, \widetilde{X}, \widetilde{A}\}$ is accompanied with appearance of standing waves. Thus, the dimension of the space of wavelet flows coincides with the number of removed grid points (i.e., $r - s$, cf. (2.1)). This result completely agrees with the representation of basis wavelets obtained in this paper (cf. Theorem 6.1).

7 Computation of the Main Flow

7.1. Statement of the problem. Realization of the decomposition algorithm involves two problems: find the main flow and find the wavelet flow. As a rule, the first problem is more important than the second one since, in the majority of cases, it is the main flow that creates an impression about the character of the original flow. We begin with the first problem.

We consider the numerical realization of the main flow $\mathbf{a}$. By (4.1)–(4.3) and (5.7),

$$a_i = c_i, \quad i \in \{-1, 0, \ldots, 2s-1\}, \tag{7.1}$$

$$a_i = \mathfrak{q}_{i,2i-2s-1} c_{2i-2s-1} + \mathfrak{q}_{i,2i-2s} c_{2i-2s}, \quad i \in \{2s, 2s+1, \ldots, s+r-1\}, \tag{7.2}$$

$$a_i = c_{i+r-s}, \quad i \in \{s+r, s+r+1, \ldots, \widetilde{N}-1\}. \tag{7.3}$$

Computation by formulas (7.1) and (7.3) is accompanied with the assignment operations without index shifting ($2s+1$ operations for formula (7.1)) and the assignment operations with index shifting by $r-s$ ($\widetilde{N}-(s+r-1) = N-2r$ operations for formula (7.3)).

The situation is more complicated in the case (7.2). To compute the coefficients $\mathfrak{q}_{i,2i-2s-1}$ and $\mathfrak{q}_{i,2i-2s}$, it is required to compute $\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q})$, $\det(\mathbf{a}_{2q+1}, \mathbf{a}_{2q})$, and $\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q+1})$. By (4.2) and (4.3), formula (7.2) is written as

$$a_{s+q} = \frac{\det(\mathbf{a}_{2q+1}, \mathbf{a}_{2q}) c_{2q-1} + \det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q+1}) c_{2q}}{\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q})}, \quad s \leqslant q \leqslant r-1. \tag{7.4}$$

Formula (7.4) consists of $r-s$ equalities. Each equality contains 3 determinants of the second order, and the computation of each determinant in the general case requires 2 multiplicative and 1 additive operations. To gather equalities in (7.4), it is required 2 multiplicative and 1 additive operations. Thus, we need 8 multiplicative and 4 additive operations. We note that computations in all equalities can be performed parallely (of course, parallelizing is possible inside each equality, but this case will not be considered here).

We assume that a computing system works synchronously (by cycles). The measured time is discrete, and for the time unit we take the time required for performing one cycle [9].

Assume that the flows under consideration consist of numbers of type `real`. The representation is denoted by f (we do not specify the structure of the representations).

7.2. Successive computations of the main flow in the first order spline-wavelet decomposition. I Let a function $\mathrm{t}_d(f, p, q)$ provide the (natural) number of time units[1)] necessary for performing p assignments with index shifting by an integer number of positions from 0 to $q-1$. Sometimes, we use the following assumption.

(B) $\mathrm{t}_d(f, p, q) = pqt_d$, where $t_d = t_d(f)$ is a natural number.

In the case of a single-processor computer system, we need to perform

(a) $2s+1$ assignments without index shifting, which takes the time $\mathrm{t}_d(f, 2s+1, 1)$,

(b) $9(r-s)$ multiplicative operations, which takes the time $9(r-s)t_m$, where $t_m = t_m(f)$ is the time required for performing one mulitplicative operation by a computing system,

[1)] Depending on the realization, the function $\mathrm{t}_d(f, p, q)$ can be linear or not (for example, in the case of extracting an information from the memory by using fragments of a certain length).

(c) $4(r-s)$ additive operations, which takes the time $4(r-s)t_a$, where $t_a = t_a(f)$ is the time required for performing one additive operation by a computing system,

(d) $N-2r$ assignments with index shifting by $r-s$, which takes the time $\mathrm{t}_d(f, N-2r, r-s+1)$.

In this variant, the total time $t_{\mathbf{a} \Leftarrow \mathbf{c}}$ of computation of the main flow **a** from **c** is equal to

$$t_{\mathbf{a} \Leftarrow \mathbf{c}} = \mathrm{t}_d(f, 2s+1, 1) + 9(r-s)t_m + 4(r-s)t_a + \mathrm{t}_d(f, N-2r, r-s+1). \tag{7.5}$$

Theorem 7.1. *If Assumption* (B) *is satisfied, then for computing the main flow* **a** *from the decomposition of a flow* **c** *by a single-processor computer system, it is required* $t_{\mathbf{a} \Leftarrow \mathbf{c},(B)}$ *time units, where*

$$t_{\mathbf{a} \Leftarrow \mathbf{c},(B)} = (2s+1+(N-2r)(r-s+1))t_d + 9(r-s)t_m + 4(r-s)t_a. \tag{7.6}$$

7.3. Successive computations of the main flow in the first order spline-wavelet decomposition. II Unlike Subsection 7.2, we deal with the differences of grid points:

$$a_{s+q} = \frac{(x_{2q+1} - x_{2q+2})c_{2q-1} + (x_{2q+2} - x_{2q})c_{2q}}{x_{2q+1} - x_{2q}}, \quad s \leqslant q \leqslant r-1. \tag{7.7}$$

In this case, for each q there are 4 additive operations and 3 multiplicative operations. Therefore, comparing with Subsection 7.2, the number of multiplicative operations changes. Thus, the total time $t^*_{\mathbf{a} \Leftarrow \mathbf{c}}$ of computation by a single-processor computer system is equal to

$$t^*_{\mathbf{a} \Leftarrow \mathbf{c}} = \mathrm{t}_d(f, 2s+1, 1) + 3(r-s)t_m + 4(r-s)t_a + \mathrm{t}_d(f, N-2r, r-s+1). \tag{7.8}$$

From (7.8) it is easy to obtain the following assertion.

Theorem 7.2. *If Assumption* (B) *is satisfied, then to compute the main flow* **a** *in the first order spline-wavelet decomposition of a flow* **c** *by a single-processor computer system, it is required* $t^*_{\mathbf{a} \Leftarrow \mathbf{c},(B)}$ *time units, where*

$$t^*_{\mathbf{a} \Leftarrow \mathbf{c},(B)} = (2s+1+(N-2r)(r-s+1))t_d + 3(r-s)t_m + 4(r-s)t_a. \tag{7.9}$$

If the grid is uniform, then the relations (7.7) take the form

$$a_{s+q} = -c_{2q-1} + 2c_{2q}, \quad s \leqslant q \leqslant r-1. \tag{7.10}$$

Thus, 3 additive and 2 multiplicative operations are removed for each q. Therefore, the following assertion holds.

Theorem 7.3. *Let Assumption* (B) *be satisfied. If the grid is uniform, then to compute the main flow* **a** *in the first order spline-wavelet decomposition of a flow* **c** *by a single-processor computer system, it is required* $\overline{t}^*_{\mathbf{a} \Leftarrow \mathbf{c},(B)}$ *time units, where*

$$\overline{t}^*_{\mathbf{a} \Leftarrow \mathbf{c},(B)} = (2s+1+(N-2r)(r-s+1))t_d + (r-s)t_m + (r-s)t_a. \tag{7.11}$$

7.4. Concurrent computations of the main flow in the first order spline-wavelet decomposition. The parallelizing process is rather varied and depends on options and variants of the usage of devices. In this paper, we deal with the simplest (in the opinion of the author) variant of parallelizing. However, based on this simple variant, it is easy to propose more efficient (and more complicated) variants of parallelizing.

We consider a concurrent computing system with a sufficiently large number of concurrent computing modules so that the realization of the algorithm is independent of the further increase of the number of modules (we refer to [9] for the conception of unrestricted parallelism).

We assume that the distribution $p\,q$ of the data of the same type `real` with representation f between p concurrent computing modules (p and q are natural numbers) is performed in accordance with the directive ALIGN [2)] by distributing q data to each concurrent computing module and this process takes $\mathrm{T}_{AL}(f,p,q)$ time units.

Sometimes, we use the following assumption.

(C) $\mathrm{T}_{AL}(f,p,q) = p\,q\,T_{AL}$, where $T_{AL} = T_{AL}(f)$ is a natural number.

We assume that the operation of parallel assignments for simple variables (of type `real`) takes $T_b = T_b(f)$ time units. Let $T_a = T_a(f)$ and $T_m = T_m(f)$ denote the time required by a concurrent computing system to perform 1 parallel additive operation and 1 parallel multiplicative operation respectively.

Using such a concurrent computing system, we perform the following action.

(1) distribute the variables (of type `real`) between $\widetilde{N}+1$ concurrent computing modules as follows:

(1a) map the variables a_i, $c_{\,i}$ to the concurrent computing module with the number $i+2$, $i=-1,0,\dots,s-1$ (cf. (7.1)), which takes $\mathrm{T}_{AL}(f,s+1,2)$ time units.

(1b) the variables a_{s+q}, $c_{\,2q-1}$, $c_{\,2q}$, $[\mathbf{a}_{2q+1}]_i$, $[\mathbf{a}_{2q})]_i$, $[\mathbf{a}_{2q-1}]_i$, $i=1,2$, are mapped to the concurrent computing module with the number $q+2$ (in accordance with the directive ALIGN), which takes $\mathrm{T}_{AL}(f,r-s,9)$ time units, $s\leqslant q\leqslant r-1$,

(1c) map the variables a_i, $c_{\,i+r-s}$ to the concurrent computing module with the number $i+2$, $i=r+s,r+s+1,\dots,\widetilde{N}-1$ (cf. (7.3)), which takes $\mathrm{T}_{AL}(f,N-2r,2)$ time units,

(2) in formula (7.4), during one action of parallel multiplication, perform 6 multiplications for computing three determinants; moreover, during this action, perform all these multiplications for all $s\leqslant q\leqslant r-1$, which takes T_m time units,

(3) in formula (7.4), during one action of parallel subtraction, compute the determinants $\det(\mathbf{a}_{2q+1},\mathbf{a}_{2q})$, $\det(\mathbf{a}_{2q+1},\mathbf{a}_{2q-1})$, $\det(\mathbf{a}_{2q-1},\mathbf{a}_{2q})$, $s\leqslant q\leqslant r-1$, which takes T_a time units,

(4) during one action of parallel multiplication, compute the product

$$\det(\mathbf{a}_{2q+1},\mathbf{a}_{2q})c_{\,2q-1},\quad \det(\mathbf{a}_{2q+1},\mathbf{a}_{2q-1})c_{\,2q},\quad s\leqslant q\leqslant r-1;$$

which takes T_m time units,

(5) in formula (7.4), during one action of parallel addition, compute the value of the expression in brackets in the numerator of formula (7.4), $s\leqslant q\leqslant r-1$, which takes T_a time units,

[2)] Here, the process performed in accordance with the directive ALIGN (cf. also the interface DVM in [9]) is understood as a process of distribution of available $p\,q$ variables (initialized or not) between p concurrent computing modules, with q variables to each module, in order to perform actions with these variables on the same concurrent computing module (without transferring between different modules). The method for defining the above relation is involved in the process and is not specified here.

(6) in formula (7.4), during one action of parallel division, compute the value of the right-hand side for all $s \leqslant q \leqslant r-1$, which takes T_m time units,

(7) perform the parallel assignments in (7.1)–(7.3), which takes T_b time units.

In this variant, the total time $T_{\mathbf{a} \Leftarrow \mathbf{c}}$ required for computation of the main flow **a** from the decomposition of a flow **c** is found by the formula[3)]

$$T_{\mathbf{a} \Leftarrow \mathbf{c}} = \mathrm{T}_{AL}(f, s+1, 2) + \mathrm{T}_{AL}(f, r-s, 9) + \mathrm{T}_{AL}(f, N-2r, 2) + 3T_m + 2T_a + T_b. \qquad (7.12)$$

The following assertion is obtained from formula (7.12).

Theorem 7.4. *If Assumption* (C) *is satisfied, then for computation of the main flow* **a** *in the first order spline-wavelet decomposition of a flow* **c** *by a concurrent computing system it is required* $T_{\mathbf{a} \Leftarrow \mathbf{c},(C)}$ *time units, where*

$$T_{\mathbf{a} \Leftarrow \mathbf{c},(C)} = \big(2N + 2 + 5r - 7s\big)T_{AL} + 3T_m + 2T_a + T_b. \qquad (7.13)$$

Remark 7.1. In formulas (7.5) and (7.8)–(7.13), one can observe terms connected with the processing of communication surrounding (cf. the terms $\mathrm{t}_d(f, 2s+1, 1)$, $\mathrm{t}_d(f, N-2r, s-r+1)$, $\mathrm{T}_{AL}(f, s+1, 2)$, $\mathrm{T}_{AL}(f, r-s, 9)$, $\mathrm{T}_{AL}(f, N-2r, 2)$). As is known, the productivity of a computing system is mainly determined by the speed of processing of communication surrounding (especially, in the case of a concurrent computing system).

Corollary 7.1. *If the processing of communication surrounding is taken into account* (*i.e.*, $t_d T_{AL} \neq 0$), *then in Assumptions* (B) *and* (C) *in the case of unrestricted parallelism for computation of the main flow*

$$\lim_{N\to+\infty} \frac{t_{\mathbf{a} \Leftarrow \mathbf{c},(B)}}{T_{\mathbf{a} \Leftarrow \mathbf{c},(C)}} = (r-s+1)\cdot\frac{t_d}{2T_{AL}},$$

i.e., the asymptotics of acceleration of computations by a concurrent computing system (*relative to the computation speed of a single-processor computer system*) *is determined by the time of processing of communication surrounding. If the time of processing of communication surrounding is ignored* (*i.e., we assume that* $t_d = T_{AL} = 0$), *then*

$$\frac{t_{\mathbf{a} \Leftarrow \mathbf{c},(B)}}{T_{\mathbf{a} \Leftarrow \mathbf{c},(C)}} = \frac{(r-s)(9t_m + 4t_a)}{3T_m + 2T_a + T_b}.$$

In both cases, the growth of acceleration is proportional to the growth of the number of removed grid points.

7.5. Simplifications in the case of concurrent computation of the main flow in the first order spline-wavelet decomposition. From (7.7) we see that, unlike the Subsection 7.4, it is not necessary to realize the operations in (2), whereas all the remaining operations should be performed. Therefore, the total computation time $T^*_{\mathbf{a} \Leftarrow \mathbf{c}}$ is equal to

$$T^*_{\mathbf{a} \Leftarrow \mathbf{c}} = \mathrm{T}_{AL}(f, s+1, 2) + \mathrm{T}_{AL}(f, r-s, 9) + \mathrm{T}_{AL}(f, N-2r, 2) + 2T_m + 2T_a + T_b. \qquad (7.14)$$

As was already mentioned, if we can modify the algorithm in a certain way, then formula (7.14) should be replaced with the formula

$$T^*_{\mathbf{a} \Leftarrow \mathbf{c}} = \mathrm{T}_{AL}(f, r-s, 9) + 2T_m + 2T_a + T_b. \qquad (7.15)$$

[3)] If our device allows us to modify the algorithm in such a way that the actions (1a) and (1c) are performed during the action (1(b), then (7.12) should be replaced with the formula $T_{\mathbf{a} \Leftarrow \mathbf{c}} = \mathrm{T}_{AL}(f, r-s, 9) + 3T_m + 2T_a + T_b$.

Theorem 7.5. *If Assumption* (C) *is satisfied, then for computing the main flow* $\mathbf{a}$ *from* $\mathbf{c}$ (*of the first order*) *by a concurrent computing system it is required* $T^*_{\mathbf{a} \Leftarrow \mathbf{c},(C)}$ *time units, where*

$$T^*_{\mathbf{a} \Leftarrow \mathbf{c},(C)} = (2N + 2 + 5r - 7s)T_{AL} + 2T_m + 2T_a + T_b. \tag{7.16}$$

8 Computation of the Wavelet Flow

8.1. Formulas. By formula (5.6),

$$b_{2q-1} = c_{2q-1} - [\mathfrak{P}\mathbf{a}]_{2q-1} \quad \forall q \in \{s+1, s+2, \ldots, r\}. \tag{8.1}$$

Taking into account (3.9), we have

$$b_{2q-1} = c_{2q-1} - a_{s+q-1} \quad \forall q \in \{s+1, s+2, \ldots, r\}. \tag{8.2}$$

Since the vector $\mathbf{b}$ is an element of the space $\mathscr{B}$, we can decompose $\mathbf{b}$ relative to the basis $\{\widetilde{\mathbf{b}}_l\}_{l=s+1\ldots,r}$. Taking into account that this basis has zero multiplicity and the odd components of basis elements are equal to 1 (cf. (5.13)), we find

$$\mathbf{b} = \sum_{l \in \{s+1, s+2, \ldots, r\}} b_{2l-1} \widetilde{\mathbf{b}}_l. \tag{8.3}$$

Thus, from (5.13) and (8.3) we obtain the following expressions for the even components of $\mathbf{b}$:

$$b_{2q} = \frac{\det(\mathbf{a}_{2q}, \mathbf{a}_{2q+1})}{\det(\mathbf{a}_{2q-1}, \mathbf{a}_{2q+1})} b_{2q-1} \quad \forall q \in \{s+1, s+2, \ldots, r-1\}. \tag{8.4}$$

For computations by formula (8.2) we need to perform $r - s$ additive operations and the same number of operations of assignment with index shifting, whereas for computations by formula (8.4) we need to compute two second order determinants, divide them by each other, and multiply the result by the number obtained from formula (8.2). Such computations take $r - s - 1$ times.

In the case of the first order spline-wavelet decomposition, formula (8.2) is preserved, whereas formula (8.4) is simplified:

$$b_{2q} = \frac{x_{2q+2} - x_{2q+1}}{x_{2q+2} - x_{2q}} b_{2q-1} \quad \forall q \in \{s+1, s+2, \ldots, r-1\}. \tag{8.5}$$

Finally, if the grid is uniform, then formula (8.2) remains valid and is completed by the equalities

$$b_{2q} = \frac{b_{2q-1}}{2} \quad \forall q \in \{s+1, s+2, \ldots, r-1\}. \tag{8.6}$$

8.2. Successive computations of the wavelet flow in the first order spline-wavelet decomposition. We suppose that the assumptions on computing systems in Section 7 are satisfied. To realize formula (8.2) by a single-processor computer system, we need to perform

(a) $r - s$ additive operations, which takes $(r - s)t_a$ time units,

(b) $r - s$ operations of assignment with shifting in the range from 1 to $r - s$, which takes $\mathrm{t}_d(f, r - s, r - s)$ time units.

For computations by formula (8.4) we need to repeat the computation of two second order determinants $r-s-1$ times, which requires

(c) $4(r-s-1)$ multiplicative operations and $2(r-s-1)$ additive operations, which takes $(r-s-1)(4t_m+2t_a)$ time units.

Finally, to complete computations by formula (8.4), it is required to divide the obtained determinants and multiply the quotient by b_{2q-1}, i.e., it is required to perform

(d) $2(r-s-1)$ multiplicative operations with the total time $2(r-s-1)t_m$ for their realization,

(e) the assignment operations in the last computation (cf. (d)), which requires $\mathrm{t}_d(f,r-s-1,r-s-1)$ time units.

Thus, the total time for computation of the wavelet flow by a single-processor computer system is equal to[4)]

$$t_{\mathbf{b}\Leftarrow\mathbf{c}} = (3r-3s-2)t_a + 6(r-s-1)t_m + \mathrm{t}_d(f,r-s,r-s) + \mathrm{t}_d(f,r-s-1,r-s-1). \tag{8.7}$$

In the case of the first order spline-wavelet decomposition, to find the wavelet flow it is not required to perform multiplications in computations of the determinants, whereas the remaining actions are preserved (cf. formula (8.5)). In this case,

$$t^*_{\mathbf{b}\Leftarrow\mathbf{c}} = (3r-3s-2)t_a + 2(r-s-1)t_m + \mathrm{t}_d(f,r-s,r-s) + \mathrm{t}_d(f,r-s-1,r-s-1). \tag{8.8}$$

Finally, in the case of a uniform grid (cf. formula (8.6)), $2(r-s-1)$ additive operations and $r-s-1$ multiplicative operations are removed. Hence the time $\bar{t}_{\mathbf{b}\Leftarrow\mathbf{c}}$ for computing the wavelet flow is equal to

$$\bar{t}_{\mathbf{b}\Leftarrow\mathbf{c}} = (r-s)t_a + (r-s-1)t_m + \mathrm{t}_d(f,r-s,r-s) + \mathrm{t}_d(f,r-s-1,r-s-1). \tag{8.9}$$

From (8.7)–(8.9) we obtain the following assertion.

Theorem 8.1. *Let Assumption* (B) *be satisfied. Then for computing the wavelet flow* $\mathbf{a}$ *from the flow* $\mathbf{c}$ *by a single-processor computer system, it is required* $t_{\mathbf{b}\Leftarrow\mathbf{c},(B)}$ *time units, where*

$$t_{\mathbf{b}\Leftarrow\mathbf{c},(B)} = (3r-3s-2)t_a + 6(r-s-1)t_m + \left[(r-s)^2+(r-s-1)^2\right]t_d.$$

Under the same assumptions, for the first order spline-wavelet decomposition we need $t^*_{\mathbf{b}\Leftarrow\mathbf{c},(B)}$ *time units, where*

$$t^*_{\mathbf{b}\Leftarrow\mathbf{c},(B)} = (3r-3s-2)t_a + 2(r-s-1)t_m + \left[(r-s)^2+(r-s-1)^2\right]t_d.$$

If, in addition, the original grid is uniform, then it is required $\bar{t}_{\mathbf{b}\Leftarrow\mathbf{c},(B)}$ *time units, where*

$$\bar{t}_{\mathbf{b}\Leftarrow\mathbf{c},(B)} = (r-s)t_a + (r-s-1)t_m + \left[(r-s)^2+(r-s-1)^2\right]t_d.$$

[4)] If the results of computations of determinants are preserved at the step of constructing the main flow, then it is possible to diminish the number of operations since it is not necessary to perform the operation indicated in (c). As a result, we have $t_{\mathbf{b}\Leftarrow\mathbf{c}} = (r-s-1)t_a + 4(r-s-1)t_m + \mathrm{t}_d(f,r-s,r-s) + \mathrm{t}_d(f,r-s-1,r-s-1)$.

8.3. Concurrent computations of the wavelet flow. For concurrent computation of a first order wavelet flow by a concurrent computing system, we perform the following actions:

(1) map the variables a_{s+q-1}, b_{2q-1}, b_{2q}, c_{2q-1} (of type `real`) to a concurrent computing module with the number $q \in \{s+1, s+2, \dots, r\}$, which requires to use $r-s$ computing modules and takes $T_{AL}(f, r-s, 4)$ time units,

(2) perform the parallel additive operation in (8.2) by using the concurrent computing modules mentioned in (1), which takes T_a time units,

(3) perform the parallel assignment operation in (8.2) by using the same concurrent computing modules as in (1), which takes T_b time units,

(4) map the components of the two-dimensional vectors $\mathbf{a}_{2q-1}$, $\mathbf{a}_{2q}$, $\mathbf{a}_{2q+1}$ (6 numbers of type `real`) to the concurrent computing module with the number $q \in \{s+1, s+2, \dots, r\}$ by using the same $r-s$ concurrent computing modules as above, which takes $T_{AL}(f, r-s, 6)$ time units,

(5) compute the determinants in (8.4) by performing 4 parallel multiplicative and 2 parallel additive operations with the use of the same computing modules, which takes $4T_m + 2T_a$ time units,

(6) complete the computations in (8.4) by performing 2 parallel multiplicative operations and 1 parallel operation of assignment, which takes $2T_m + T_b$ time units.

Thus, the computation of the wavelet flow takes $T_{\mathbf{b} \Leftarrow \mathbf{c}}$ time units[5)], where

$$T_{\mathbf{b} \Leftarrow \mathbf{c}} = T_{AL}(f, r-s, 4) + T_{AL}(f, r-s, 6) + 3T_a + 6T_m + 2T_b.$$

In the case of the first order spline-wavelet decomposition, the actions (4) and (5) should be replaced with the following.

(4*) map the numbers x_{2q}, x_{2q+1}, x_{2q+2} (3 numbers of type `real`) to the computing module with the number $q \in \{s+1, s+2, \dots, r\}$ by using the same $r-s$ computing modules as above, which takes $T_{AL}(f, r-s, 3)$ time units,

(5*) instead of computation of the determinants in (8.4), compute two differences in (8.5), i.e., perform 2 parallel additive operations by using the same computing modules, which takes $2T_a$ time units.

Thus, to compute the wavelet flow in the first order spline-wavelet decomposition, we need $T^*_{\mathbf{b} \Leftarrow \mathbf{c}}$ time units[6)], where

$$T^*_{\mathbf{b} \Leftarrow \mathbf{c}} = T_{AL}(f, r-s, 4) + T_{AL}(f, r-s, 3) + 3T_a + 2T_m + 2T_b.$$

Finally, computing the wavelet flow in the first order spline-wavelet decomposition on a uniform grid, we can replace (8.5) with (8.6) so that it is not necessary to perform 2 parallel additive operations and 1 parallel multiplicative operation. Hence it is required only $\overline{T}_{\mathbf{b} \Leftarrow \mathbf{c}}$

5) If the results of computation of determinants are preserved at the step of constructing the main flow, then it is possible to diminish the number of operations since it is not necessary to perform the actions described in (4) and (5). In this case, $T_{\mathbf{b} \Leftarrow \mathbf{c}} = T_{AL}(f, r-s, 4) + T_a + 2T_m + 2T_b$.

6) If the results of computations of differences are preserved at the stage of constructing the main flow, then it is possible to diminish the number of operations since it is not required to perform the actions described in (4*) and (5*). In this case, $T^*_{\mathbf{b} \Leftarrow \mathbf{c}} = T_{AL}(f, r-s, 4) + T_a + 2T_m + 2T_b$.

time units, where

$$\overline{T}_{\mathbf{b} \Leftarrow \mathbf{c}} = T_{AL}(f, r-s, 4) + T_{AL}(f, r-s, 3) + T_a + T_m + 2T_b.$$

From the above consideration we obtain the following assertion.

Theorem 8.2. *If Assumption* (C) *is satisfied, then to compute the wavelet flow* $\mathbf{a}$ *in the first order spline-wavelet decomposition of a flow* $\mathbf{c}$ *by a concurrent computing system, it is required* $T_{\mathbf{b} \Leftarrow \mathbf{c},(C)}$ *time units, where*

$$T_{\mathbf{b} \Leftarrow \mathbf{c},(C)} = 10(r-s)T_{AL} + 3T_a + 6T_m + 2T_b.$$

Under the same assumptions, in the case of the first order spline-wavelet decomposition, it is required $T^*_{\mathbf{b} \Leftarrow \mathbf{c},(C)}$ *time units, where*

$$T^*_{\mathbf{b} \Leftarrow \mathbf{c},(C)} = 7(r-s)T_{AL} + 3T_a + 2T_m + 2T_b.$$

Finally, if the grid is uniform, then it is required $\overline{T}_{\mathbf{b} \Leftarrow \mathbf{c},(C)}$ *time units, where*

$$\overline{T}_{\mathbf{b} \Leftarrow \mathbf{c},(C)} = 7(r-s)T_{AL} + T_a + T_m + 2T_b.$$

Corollary 8.1. *Let the processing of communication surrounding be taken into account* (*i.e., we assume that* $t_d T_{AL} \neq 0$). *In the case of unrestricted parallelism, if Assumptions* (B) *and* (C) *are satisfied, then for computation of the wavelet flow we have the asymptotics*

$$\frac{t_{\mathbf{b} \Leftarrow \mathbf{c},(B)}}{T_{\mathbf{b} \Leftarrow \mathbf{c},(C)}} = (r-s)\frac{t_d}{5T_{AL}} + o(r-s), \quad r-s \to +\infty.$$

If the time of processing of communication surrounding is ignored (*i.e., we assume that* $t_d T_{AL} = 0$), *then the asymptotics has the form*

$$\frac{t_{\mathbf{b} \Leftarrow \mathbf{c},(B)}}{T_{\mathbf{b} \Leftarrow \mathbf{c},(C)}} = (r-s)\frac{6t_m + t_a}{6T_m + 3T_a + 2T_b} + o(r-s), \quad r-s \to +\infty,$$

i.e., the asymptotics of acceleration is proportional to the growth of the number of removed grid points.

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