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PERIODIC RESIDUALLY FINITE GROUPS WITH ABELIAN SUBGROUPS OF FINITE RANKS

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Abstract. In this paper, the investigation of groups with additional finiteness conditions is continued. Criteria of residual finiteness and of almost local solvability of a periodic F^*-group are obtained.

In the present paper, the author continues the investigation of groups with additional finiteness conditions. One can find the definitions of well-known concepts of group theory such as periodic group, residually finite group, Sylow subgroup, Chernikov group, etc. in [3, 4].

Let us give some definitions and notation of some objects as well as statements of some propositions used in this paper.

Definition 1. The set of all prime divisors of orders of elements of a periodic group G is denoted by $\pi(G)$.

Definition 2. A group G is called an F^*-group if for any of its subgroups K and B, $K \subset B$, $|K| < \infty$, and for any elements $a, b \in H$ of the same prime order there is an element $c \in H$ such that the group $gp\langle a, b^c = c^{-1}ac\rangle$ is finite.

Definition 3. A periodic F^*-group is called a KF^*-group if for any $p \in \pi(G)$ at least one of its Sylow p-subgroups is finite. If this is true for a given prime number $p \in \pi(G)$, then the group is called a KF^*_p-group.

Proposition 1 (Gorchakov's theorem [2]). *A periodic locally solvable group has finite rank if and only if ranks of its Abelian subgroups are finite.*

Proposition 2 (Thompson's theorem). *A finite binary solvable group is solvable.*

Proposition 3 ([10]). *An infinite biprimitively finite group possesses an infinite Abelian subgroup.*

Proposition 4 (Myagkova's theorem [5]). *A locally finite p-group has finite rank if and only if it is a Chernikov group.*

Proposition 5 (Shunkov's theorem [9]). *A locally finite group with Abelian subgroups of finite ranks is an almost locally solvable group of a finite rank.*

Proposition 6 (Frattini's lemma). *Let G be a group, H be a normal subgroup in G, S be a Sylow p-subgroup from H, and Sylow p-subgroups of H be conjugated in H. Then $G = N_G(S)H$.*

Proposition 7. *An infinite F^*-group that is a residually finite p-group possesses an infinite locally finite subgroup.*

Proposition 8. *Let H be a normal subgroup of a KF^*_p-group G. If G is p-residually finite, then the factor group G/H is a p-residually finite KF^*_p-group.*

Proposition 9. *Let G be a periodic KF^*-group, and a be an element of a prime order p of the group G for which the following conditions are satisfied:*

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(1) $\pi(G) \setminus \pi(C_G(a)$ *is finite;*

(2) *for almost every* $q \in \pi\big(C_G(a)\big)$ *the Sylow* q*-subgroup from* $C_G(a)$ *is a Sylow* q*-subgroup in* G*.*

If the group G *is residually finite, then the closure of the element* a *in the group* G *is finite.*

Proposition 10 ([8])**.** *Let* G *be a periodic residually finite* KF^**-group,* M *be an infinite set of nonisomorphic Sylow subgroups, and* P_1 *be some subgroup chosen in it. Then there is an infinite subgroup* T *that is the union of finite Hall subgroups possessing Sylow series:* $B_1 < B_2 < \cdots < B_n < \cdots < T$ *with* $B_1 \cong P_1$ *and* $\pi(T) \subseteq \pi(M)$*.*

Proposition 11 ([8])**.** *Every periodic binary solvable group with Chernikov Sylow* p*-subgroups for all primes* p *possesses a full part* R*, and the factor group* G/R *is a residually finite binary solvable group with finite Sylow* p*-subgroups on all* p*.*

Let us begin with the theme of the paper.

Theorem 1. *A periodic* F^**-group with Abelian subgroups of finite ranks is a locally finite group with finite Sylow* p*-subgroups on all primes* p *if and only if it is residually finite.*

Proof. Necessity. Let G be a locally finite group with finite Sylow p-subgroups for all p and let the ranks of its Abelian subgroups be finite. By Proposition 4, the group G is an almost locally solvable group, and by Proposition 11 it is residually finite.

Sufficiency. Let G be a periodic residually finite group with Abelian subgroups of finite ranks. First, we will prove that G is a KF^*-group. By contradiction, let us assume that G possesses an infinite p-subgroup $(p \in \pi(G))$. As P is residually finite by the hypothesis of the theorem, then by Proposition 6 it possesses an infinite locally finite -subgroup K of finite rank (Proposition 3). According to Proposition 8, the subgroup K is a Chernikov group. We obtain a contradiction with the fact that K is residually finite. Let us denote by $R(G)$ the locally finite radical of the group G. If $G = R(G)$, then the theorem is proved.

Let $R(G) \neq 1$. The factor-group $G/R(G)$ is a residually finite group with finite Sylow p-subgroups on all p by Proposition 8; moreover, using Proposition 5, it is easy to prove the finiteness of ranks of its Abelian subgroups. Thus, $G/R(G)$ meets all conditions of the theorem and has a trivial locally finite radical. Without loss of generality, we can assume that $R(G) = 1$.

Lemma 1. *In the group* G*, ranks of Sylow subgroups are bounded in totality.*

Proof. Let us suppose that this is not so. Then in the group G there is an infinite sequence of Sylow p_i-subgroups $P_1, P_2, \ldots, P_n, \ldots$, whose rank grows together with the index i $(i = 1, 2, \ldots, n, \ldots)$. Using Proposition 10, we will construct in G an infinite locally finite subgroup of T that is the union of finite Hall subgroups $B_1 < B_2 < \cdots < B_n < \cdots < T$, where $\pi(T) \subseteq \pi(M)$, $M = \{P_1, P_2, \ldots\}$. By construction of T and by the choice of the set M, the rank of the group T is infinite. We obtained a contradiction with Proposition 5. The lemma is proved. □

By Lemma 1, there is a natural number k such that the rank $r\big(P/\Phi(P)\big) \leq k$ for any Sylow p-subgroup P and any prime number $p \in \pi(G)$. Let p be any prime number from $\pi(G)$. As the group G is residually finite, it possesses a normal subgroup N of finite index containing no p-elements. Let us introduce designations: $H_p = G/N$, $n_p = |H_p|$, and Q is a Sylow q-subgroup from N, for which $(q, n_p) = 1$. According to Proposition 6, the group G is representable as $G = N_G(Q)N$. Using the theorem on isomorphism, we get that

$$H_p = G/N = N_G(Q)N/N \cong N_G(Q)/N \cap N_G(Q),$$

where $T = N \cap N_G(Q)$ is p-group. The Frattini subgroup $\Phi(Q)$ of the group Q is characteristic and, therefore, $\Phi(Q) \lhd Q$. Obviously, $Q/\Phi(Q) = \bar{Q}$ is a normal elementary Abelian q-subgroup of the factor group $N_G(Q)/\Phi(Q) = B$ and the subgroup $C_B(Q)$ is normal in B. Let us denote by C the inverse image of $C_B(\bar{Q})$ in $N_G(Q)$. The subgroup $C \lhd N_G(Q)$ and $N_G(Q)/C = V_p$ is a linear group over a field of the characteristic q not dividing the order of V_p. According to the theorem by Jordan–Brauer–Feit, the group V_p possesses an Abelian normal subgroup L such that $|V_p : L| \leq f(k)$, where $f(k)$ is a function depending only on k. Using theorems on isomorphisms, we get $A_q = CN/N = CN/T \lhd N_G(Q)/T$;

$$\overline{H_p} = H_p/A_q \cong N_G(Q)/CT = N_G(Q)C/CT \cong (N_G(Q)C/C)/(CT/C) = V_p/F,$$

where $F = CT/T$. In view of the structure of the subgroup V_p, the factor group $V_p/F \cong \overline{H_p}$ also possesses an Abelian normal subgroup of the index bounded above with the number $f(k)$.

For the proof of the theorem it is necessary to consider two cases:

(1) $\bigcap\limits_{q \in \pi(N) \setminus \pi(H_p)} A_q = D \neq 1$;

(2) $D = 1$.

Let us consider the first case. Let d be a nontrivial element of a prime order from the group D. In view of definition of the subgroup C, we have $(|d|, q) = 1$, and the element d centralizes the subgroup Q according to Proposition 4. As $\pi(N) \setminus \pi(H)$ is finite and the choice of q is arbitrary, the element d meets all requirements of Proposition 9, and the locally finite radical of the group G is nontrivial, a contradiction. Therefore, the second case takes place: $D = 1$. According to Remak's theorem [3, p. 54], the group H_p is isomorphically embedded in the full direct product of subgroups of type H_p/A_q. Any such group, as was shown above, possesses an Abelian normal subgroup of finite index not exceeding $f(k)$. Therefore, the group H_p also possesses an Abelian normal subgroup Y_p such that the period of the group is bounded by the number $f(k)$, not depending on the choice of $p \in \pi(G)$. Again using Remak's theorem, we embed the group G into full direct product of subgroups H_p:

$$G \cong G_1 < H = \widetilde{\prod_{p \in \pi(G)}} H_p.$$

Let us denote by m the number divisible with the period of factor groups H_p/Y_p. As is shown above, such a number exists. Let us consider a subgroup $H^m = gp\langle h^m | h \in H\rangle$. As Y_p is an Abelian group for any $p \in \pi(G)$, the subgroup H^m is also Abelian, and the factor group H/H^m is the group of the period m. In view of the embedding $G \cong G_1 < H$, a similar statement is valid for the group G. Let A be an Abelian normal subgroup of the group G such that the period of the factor group equals m. According to Proposition 8, G/A is a residually finite KF^*-group, and as its period is finite, the index $|G : A|$ is finite. Obviously, the group G is also locally finite, and we obtained a contradiction with the assumption $R(G) = 1$. Theorem 1 is proved. $\square$

Theorem 2. *A periodic group is a locally solvable group of finite rank if and only if it is binary solvable and ranks of its Abelian subgroups are finite.*

Proof. The necessity of the conditions of the theorem is obvious.

Let G be a periodic binary solvable group with Abelian subgroups of finite ranks. Using Proposition 3, it is easy to show that Sylow p-subgroups of G are Chernikov groups for any $p \in \pi(G)$. By Proposition 11, G possesses the full part $R(G)$ and the factor group $G/R(G)$ is a periodic binary solvable group with finite Sylow subgroups on all $p \in \pi\big(G/R(G)\big)$. Moreover, the group $G/R(G)$ is residually finite and, as easily follows from Proposition 5, the ranks of its Abelian subgroups are finite. Thus, $G/R(G)$ meets all conditions of Theorem 1 and is a locally finite group. From Schmidt's theorem follows the local finiteness of the group G, and from Thompson's theorem (Proposition 2) its local solvability. According to Gorchakov's theorem (Proposition 3), the rank of the group G is finite. Theorem 2 is proved. $\square$

As a result of Theorem 2, we get an analog of a known result of Yu. M. Gorchakov (see [1]).

Corollary. *A periodic binary solvable group of an infinite rank possesses an Abelian subgroup of infinite rank.*

Theorem 3. *A periodic F^*-group with Abelian subgroups of finite ranks is an almost locally solvable group of finite rank if and only if any two of its elements generate the residually finite group.*

Proof. The necessity of the conditions of the theorem is obvious.

Let G be a periodic F^*-group with Abelian subgroups of finite ranks in which any two elements generate a residually finite subgroup. Let a and b be arbitrary elements of the group G. The subgroup $gp\langle a, b\rangle$ is a periodic residually finite KF^*-group with Abelian subgroups of finite ranks. According to Theorem 1, it is residually finite and, as $gp\langle a, b\rangle$ is finitely generated, $|gp\langle a, b\rangle| < \infty$. The choice of elements a and b from the group is arbitrary; therefore, G is a binary finite group. A binary finite group with Abelian subgroups of finite ranks is locally finite [7] and, according to Shunkov's theorem (Proposition 5), the group G is an almost locally solvable group of finite rank. Theorem 3 is proved. □

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