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AFFINE DIAMETERS OF A CONVEX BODY

V. V. Makeev* **and M. Yu. Zvagel'skii*** UDC 514.172

We prove that each convex body in $\mathbb{R}^n$ has n pairwise orthogonal affine diameters $d_1, \dots, d_n$ such that it is possible to shift each of them through a linear combination of direction vectors of the diameters with smaller numbers so that their translates will intersect at their common midpoint. Bibliography: 5 *titles.*

By a convex body we mean a compact convex set in $\mathbb{R}^n$ with nonempty interior. An affine diameter is a segment in a convex body with endpoints at parallel planes of support.

It is well known that each strictly convex body has exactly one diameter passing in any preassigned direction. Furthermore, the diameter continuously depends on the direction.

Affine diameters of a convex body have many mutual intersections. A number of results and problems on affine diameters are stated in [1, p. 42].

It is proved in [2] that each three-dimensional convex body has three mutually perpendicular affine diameters intersecting at one point. This assertion cannot be directly transferred to higher dimensions. It is proved in [3] that a convex body in $\mathbb{R}^4$ has four pairwise orthogonal affine diameters one of which intersects the other three. It is proved in [4] that for each affine diameter of a convex body in $\mathbb{R}^n$ there are two affine diameters intersecting it and making with each other any preassigned angle. See [5] for a survey of papers devoted to affine diameters of convex bodies.

Theorem 1. *Each convex body in $\mathbb{R}^n$ has n pairwise orthogonal affine diameters $d_1, \dots, d_n$ such that we can shift each of the diameters through a linear combination of direction vectors of the diameters with smaller indices so that the translates intersect at their common midpoint.*

The proof involves several auxiliary assertions.

In what follows, $F(\mathbb{R}^n)$ denotes the manifold of flags $(L_1, \dots, L_{n-1})$ in $\mathbb{R}^n$, where $L_1 \subset \dots \subset L_{n-1}$ and $\dim L_i = i$.

Lemma 1. *Assume that to each flag in $F(\mathbb{R}^n)$ an ordered n-tuple of points in $\mathbb{R}^n$ is assigned that depends continuously on the flag. Then there is a flag $(L_1, \dots, L_{n-1})$ such that the n-tuple $(A_1, \dots, A_n)$ assigned to this flag has the following property: The vector $\overline{A_1A_{i+1}}$ is parallel to L_i for $1 \le i \le n-1$.*

Proof. Assigning to each flag $(L_1, \dots, L_{n-1})$ the n-tuple of projections of the vectors $\overline{A_1A_2}, \dots, \overline{A_1A_n}$ to the orthogonal complements $L_1^\perp, \dots, L_{n-1}^\perp$ of the spaces $L_1, \dots, L_{n-1}$, respectively, we obtain a section of the Whitney sum ξ of $n-1$ vector bundles over $F(\mathbb{R}^n)$, where the fiber of the bundle with index i over the flag $(L_1, \dots, L_{n-1})$ is the orthogonal complement of L_i.

It suffices to prove that each section of ξ intersects the zero section. For this purpose, we easily present a smooth section of ξ that has exactly one nondegenerate zero.

We assign to all the flags in $F(\mathbb{R}^n)$ (one and the same) n-tuple of points $A_1, \dots, A_n$ such that the vectors $\overline{A_1A_2}, \dots, \overline{A_1A_n}$ are linearly independent. The section induced by the n-tuple has a zero only over the flag $(L_1, \dots, L_{n-1})$, where the space L_i is the linear span of the vectors $\overline{A_1A_2}, \dots, \overline{A_1A_{i+1}}$, and this zero is nondegenerate. The lemma is proved. $\square$

*St. Petersburg State University, St. Petersburg, Russia, e-mail: mvv57@inbox.ru, mikhailzvagelsky@gmail.com.

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There is a one-to-one correspondence between the considered flags and ordered n-tuples of pairwise orthogonal lines $(l_1, \dots, l_n)$ passing through the origin in $\mathbb{R}^n$: The n-tuple of lines $(l_1, \dots, l_n)$ corresponds to the flag $(L_1, \dots, L_{n-1})$, where L_i is the affine hull of $(l_1, \dots, l_i)$ for $1 \le i \le n-1$. Taking into account the above lemma, we obtain a corollary.

Proposition 1. *Let G be the Grassmann manifold of lines in $\mathbb{R}^n$ passing through a fixed point, and let $f\ G \to \mathbb{R}^n$ be a continuous mapping. Then there exist n mutually perpendicular lines $l_1, \dots, l_n \in G$ with direction vectors $e_1, \dots, e_n$, respectively, such that we can shift each of the points $f(l_1)$, ..., $f(l_n)$ through a linear combination of direction vectors of the lines with smaller indices so that the translates coincide.*

Proof of Theorem 1. In the case of strictly convex bodies, it suffices to apply Proposition 1 to the mapping f taking a line ℓ to the midpoint of the affine diameter of the body parallel to ℓ. In the general case, the proof is obtained by passage to the limit when we approximate convex bodies by strictly convex ones. $\square$

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