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LATTICE PACKINGS OF MIRROR SYMMETRIC OR CENTRALLY SYMMETRIC THREE-DIMENSIONAL CONVEX BODIES

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We prove a number of statements concerning lattice packings of mirror symmetric or centrally symmetric convex bodies. This enables one to establish the existence of sufficiently dense lattice packings of any three-dimensional convex body of such type.

The main result states that each three-dimensional, mirror symmetric, convex body admits a lattice packing with density at least $8/27$. *Furthermore, two basis vectors of the lattice generating the packing can be chosen parallel to the plane of symmetry of the body.*

The best result for centrally symmetric bodies was obtained by Edwin Smith (2005)*: Each three-dimensional, centrally symmetric, convex body admits a lattice packing with density greater than* 0.53835*. In the present paper, it is only proved that each such body admits a lattice packing with density* $(\sqrt{3}+\sqrt[4]{3/4}+1/2)/6 > 0.527$*. Bibliography:* 5 *titles.*

1. On lattice packings of mirror symmetric, three-dimensional, convex bodies

By a *convex body* K in $\mathbb{R}^n$ we mean a compact convex subset with nonempty interior, and $V(K)$ denotes the volume of K ($S(K)$ is the area for $n = 2$).

A *lattice arrangement* of a subset A in $\mathbb{R}^n$ is defined as the union of images of A under shifts through the vectors of a certain lattice. The vectors are integer linear combinations of linearly independent vectors $a_1, \dots, a_k$. A lattice arrangement of a subset A is called a *packing* if the images of A have no common interior points.

Theorem 1. *Each mirror symmetric, three-dimensional, convex body* K *admits a lattice packing of density at least* $8/27$*. Furthermore, the two basis vectors of the vector lattice generating the packing can be chosen to be parallel to the plane of symmetry of* K*.*

The proof involves several auxiliary assertions, which are also useful in the case of a centrally symmetric convex body.

Lemma 1. *Let* A *and* B *be two subsets in* $\mathbb{R}^n$*, and let* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *be two lattice arrangements of* A *and* B *in* $\mathbb{R}^n$ *constructed with the help of one and the same vector lattice. Assume that the difference set* $A - B = \{x - y \mid x \in A, y \in B\}$ *is measurable and the volume of* $A - B$ *does not exceed the volume of the fundamental domain of the lattice. Then we can shift, say, the arrangement* $\mathcal{R}_1$ *to an arrangement* $\mathcal{R}_1'$ *so that sets from the arrangements* $\mathcal{R}_1'$ *and* $\mathcal{R}_2$ *have no common interior points.*

Proof. Consider all possible difference sets of the form $A_i - B_j$, where A_i and B_j are taken from the lattice arrangements $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. They constitute a lattice arrangement of the difference set $A - B$ constructed with the help of the same lattice. In view of our assumption about the volume of the difference set, the interiors of their images cannot cover the entire space, and the required shift can be performed through the position vector of a point not covered by the interiors of the difference sets. Lemma 1 is proved. □

Lemma 2. *Let* $\mathcal{R}$ *be a lattice arrangement of a convex body* K *in* $\mathbb{R}^n$ *and let the volume of the fundamental domain of the lattice be at least* $C_{2n}^n V(K)$*. Then the arrangement* $\mathcal{R}$ *can be*

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shifted to an arrangement $\mathcal{R}'$ so that the bodies of the arrangements $\mathcal{R}$ and $\mathcal{R}'$ have no common interior points.

To prove this, it suffices to apply Lemma 1 and take into account the Rogers–Shephard inequality $V(K-K) \le C_{2n}^{n} V(K)$ (see [1, Chap. 2]).

Lemma 3. *Let K be a convex body in $\mathbb{R}^n$, and let $-K$ be the body symmetric to K with respect to the origin. Consider two lattice arrangements of K and $-K$ in $\mathbb{R}^n$ constructed with the help of one and the same lattice such that the volume of the fundamental domain of the lattice is at least $2^n V(K)$. Then one of the two lattice arrangements can be shifted so that the bodies from different lattice arrangements have no common interior points.*

To prove this, it suffices to apply Lemma 1 and take into account that $V(K-(-K)) = V(2K) = 2^n V(K)$.

Consider a lattice packing of a mirror symmetric convex body K in $\mathbb{R}^n$, where the lattice L is generated by $n-1$ linearly independent vectors $a_1, \dots, a_{n-1}$ parallel to a hyperplane Q. We denote by V the $(n-1)$-volume of the fundamental domain of L (in its hyperplane parallel to Q). Let P_1 and P_2 be common hyperplanes of support of all bodies of the packing parallel to Q. Let Q_1 be a hyperplane parallel to the hyperplanes P_1 and Q and lying between them and such that the section by Q_1 of the body of the initial packing has the $(n-1)$-dimensional volume not greater than V/C_{2n-2}^{n-1}.

Lemma 4. *Under the above assumptions, there exists a lattice packing of the body K in $\mathbb{R}^n$ such that the lattice is generated by the vectors $a_1, \dots, a_n$, and the projection of the vector a_n to a line perpendicular to all the hyperplanes indicated above has length $2d - d_1$, where d is the distance between Q and P_1 and d_1 is the distance between Q_1 and P_1.*

Proof. We denote by Q_2 the hyperplane symmetric to Q_1 with respect to Q. We shift the initial lattice packing so that the hyperplane Q_2 coincides with the hyperplane P_1. Below we denote the initial lattice packing by $\mathcal{R}_1$, and the shifted one is denoted by $\mathcal{R}_2$. We must prove that the second packing can be shifted in parallel to all the hyperplanes under consideration so that bodies of the second packing have no common interior points with bodies of the first packing.

In each hyperplane parallel to Q_1 and lying between Q_1 and P_1, we consider two lattice packings of $(n-1)$-dimensional convex bodies consisting of the intersections of the hyperplane with bodies from the first and second lattice packing, respectively. These lattice packings are also generated by the vectors $a_1, \dots, a_{n-1}$. Since the body K is mirror symmetric, all the orthogonal projections of the sections of K by the hyperplanes from the considered family to Q_1 are contained in the section $K \cap Q_1$. Applying Lemmas 1 and 2 to two copies of the lattice of the sections of K by the hyperplane Q_1, we obtain the assertion of Lemma 4. □

Proof of Theorem 1. Let Q be the plane of symmetry of the three-dimensional body K. For the sake of convenience, below we assume that the maximum distance from points of K to the plane Q is equal to 1. Let p be the maximum density of a lattice packing $\mathcal{R}$ of the section $K_1 = K \cap Q$ in the plane Q. (We have $p \ge 2/3$; see [2].) We also assume that the area of the fundamental domain of the densest lattice packing $\mathcal{R}$ of the figure K_1 is equal to 1 (consequently, $S(K_1) = p$). We denote by t the distance from the plane of symmetry Q to the plane of the section of the body K parallel to Q and having area $1/6$. (If there is no such section, we put $t = 1$.) We observe that $t \ge 1/2$ because we have $p \ge 2/3$, and the area of the section passing at a distance $1/2$ from the plane Q is at least $p/4 \ge 1/6$.

Let P_1 and P_2 be the planes of support of the body K parallel to the plane Q, and let U be the layer of unit thickness between the planes Q and P_1. Consider the lattice packing of the layer U by the intersections of U with right cylinders having figures from the packing $\mathcal{R}$ as bases. These cylinders yield a packing of U with density p. We perform the Schwarz symmetrization of the body K with respect to a line perpendicular to the bases of the cylinder (see [3, p. 224]). Since the result of the symmetrization is a convex body, it follows that the half of its volume is not less than the volume of the frustum of a right circular cone with height t, with lower base of area p, and with top base of area $1/6$, plus the volume of a right circular cone with height $1-t$ and with base of area $1/6$. Simple computations show that the volume of the indicated body is equal to $((6p+\sqrt{p})t+1)/18$. Applying Lemma 4, we can consider two layers between two parallel planes lying at distance 2 from each other. The layers contain lattice packings consisting of cylinders found above, and we can push them into each other through $1-t$. This yields a lattice packing of the body K of density not less than

$$\frac{(6p+\sqrt{6p})t+1}{9(1+t)} = \frac{1}{9}\left(6p+\sqrt{6p}-\frac{6p+\sqrt{6p}-1}{1+t}\right).$$

In view of the above restrictions, the latter expression attains the minimum for $p = 2/3$ and $t = 1/2$, which coincides with the bound stated in the theorem. □

Remarks. 1. It is proved in [2, Chap. 6] that the density of the packing in $\mathbb{R}^n$ of the simplices obtained by shifts from a certain simplex (not necessarily a lattice packing) does not exceed $2^n(n!)^2/(2n)!$. For $n = 3$, this yields the upper bound $2/5$.

2. It is proved in [4, Sec. 10] that each three-dimensional convex body admits a lattice packing of density at least $1/4$.

2. On lattice packings of a centrally symmetric, three-dimensional, convex body

The following lemma about lattice packings of centrally symmetric convex bodies is an analog of Lemma 4 for the mirror symmetric case.

Consider a lattice packing of a centrally symmetric convex body K in $\mathbb{R}^n$, where the lattice L is generated by $n-1$ linearly independent vectors $a_1, \dots, a_{n-1}$. We denote by V the $(n-1)$-volume of the fundamental domain of L (in its hyperplane H). Let P_1 and P_2 be two common hyperplanes of support of all the bodies of the packing (they are parallel to the hyperplane H). Let Q be the hyperplane parallel to H and containing the centers of all the bodies of the packing. Let Q_1 be the hyperplane parallel to H and lying between P_1 and Q such that the $(n-1)$-dimensional volume of the intersection $K_1 = K \cap Q_1$ is not greater than $(1/2)^{n-1}V$.

Lemma 5. *Under the assumptions indicated above, there exists a lattice packing of the body K in $\mathbb{R}^n$ such that the lattice is generated by the vectors $a_1, \dots, a_n$, and the projection of the vector a_n to a line perpendicular to all the hyperplanes described above has length $2(d-d_1)$, where d is the distance between Q and P_1 and d_1 is the distance between Q_1 and P_1. (We assume that $d_1 \le d/2$.)*

Proof. We denote by Q_2 the hyperplane symmetric to Q_1 with respect to Q. We shift the initial lattice packing $\mathcal{R}_1$ to a packing $\mathcal{R}_2$ so that the hyperplane Q_2 coincides with the hyperplane Q_1. We must prove that the packing $\mathcal{R}_2$ can be shifted in parallel to all the considered hyperplanes into a packing $\mathcal{R}_2'$ so that the shifted bodies have no common interior points with bodies of the packing $\mathcal{R}_1$.

We choose a Cartesian coordinate system such that the last axis is perpendicular to the considered hyperplanes. Rotating the packing $\mathcal{R}_2$ at 180° around the last coordinate axis, we obtain the third lattice packing $\mathcal{R}_3$. It is clear that the lattice packings of $(n-1)$-dimensional bodies obtained as intersections of the bodies from the packings $\mathcal{R}_1$ and $\mathcal{R}_3$ with Q_1 differ by a shift. Performing this shift, below we assume that the bodies from $\mathcal{R}_1$ and $\mathcal{R}_3$ have the same intersections with Q_1.

In each hyperplane Q' parallel to Q_1 and passing at a distance of $\le d_1$ to Q_1, we consider two lattice packings of $(n-1)$-dimensional convex bodies formed by the sections by Q' of the bodies from $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. These lattice packings are also generated by the vectors $a_1, \dots, a_{n-1}$. We consider the pairwise differences of bodies from different lattice packings (the first body is taken from the first packing) and take their orthogonal projections to the hyperplane Q_1. We show that all the projections are contained in the difference sets which were initially obtained in Q_1.

It is clear that the considered difference sets are congruent to the sums of the corresponding sections by the hyperplane Q_1 of the bodies from the packings $\mathcal{R}_1$ and $\mathcal{R}_3$ and differ from them by a shift through one and the same vector. Let us prove that the orthogonal projection to Q_1 of the sum of sections of any two bodies from $\mathcal{R}_1$ and $\mathcal{R}_3$, respectively, by any hyperplane from the family described above is contained in the sum of the sections obtained in Q_1.

In view of periodicity, it suffices to verify this for arbitrary two bodies from $\mathcal{R}_1$ and $\mathcal{R}_3$. We verify this for a pair of bodies that have the same intersection with the hyperplane Q_1. These bodies are mirror symmetric with respect to Q_1. The orthogonal projection to Q_1 of the sum of sections of two bodies symmetric with respect to Q_1 by a hyperplane H from the family described above is congruent to the sum of the section of one of these bodies by H with the section of the body by the hyperplane symmetric to H with respect to Q_1. Since the body K is convex, the half-sum of sections of K by parallel hyperplanes symmetric with respect to Q_1 is contained in the section $K_1 = K \cap Q_1$. Consequently, the sum of sections of K by hyperplanes symmetric with respect to Q_1 is contained in the sum of K_1 with the same K_1. The sum is homothetic to K_1 with coefficient 2 and has volume $2^{n-1}V(K_1)$. In view of the assumption about the volume of K_1 (made before the formulation of Lemma 5), the interiors of the bodies $2K_1$ arranged in a lattice do not cover Q_1, which allows us to find a vector v such that after shift through v, the interiors of the bodies from the packing $\mathcal{R}_2$ do not intersect the interiors of the bodies from the packing $\mathcal{R}_1$. Lemma 5 is proved. □

Lemma 6. *Let K be a centrally symmetric convex figure K and let A be an affine-regular hexagon inscribed in K. Consider two centrally symmetric hexagons B and C circumscribed about K. The hexagon B is bounded by lines of support of K passing through the vertices of A. The hexagon C is bounded by lines of support of K parallel to sides of A. Then one of the quotients of the areas*

$$p = S(K)/S(B) \quad \text{and} \quad q = S(K)/S(C)$$

is at least $\sqrt{3}/2$.

Proof. Consider the quotient $r = S(A)/S(B)$. The easy inequality $r \ge 3/4$ is well known. Since the assertion of the lemma is affine, we can assume (after an affine transformation) that the hexagon A is regular and circumscribed about a circle of unit radius.

Let x be the arithmetic mean of the distances from the common center of symmetry of the figures K, A, B, and C to the sides of the hexagon C. The figure K contains a centrally symmetric dodecagon D which is the convex hull of the vertices of the hexagon A and the points where the sides of C touch the figure K. It is obvious that $S(K) \ge S(D) = x * S(A)$, whence $p \ge 3x/4$.

For a fixed x, the hexagon C has the maximum area only if C is regular. Otherwise, we subject the pair of the symmetric shortest sides of C to a small parallel shift inwards through equal distances, and subject the pair of the symmetric longest sides of C to a small parallel shift outwards through the same equal distances. This does not change x, and the area of C increases. Thus, we have $q \ge 1/x$.

In view of the estimates $p \geq 3x/4$ and $q \geq 1/x$ established above, we have $\max(p, q) \geq \max(3x/4, 1/x)$, $x > 1$. The expression on the right-hand side of the inequality attains the minimum for $x = \sqrt{4/3}$, and the minimum is equal to $\sqrt{3}/2$. □

Theorem 2. *Let K be a centrally symmetric three-dimensional body. Then K is contained in a centrally symmetric prism Π with the same center of symmetry such that the base of Π is a polygon with at most 6 sides, and the area of the central section of K parallel to the base of Π is not less than $\sqrt{3}/2$ of the area of the base of Π.*

Proof. It suffices to prove the theorem for a smooth strictly convex body K. In the general case, the result is obtained by passage to the limit.

The central section K_1 of K by an oriented plane is a centrally symmetric strictly convex figure. For this reason, each boundary point A_1 of K_1 is a vertex of a unique affine-regular hexagon A inscribed in K_1 and continuously depending on A_1. Let $A_1, \dots, A_6$ be the vertices of A indexed counter-clockwise. The foregoing implies that the configuration space M of the hexagons considered is homeomorphic to the orthogonal group SO(3).

Let A be an affine-regular hexagon inscribed in the oriented central section K_1. Consider the two centrally symmetric hexagons B and C circumscribed about K_1 that are described in the formulation of Lemma 6. Let a_1, a_2, and a_3 be the external unit normals of the body K at the points A_1, A_2, and A_3, respectively, and let c_1, c_2, and c_3 be the external unit normals of K at the points of tangency of adjacent sides of the hexagon C with the section K_1. The normals are also indexed counter-clockwise.

We prove that for a certain central section and an affine-regular hexagon A inscribed in the section, the vectors in each of the indicated triples are coplanar. In this case, Lemma 6 implies that the requirements of Theorem 2 are satisfied by one of the two centrally symmetric hexagonal prisms circumscribed about K, the bases of which are parallel to the above section and touch the body K, while the planes of the lateral faces of the prisms are orthogonal to the vectors a_i and to the vectors c_i, respectively.

Consider the continuous mapping $f \colon M \to \mathbb{R}^2$ taking a hexagon A to the pair $((a_1, a_2, a_3)$, $(c_1, c_2, c_3))$ of mixed products. We show that $f(M)$ contains the point $(0, 0)$.

If this is not so, then we consider the mapping $g = f/\|f\| \colon M \to S^1$ to the standard unit circle S^1. The cyclic group $\mathbb{Z}_2$ freely acts on the configuration space M of hexagons by taking a hexagon into the centrally symmetric hexagon. Also, $\mathbb{Z}_2$ acts on the circle S^1 by symmetries with respect to the center. By construction, the mapping g is $\mathbb{Z}_2$-equivariant. Consider a path γ in M joining two points of an orbit of the group $\mathbb{Z}_2$. Then the path $g \circ \gamma$ also joins two points of an orbit of $\mathbb{Z}_2$. The projections of these paths to the quotient spaces by the action of $\mathbb{Z}_2$ are non-null-homotopic loops because their covering paths are not closed. The quotient mapping g takes the first loop to the second one and induces a nontrivial homomorphism of fundamental groups of the quotient spaces. However, the fundamental group of the first space $M/\mathbb{Z}_2$ has order 4, while the fundamental group of the second space $S^1/\mathbb{Z}_2$ is isomorphic to $\mathbb{Z}$. This contradiction completes the proof of Theorem 2.

The assertions proved above allow us to prove that there exist sufficiently dense lattice packings of a centrally symmetric, three-dimensional, convex body. The best result in this direction is presented in [5], where it is proved that each centrally symmetric, three-dimensional, convex body admits a lattice packing of density greater than 0.53835. □

Proposition. *Each centrally symmetric, three-dimensional, convex body K admits a lattice packing of density $(\sqrt{3} + \sqrt[4]{3/4} + 1/2)/6 > 0.527$.*

Proof. For the sake of convenience, below we assume that the prism existing in view of Theorem 2 has height 2 and the area of the base of the prism is equal to 1. We denote by p the area of of the central section K_0 of the body K parallel to the base (we have $p \geq \sqrt{3}/2$), and we denote by t the distance from the plane of K_0 to the plane of the section of K parallel to K_0 and having area 1/4. (If there is no such section, then we put $t = 1$.)

We perform the Schwarz symmetrization of the body K with respect to a line perpendicular to the bases of the prism. Since the result of the symmetrization is a convex body, it follows that the quantity $V(K)/2$ is not less than the sum of the volume of a frustum of a right circular cone with height t, with lower base of area p, and with top base of area 1/4, and the volume of the right circular cone with height $(1 - t)$ and with base of area 1/4.

Applying Lemma 6, we can consider two layers between two parallel planes lying at distance 2 from each other. The layers contain lattice packings consisting of the prisms found above, which cover the layers, and we can push the packings into each other through $2(1 - t)$. This yields a lattice packing of the body K of density not less than $((2p + \sqrt{p})t + 1/2)/(6t) = ((2p + \sqrt{p}) + 1/(2t))/6$. In view of the restrictions indicated above, the latter expression attains the minimum for $p = \sqrt{3}/2$ and $t = 1$, which coincides with the declared bound. □

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