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ESTIMATING THE SURFACE AREA OF SPHERES IN NORMED SPACES

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The surface area of a polyhedron in a normed space is defined as the sum of the areas of its faces, each divided by the area of the central section of the unit ball, parallel to the face. This functional naturally extends to convex bodies.

In this paper, it is proved, in particular, that the surface area of the unit sphere in any three-dimensional normed space does not exceed 8. *Bibliography: 2 titles.*

The surface area of a polyhedron in a normed space is defined as the sum of the areas of its faces, each divided by the area of the central section of the unit ball parallel to the face. This functional naturally extends to convex bodies.

Let c_n be the smallest number such that each centrally symmetric convex body K in $\mathbb{R}^n$ is contained in a parallelepiped of volume not greater than $c_n V(K)$, where $V(K)$ is the volume of K. It is proved in [1] that $c_2 = 4/3$, and it is proved in [2] that $c_3 < 3.2082$.

Lemma 1. *The surface area of a parallelepiped of smallest volume that contains the unit ball in an n-dimensional normed space does not exceed $2nc_{n-1}$.*

Proof. Let us consider a parallelepiped P of smallest volume that contains the unit ball K. It is well known that the centers of faces of any parallelepiped of smallest volume that contains a centrally symmetric convex body K belong to K. Let L be the hyperplane through the center of K parallel to a face F of the parallelepiped P. Clearly, $L \cap P$ is an $(n-1)$-dimensional parallelepiped Q congruent to F. The parallelepiped Q is a parallelepiped of smallest volume that contains the section $L \cap K$. Indeed, if there exists a parallelepiped Q' of smaller volume that contains the section $L \cap K$, then the hyperfaces of Q' lie in hyperplanes of support of K, which, together with two hyperplanes of the faces of P parallel to L, bound a parallelepiped with volume smaller than the volume of P. A contradiction.

The above implies that the area of any face of the parallelepiped P does not exceed c_{n-1}, which implies the assertion of the lemma. □

Theorem 1. *The surface area of a parallelepiped of smallest volume that contains the unit ball K in a three-dimensional normed space does not exceed* 8.

Proof. This follows from the lemma and the fact that $c_2 = 4/3$. □

Corollary 1. *The surface area of the unit ball in a three-dimensional normed space does not exceed* 8.

The proof follows from the monotonicity of the functional of the surface area.

Remarks. 1. In order to obtain the sharp bound 4 for the perimeter of a disk with unit diameter in a two-dimensional normed space, different authors considered a parallelogram of smallest area circumscribed around the disk.

2. If the norm in a three-dimensional normed space is Euclidean, i.e., the ball is standard, then the circumscribed parallelepiped of smallest volume is a cube, and the surface area of the cube is equal to $24/\pi > 7.6394$.

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3. Taking into account the bound for c_3 presented above, we see that the surface area of the parallelepiped of smallest volume circumscribed around a ball of unit diameter in a four-dimensional normed space does not exceed $8 \times 3.2082 = 25.6656$. It follows that the same holds true for the surface area of the ball itself.

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