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EXPONENTIAL DICHOTOMY AND EXISTENCE OF ALMOST PERIODIC SOLUTIONS OF IMPULSIVE DIFFERENTIAL EQUATIONS

V. I. Tkachenko

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We establish conditions for the existence of piecewise continuous almost periodic solutions of a system of impulsive differential equations with exponentially dichotomous linear part. The robustness of exponential dichotomy and exponential contraction are investigated for linear systems with small perturbations of the right-hand sides and the points of impulsive action.

1. Introduction

We investigate the problem of existence of a piecewise continuous almost periodic solution for the semilinear impulsive differential equation

$$\frac{du}{dt} = A(t)u + f(t,u), \quad t \neq \tau_j, \tag{1}$$

$$\Delta u|_{t=\tau_j} = u(\tau_j + 0) - u(\tau_j) = B_j u(\tau_j) + g_j(u(t_j)), \quad j \in \mathbb{Z}, \tag{2}$$

where $u : \mathbb{R} \to \mathbb{R}^n$. We use the concept of discontinuous almost periodic functions in a sense of [1, 2]. There are numerous works (see, e.g., [3–6] and the references therein) devoted to the investigation of almost periodic solutions of impulsive systems.

We assume that the corresponding linear homogeneous equation (with $f \equiv 0$ and $g_j \equiv 0$) has exponential dichotomy. The matrices $(I + B_j)$ may be degenerated, i.e., $\det(I + B_j) = 0$, for some (or all) $j \in \mathbb{Z}$. Therefore, the solutions of the system are not extendable to the negative semiaxis or are ambiguously extendable. To define exponential dichotomy, we require that only solutions of the linear system from the unstable manifold can be unambiguously extended to the negative semiaxis. This corresponds to the definition of exponential dichotomy for the evolution equations in infinite-dimensional Banach spaces [7–9].

Robustness is an important property of exponential dichotomy [8–10]. We mention papers [11–14], where the robustness of exponential dichotomy is proved for impulsive systems by small perturbations on the right-hand sides. In the present paper, we prove the robustness of exponential dichotomy also by small perturbations of points of the impulsive action. We use the change of time in the system. Then the approximation of the impulsive system by difference systems (see [7]) can be used.

2. Preliminaries and Main Results

Let X be an abstract Banach space and let $\mathbb{R}$ and $\mathbb{Z}$ be the sets of real and integer numbers, respectively.

We consider the space $\mathcal{PC}(J, X)$, $J \subset \mathbb{R}$, of all piecewise continuous functions $x : J \to X$ such that

(i) $T = \{\tau_j \in J : \tau_{j+1} > \tau_j, \, j \in \mathbb{Z}\}$ is the set of discontinuities of x;

Institute of Mathematics, Ukrainian National Academy of Sciences, Tereshchenkivska str., 3, Kyiv, 01601, Ukraine.

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(ii) $x(t)$ is left-continuous, $x(t_j - 0) = x(t_j)$, and the following limit exists:

$$\lim_{t \to t_j + 0} x(t) = x(t_j + 0) < \infty.$$

Definition 1. *A strictly increasing sequence $\{\tau_k\}$ of real numbers has uniformly almost periodic sequences of differences if, for any $\varepsilon > 0$, there exists a relatively dense set of ε-almost periods common for all sequences $\{\tau_k^j\}$, where $\tau_k^j = \tau_{k+j} - \tau_k$, $j \in \mathbb{Z}$.*

Recall that an integer p is called an ε-almost period of a sequence $\{x_k\}$ if $\|x_{k+p} - x_k\| < \varepsilon$ for any $k \in \mathbb{Z}$. A sequence $\{x_k\}$ is almost periodic if, for any $\varepsilon > 0$, there exists a relatively dense set of its ε-almost periods.

Definition 2. *A function $\varphi(t) \in \mathcal{PC}(\mathbb{R}, X)$ is called W-almost periodic if*

(i) the sequence $\{\tau_k\}$ of discontinuities of $\varphi(t)$ has uniformly almost periodic sequences of differences;

(ii) for any $\varepsilon > 0$, there exists a positive number $\delta = \delta(\varepsilon)$ such that if the points t' and t'' belong to the same interval of continuity and $|t' - t''| < \delta$, then

$$\|\varphi(t') - \varphi(t'')\| < \varepsilon;$$

(iii) for any $\varepsilon > 0$, there exists a relatively dense set Γ of ε-almost periods such that if $\tau \in \Gamma$, then

$$\|\varphi(t + \tau) - \varphi(t)\| < \varepsilon$$

for all $t \in \mathbb{R}$ satisfying the condition $|t - \tau_k| \geq \varepsilon$, $k \in \mathbb{Z}$.

We consider the impulsive equation (1), (2) under the following assumptions:

(H_1) the matrix-valued function $A(t)$ is Bohr almost periodic,

(H_2) the sequence of real numbers τ_k has uniformly almost periodic sequences of differences and there exists $\theta > 0$ such that $\inf_k \tau_k^1 = \theta > 0$,

(H_3) the sequence $\{B_j\}$ of $(n \times n)$-matrices is almost periodic,

(H_4) denote

$$U_\rho = \{x \in \mathbb{R}^n : \|x\| \leq \rho\};$$

the function $f(t, u) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in u and W-almost periodic in t uniformly with respect to $u \in U_\rho$ for some $\rho > 0$,

(H_5) the sequence $\{g_j(u)\}$ of continuous functions $U_\rho \to \mathbb{R}^n$ is almost periodic uniformly with respect to $u \in U_\rho$.

By Lemma 22 [9, p. 192], for a sequence $\{\tau_j\}$ with uniformly almost periodic sequences of differences, the limit

$$\lim_{T \to \infty} \frac{\mathrm{i}(t, t + T)}{T} = p$$

exists uniformly with respect to $t \in \mathbb{R}$, where $\mathrm{i}(s, t)$ is the number of points τ_k lying in the interval (s, t).

The following lemma is proved in [9]:

Lemma 1. *Assume that the sequence of real numbers $\{\tau_j\}$ has uniformly almost periodic sequences of differences, the sequence $\{B_j\}$ is almost periodic, and the function $f(t) : \mathbb{R} \to \mathbb{R}^n$ is W-almost periodic. Then, for any $\varepsilon > 0$, there exist a real number ν, $0 < \nu < \varepsilon$, and relatively dense sets of real numbers Γ and integers Q such that the following relations are true:*

$$\|f(t+r) - f(t)\| < \varepsilon, \quad t \in \mathbb{R}, \quad |t - \tau_j| > \varepsilon, \quad j \in \mathbb{Z},$$

$$\|B_{k+q} - B_k\| < \varepsilon, \quad \|\tau_k^q - r\| < \nu,$$

for $k \in \mathbb{Z}$, $r \in \Gamma$, and $q \in Q$.

Definition 3. *A function $x(t) : [t_0, t_1] \to \mathbb{R}^n$ is called a solution of the initial problem $u(t_0) = u_0 \in \mathbb{R}^n$ for equation (1), (2) on $[t_0, t_1]$ if*

(i) it is continuous in $[t_0, \tau_k], (\tau_k, \tau_{k+1}], \dots, (t_{k+s}, t_1]$ with discontinuities of the first kind at the times $t = \tau_j$;

(ii) $x(t)$ is continuously differentiable in each interval $(t_0, \tau_k), (\tau_k, \tau_{k+1}), \dots, (t_{k+s}, t_1)$ and satisfies the equations (1) and (2) if $t \in (t_0, t_1)$, $t \neq \tau_j$, and $t = \tau_j$, respectively,

(iii) the initial condition $u(t_0) = u_0$ is satisfied.

Together with equation (1), (2), we consider the following linear homogeneous equation;

$$\frac{du}{dt} = A(t)u, \quad t \neq \tau_j, \tag{3}$$

$$\Delta u|_{t=\tau_j} = u(\tau_j + 0) - u(\tau_j) = B_j u(\tau_j), \quad j \in \mathbb{Z}. \tag{4}$$

By $X(t, s)$, we denote the evolution operator of the linear equation without pulses (3). It satisfies the equalities $X(\tau, \tau) = I$ and $X(t, s)X(s, \tau) = X(t, \tau)$, $t, s, \tau \in \mathbb{R}$.

We define the evolution operator for equation (3), (4) as follows:

$$U(t, s) = X(t, s) \quad \text{if} \quad \tau_k < s \le t \le \tau_{k+1}$$

and

$$U(t, s) = X(t, \tau_k)(I + B_k)X(\tau_k, \tau_{k-1}) \dots (I + B_m)X(\tau_m, s), \tag{5}$$

if $\tau_{m-1} < s \le \tau_m < \tau_{m+1} < \dots < \tau_k < t \le \tau_{k+1}$.

Definition 4. *We say that system (3), (4) has an exponential dichotomy on $\mathbb{R}$ with exponent $\beta > 0$ and bound $M \ge 1$ if there exist projections $P(t)$, $t \in \mathbb{R}$, such that*

(i) $U(t, s)P(s) = P(t)U(t, s)$, $t \ge s$;

(ii) $U(t, s)|_{\mathrm{Im}\,(P(s))}$ for $t \ge s$ is an isomorphism on $\mathrm{Im}\,(P(s))$; then $U(s, t)$ is defined as an inverse map from $\mathrm{Im}\,(P(t))$ to $\mathrm{Im}\,(P(s))$;

(iii) $\|U(t,s)(1-P(s))\| \le Me^{-\beta(t-s)}$, $t \ge s$;

(iv) $\|U(t,s)P(s)\| \le Me^{\beta(t-s)}$, $t \le s$.

We now formulate our main result:

Theorem 1. *Suppose that system (1), (2) satisfies the assumptions (H_1)–(H_5) and the linear system (3), (4) is exponentially dichotomous with constants β and $M \ge 1$.*

Assume that the functions $f(t,u)$ and $g_j(u)$ satisfy the Lipschitz condition

$$\|f(t,u_1) - f(t,u_2)\| \le L\|u_1 - u_2\|, \quad \|g_j(u_1) - g_j(u_2)\| \le L\|u_1 - u_2\|, \quad j \in \mathbb{Z},$$

with a positive constant L and that they are uniformly bounded in the region $t \in \mathbb{R}$, $u \in U_\rho$:

$$\sup_{(t,u)} \|f(t,u)\| \le H < \infty, \quad \sup_u \|g_j(u)\| \le H < \infty, \quad j \in \mathbb{Z}.$$

Then, for a sufficiently small L, system (1), (2) has a unique piecewise continuous W-almost periodic solution.

3. Robustness of Exponential Dichotomy

If system (3), (4) possesses an exponential dichotomy on $\mathbb{R}$, then the inhomogeneous equation

$$\frac{du}{dt} = A(t)u + f(t), \quad t \ne \tau_j, \tag{6}$$

$$\Delta u|_{t=\tau_j} = u(\tau_j + 0) - u(\tau_j) = B_j u(\tau_j) + g_j, \quad j \in \mathbb{Z}, \tag{7}$$

has a unique solution bounded on $\mathbb{R}$:

$$u_0(t) = \int_{-\infty}^{\infty} G(t,s)f(s)(x)ds + \sum_{j\in\mathbb{Z}} G(t,\tau_j)g_j, \tag{8}$$

where

$$G(t,s) = \begin{cases} U(t,s)(I - P(s)), & t \ge s, \\ -U(t,s)P(s), & t < s, \end{cases}$$

is a Green function such that

$$\|G(t,s)\| \le Me^{-\beta|t-s|}, \quad t,s \in \mathbb{R}. \tag{9}$$

By analogy with [7, p. 250], it can be shown that a function $u(t)$ is a bounded solution on the semiaxis $[t_0, +\infty)$ if and only if

$$u(t) = U(t,t_0)(I - P(t_0))u(t_0) + \int_{t_0}^{+\infty} G(t,s)f(s)ds + \sum_{t_0 \le \tau_j} G(t,\tau_j)g_j, \quad t \ge t_0. \tag{10}$$

A function $u(t)$ is a bounded solution on the semiaxis $(-\infty, t_0]$ if and only if

$$u(t) = U(t,t_0)P(t_0)u(t_0) + \int_{-\infty}^{t_0} G(t,s)f(s)ds + \sum_{t_0>\tau_j} G(t,\tau_j)g_j, \quad t \le t_0. \tag{11}$$

Lemma 2. *Let the impulsive system (3), (4) be exponentially dichotomous with positive constants β and M. Then there exists $\delta_0 > 0$ such that the perturbed systems*

$$\frac{du}{dt} = \tilde{A}(t)u, \quad t \ne \tilde{\tau}_j, \tag{12}$$

$$\Delta u|_{t=\tilde{\tau}_j} = u(\tilde{\tau}_j + 0) - u(\tilde{\tau}_j) = \tilde{B}_j u(\tilde{\tau}_j), \quad j \in \mathbb{Z}, \tag{13}$$

with

$$\sup_j |\tau_j - \tilde{\tau}_j| \le \delta_0, \quad \sup_j \|B_j - \tilde{B}_j\| \le \delta_0, \quad \textit{and} \quad \sup_t \|A(t) - \tilde{A}(t)\| \le \delta_0,$$

are also exponentially dichotomous with some constants $\beta_1 \le \beta$ and $M_1 \ge M$.

Proof. In system (3), (4), we introduce the change of time $t = \alpha(t')$ such that $\tau_j = \alpha(\tilde{\tau}_j)$, $j \in \mathbb{Z}$, and the function α is continuously differentiable and monotone in each interval $(\tilde{\tau}_j, \tilde{\tau}_{j+1})$.

The function α can be chosen in a piecewise linear form,

$$t = a_j t' + b_j, \quad a_j = \frac{\tau_{j+1} - \tau_j}{\tilde{\tau}_{j+1} - \tilde{\tau}_j}, \quad b_j = \frac{\tau_{j+1}\tilde{\tau}_j - \tau_j\tilde{\tau}_{j+1}}{\tilde{\tau}_{j+1} - \tilde{\tau}_j} \quad \text{if} \quad t' \in (\tilde{\tau}_j, \tilde{\tau}_{j+1}).$$

The function $\alpha(t')$ satisfies the conditions

$$|\alpha(t') - t'| \le \delta_0, \quad \left| \frac{d\alpha(t')}{dt'} - 1 \right| \le a\delta_0$$

with a positive constant a independent of j and δ_0.

In the new coordinates $v(t') = u(\alpha(t'))$, system (3), (4) takes the form

$$\frac{dv}{dt'} = A_1(t')v, \quad t \ne \tilde{\tau}_j, \tag{14}$$

$$\Delta v|_{t'=\tilde{\tau}_j} = v(\tilde{\tau}_j + 0) - v(\tilde{\tau}_j) = B_j v(\tilde{\tau}_j), \quad j \in \mathbb{Z}, \tag{15}$$

where

$$A_1(t') = \frac{d\alpha(t')}{dt'} A(\alpha(t')).$$

System (14), (15) has the evolution operator

$$U_1(t', s') = U(\alpha(t'), \alpha(s')).$$

If system (3), (4) has exponential dichotomy with a projection $P(t)$ at the point t, then system (14), (15) has exponential dichotomy with the projection $P_1(t') = P(\alpha(t'))$ at the point t'. Indeed,

$$\|U_1(t', s')(1 - P_1(s'))\| = \|U(\alpha(t'), \alpha(s'))(1 - P(\alpha(s'))\|$$

$$\leq Me^{-\beta(\alpha(t')-\alpha(s'))} \leq M_1 e^{-\beta(t'-s')}, \quad t' \geq s',$$

where $M_1 = Me^{2\delta_0}$. The inequality for an unstable manifold is proved similarly.

The linear systems (14), (15) and (12), (13) have the same points of impulsive actions $\tilde{\tau}_j$, $j \in \mathbb{Z}$, and

$$\|A_1(t') - \tilde{A}(t')\| \leq \left\| \frac{d\alpha(t')}{dt'} A(\alpha(t')) - A(\alpha(t')) \right\| + \|A(\alpha(t')) - A(t')\|$$

$$+ \|A(t') - \tilde{A}(t')\| \leq \delta_0 \left(a \sup_t \|A\| + \sup_t \left\| \frac{dA}{dt} \right\| + 1 \right).$$

Let $\tilde{U}(t', s')$ be an evolution operator for system (12), (13). To show that system (12), (13) is exponentially dichotomous for sufficiently small δ_0, we use the following version of Theorem 7.6.10 [7]:

Assume that the evolution operator $U_1(t', s')$ has an exponential dichotomy on $\mathbb{R}$ and satisfies the inequality

$$\sup_{0 \leq t'-s' \leq d} \|U_1(t', s')\| < \infty \tag{16}$$

for some positive d. Then there exists $\eta > 0$ such that

$$\|\tilde{U}(t', s') - U_1(t', s')\| < \eta, \quad \text{whenever} \quad t - s \leq d,$$

and the evolution operator $\tilde{U}(t', s')$ has an exponential dichotomy on $\mathbb{R}$.

To prove this statement, we set

$$t_n = s' + dn, \;\; T_n = U_1(s' + d(n+1), s' + dn), \;\; \tilde{T}_n = \tilde{U}(s' + d(n+1), s' + dn) \;\; \text{for} \;\; n \in \mathbb{Z}.$$

If the evolution operator $U_1(t, s)$ has an exponential dichotomy, then $\{T_n\}$ has a discrete dichotomy in a sense of [7] (Definition 7.6.4).

By [7] (Theorem 7.6.7), there exists $\eta > 0$ such that $\{\tilde{T}_n\}$ with $\sup_n \|T_n - \tilde{T}_n\| \leq \eta$ has a discrete dichotomy.

We are now in the conditions of Exercise 10 from [7, p. 229, 230] (see also a more general statement [8, Theorem 4.1]). This completes the proof.

The exponentially dichotomous system (12), (13) possesses the Green function

$$\tilde{G}(t, s) = \begin{cases} \tilde{U}(t, s)(I - \tilde{P}(s)), & t \geq s, \\ -\tilde{U}(t, s)\tilde{P}(s), & t < s, \end{cases}$$

such that

$$\|\tilde{G}(t, s)\| \leq M_1 e^{-\beta_1 |t-s|}, \quad t, s \in \mathbb{R}.$$

Lemma 3. *The difference between the Green functions of the exponentially dichotomous linear systems (12), (13) and (14), (15) satisfies the relation*

$$\tilde{G}(t,\tau) - G_1(t,\tau) = \int_{-\infty}^{\infty} G_1(t,s)(\tilde{A}(s) - A_1(s))\tilde{G}(s,\tau)ds$$

$$+ \sum_j G_1(t,\tilde{\tau}_j)(\tilde{B}_j - B_j)\tilde{G}(\tilde{\tau}_j,\tau), \quad t,\tau \in \mathbb{R}, \tag{17}$$

where $G_1(t,\tau) = G(\alpha(t),\alpha(\tau))$.

Proof. $\tilde{G}(t,\tau)$ satisfies the equation

$$\frac{du}{dt} = A_1(t)u + (\tilde{A}(t) - A_1(t))\tilde{G}(t,\tau),$$

$$\Delta u|_{t=\tilde{\tau}_j} = B_j u + (\tilde{B}_j - B_j)\tilde{G}(\tilde{\tau}_j,\tau).$$

By virtue of (10), we conclude that, for $t \geq \tau$,

$$\tilde{G}(t,\tau) = U_1(t,\tau)(I - P_1(\tau))\tilde{G}(\tau,\tau) + \int_{\tau}^{+\infty} G_1(t,s)(\tilde{A}(s) - A_1(s))\tilde{G}(s,\tau)ds$$

$$+ \sum_{\tau \leq \tilde{\tau}_j} G_1(t,\tilde{\tau}_j)(\tilde{B}_j - B_j)\tilde{G}(\tilde{\tau}_j,\tau). \tag{18}$$

Similarly, by (11), we get

$$\tilde{G}(t,\tau) = U_1(t,\tau)P_1(\tau)\tilde{G}(\tau - 0,\tau) + \int_{-\infty}^{\tau} G_1(t,s)(\tilde{A}(s) - A_1(s))\tilde{G}(s,\tau)ds$$

$$+ \sum_{\tau > \tilde{\tau}_j} G_1(t,\tilde{\tau}_j)(\tilde{B}_j - B_j)\tilde{G}(\tilde{\tau}_j,\tau) \tag{19}$$

for $t < \tau$.

Setting $t = \tau$ in (18), we find

$$P_1(\tau)\tilde{G}(\tau,\tau) = \int_{\tau}^{+\infty} G_1(\tau,s)(\tilde{A}(s) - A_1(s))\tilde{G}(s,\tau)ds + \sum_{\tau \leq \tilde{\tau}_j} G_1(\tau,\tilde{\tau}_j)(\tilde{B}_j - B_j)\tilde{G}(\tilde{\tau}_j,\tau).$$

Since

$$\tilde{G}(\tau,\tau) - \tilde{G}(\tau - 0,\tau) = I,$$

by (19), we conclude that, for $t < \tau$,

$$\tilde{G}(t,\tau) = U_1(t,\tau)\Bigg(\int\limits_{\tau}^{+\infty} G_1(\tau,s)(\tilde{A}(s) - A_1(s))\tilde{G}(s,\tau)ds$$

$$+ \sum_{\tau \le \tilde{\tau}_j} G_1(\tau,\tilde{\tau}_j)(\tilde{B}_j - B_j)\tilde{G}(\tilde{\tau}_j,\tau)\Bigg) - U_1(t,\tau)P_1(\tau)$$

$$+ \int\limits_{-\infty}^{\tau} G_1(t,s)(\tilde{A}(s) - A_1(s))\tilde{G}(s,\tau)ds + \sum_{\tau > \tilde{\tau}_j} G_1(t,\tilde{\tau}_j)(\tilde{B}_j - B_j)\tilde{G}(\tilde{\tau}_j,\tau)$$

$$= G_1(t,\tau) + \int\limits_{-\infty}^{\infty} G_1(t,s)(\tilde{A}(s) - A_1(s))\tilde{G}(s,\tau)ds + \sum_{j} G_1(t,\tilde{\tau}_j)(\tilde{B}_j - B_j)\tilde{G}(\tilde{\tau}_j,\tau).$$

The case $t \ge \tau$ is analyzed similarly.
Lemma 3 is proved.

By virtue of (17), we arrive at the estimate

$$\|\tilde{G}(t,\tau) - G_1(t,\tau)\| \le \delta_0 M_2 e^{-\beta_2|t-\tau|}, \quad t,\tau \in \mathbb{R}, \tag{20}$$

with some $\beta_2 \le \beta_1$ and $M_2 \ge 1$.

Lemma 4. *Assume that systems (3), (4) and (12), (13) satisfy the assumptions of Lemma 2 with sufficiently small* $\delta_0 > 0$. *Then the corresponding Green functions of these systems satisfy the inequality*

$$\|\tilde{G}(t,\tau) - G(t,\tau)\| \le \delta_0 \tilde{M}_2 e^{-\beta_2|t-s|}, \tag{21}$$

for all t and s such that $|t-\tau_j| > \delta_0$ *and* $|s-\tau_j| > \delta_0$ *for all* $j \in \mathbb{Z}$.

Proof. We have

$$\|G(t,\tau) - \tilde{G}(t,\tau)\| \le \|G(t,\tau) - G(\alpha(t),\alpha(\tau))\| + \|G(\alpha(t),\alpha(\tau)) - \tilde{G}(t,\tau)\|.$$

Let $t > s$ (the case $t < s$ is studied similarly). Then

$$\|G(t,s) - G(\alpha(t),\alpha(s))\| = \|U(t,s)P(s) - U(\alpha(t),\alpha(s))P(\alpha(s))\|$$

$$\le \|U(t,s)P(s) - U(\alpha(t),s)P(s)\|$$

$$+ \|U(\alpha(t),s)P(s) - U(\alpha(t),\alpha(s))P(\alpha(s))\|$$

$$\le \|U(t,s)P(s) - U(\alpha(t),t)U(t,s)P(s)\| +$$

$$+ \|U(\alpha(t), s)P(s) - U(\alpha(t), s)U(s, \alpha(s))P(\alpha(s))\|$$

$$\le \|I - U(\alpha(t), t)\| \|U(t, s)P(s)\|$$

$$+ \|U(\alpha(t), s)P(s)\| \|I - U(s, \alpha(s))\|$$

$$\le Me^{-\gamma(t-s)}\|I - U(\alpha(t), t)\| + Me^{-\gamma(\alpha(t)-s)}\|I - U(s, \alpha(s))\|.$$

Here, for the sake of definiteness, $s \ge \alpha(s)$ and $t \le \alpha(t)$. If $t \in (\tau_j + \delta_0, \tau_{j+1} - \delta_0)$, then $\alpha(t) \in (\tau_j, \tau_{j+1})$. Therefore, by continuity, there exists a positive constant C_1 independent of t such that

$$\|I - U(\alpha(t), t)\| \le C_1\delta_0.$$

As a result, we obtain

$$\|G(t, s) - G(\alpha(t), \alpha(s))\| \le \delta_0 M_3 e^{-\beta|t-s|}$$

with a positive constant M_3 independent of $t, s \in \mathbb{R}$ and δ_0. Further, in view of (20), we obtain (21).

Lemma 4 is proved.

Corollary 1. *Assume that system (3), (4) satisfies the conditions (H_1)–(H_3) and is exponentially dichotomous with constants β and M. Then, for any $\varepsilon > 0$, $t, s \in \mathbb{R}$, $|t - \tau_j| > \varepsilon$, $|s - \tau_j| > \varepsilon$, $j \in \mathbb{Z}$, there exists a relatively dense set of ε-almost periods r such that*

$$\|G(t + r, s + r) - G(t, s)\| \le \varepsilon C_1 \exp\left(-\frac{\beta}{2}|t - s|\right), \tag{22}$$

where C_1 is a positive constant independent of ε.

Proof. If

$$u(t) = U(t, s)u_0, \quad u(s) = u_0,$$

is a solution of the impulsive equation (3), (4), then

$$u_1(t) = U(t + r, s + r)u_0$$

is a solution of the equation

$$\frac{du}{dt} = A(t + r)u, \quad t \ne \tau_j, \tag{23}$$

$$\Delta u|_{t=\tau_{j+q}} = u(\tau_{j+q} + 0) - u(\tau_{j+q}) = B_{j+q}u(\tau_{j+q}), \quad j \in \mathbb{Z}. \tag{24}$$

By Lemma 1, there exists a positive integer q such that $\tau_{j+q} \in (s + r, t + r)$ if $\tau_j \in (s, t)$. Finally, we apply Lemma 4.

4. Almost Periodic Solutions of Linear Inhomogeneous System

We prove the existence of almost periodic solutions of a linear inhomogeneous system.

Lemma 5. *Assume that a linear homogeneous system satisfies the assumptions (H_1)–(H_3) and is exponentially dichotomous on the axis. If the function $f(t)$ is W-almost periodic and the sequence $\{g_j\}$ is almost periodic, then the linear inhomogeneous system (6), (7) has a unique solution bounded on $\mathbb{R}$ and W-almost periodic.*

Proof. The unique solution of system (6), (7) bounded on $\mathbb{R}$ is given by relation (8). We show that it is W-almost periodic.

We take an ε-almost period r for the right-hand side of the equation. Then

$$u_0(t+r) - u_0(t) = \int_{-\infty}^{+\infty} G(t+r,s)f(s)ds + \sum_j G(t+r,\tau_j)g_j - \int_{-\infty}^{+\infty} G(t,s)f(s)ds$$

$$- \sum_j G(t,\tau_j)g_j = \int_{-\infty}^{+\infty} (G(t+r,s+r) - G(t,s))f(s+h)ds$$

$$+ \int_{-\infty}^{+\infty} G(t,s)(f(s+r) - f(s))ds + \sum_j (G(t+r,\tau_{j+q}) - G(t,\tau_j))g_{j+q}$$

$$+ \sum_j G(t,\tau_j)(g_{j+q} - g_j).$$

We estimate the first integral,

$$\int_{-\infty}^{\infty} \|(G(t+r,s+r) - G(t,s))f(s+r)\|ds$$

$$\leq \sum_{k\in\mathbb{Z}} \int_{\tau_k+\varepsilon}^{\tau_{k+1}-\varepsilon} \|(G(t+r,s+r) - G(t,s))f(s+r)\|ds$$

$$+ \sum_{k\in\mathbb{Z}} \int_{\tau_k-\varepsilon}^{\tau_k+\varepsilon} \|(G(t+r,s+r) - G(t,s))f(s+r)\|ds$$

$$\leq \int_{-\infty}^{\infty} \varepsilon C_1 e^{-\frac{\beta}{2}|t-s|}\|f(s)\|ds + \sum_{k\in\mathbb{Z}} \int_{\tau_k-\varepsilon}^{\tau_k+\varepsilon} \|G(t+r,s+r)f(s+r)\|ds$$

$$+\sum_{k\in\mathbb{Z}}\int_{\tau_k-\varepsilon}^{\tau_k+\varepsilon}\|G(t,s)f(s+r)\|ds.$$

By Lemma 1, $|\tau_{j+q}-\tau_j-r|<\varepsilon$. Therefore, $\tau_j+r+\varepsilon>\tau_{j+q}$ (for definiteness, we assume that $r>0$). The difference $G(t,\tau_j)-G(t+r,\tau_{j+q})$ can be estimated as follows:

Let $t-\tau_j\geq\varepsilon$. Then

$$\begin{aligned}\|G(t,\tau_j)-G(t+r,\tau_{j+q})\| &= \|U(t,\tau_j)(I-P(\tau_j))-U(t+r,\tau_{j+q})(I-P(\tau_{j+q}))\|\\ &\leq \|U(t,\tau_j)(I-P(\tau_j))-U(t,\tau_j+\varepsilon)(I-P(\tau_j+\varepsilon))\|\\ &\quad+\|U(t,\tau_j+\varepsilon)(I-P(\tau_j+\varepsilon))-U(t+r,\tau_j+\varepsilon+r)(I-P(\tau_j+\varepsilon+r))\|\\ &\quad+\|U(t+r,\tau_j+\varepsilon+r)(I-P(\tau_j+\varepsilon+r))-U(t+r,\tau_{j+q})(I-P(\tau_{j+q}))\|.\end{aligned}\tag{25}$$

The first and the third differences are small in view of the continuity of the function $U(t,s)$ inside the intervals between the points of pulses:

$$\begin{aligned}&\|U(t,\tau_j)(I-P(\tau_j))-U(t,\tau_j+\varepsilon)(I-P(\tau_j+\varepsilon))\|\\ &\qquad\leq\|U(t,\tau_j+\varepsilon)(I-P(\tau_j+\varepsilon))(U(\tau_j+\varepsilon,\tau_j)-I)\|\\ &\qquad\leq\varepsilon C_2e^{-\beta(t-\tau_j-\varepsilon)},\\ &\|U(t+r,\tau_j+\varepsilon+r)(I-P(\tau_j+\varepsilon+r))-U(t+r,\tau_{j+q})(I-P(\tau_{j+q}))\|\\ &\qquad=\|U(t+r,\tau_j+\varepsilon+r)(I-P(\tau_j+\varepsilon+r))(U(\tau_j+\varepsilon+r,\tau_{j+q})-I)\|\\ &\qquad\leq\varepsilon C_2e^{-\gamma(t-\tau_j-\varepsilon)}.\end{aligned}$$

The second difference in (25) is small in view of (22).

5. Proof of Theorem 1

Denote by $\mathfrak{M}$ the space of all W-almost periodic functions with discontinuities at points of the same sequence $\{\tau_j\}$. The norm in the space $\mathfrak{M}$ is introduced by the formula

$$\|\varphi\|_0=\sup_{t\in\mathbb{R}}\|\varphi(t)\|,\qquad\varphi\in\mathfrak{M}.$$

We define an operator T on $\mathfrak{M}$ as follows: If $\varphi(t)\in\mathfrak{M}$, then

$$(T\varphi)(t)=\int_{-\infty}^{\infty}G(t,s)f(s,\varphi(s))ds+\sum_j G(t,\tau_j)g(\varphi(\tau_j)).$$

First, we prove that $T(D_h) \subseteq D_h$ for some $h > 0$, where

$$D_h = \{\varphi \in \mathfrak{M}, \|\varphi\|_0 \leq h\}.$$

Indeed, if $\|\varphi\|_0 \leq h$, then

$$\|T\varphi\| \leq \int_{-\infty}^{\infty} \|G(t,s)\| \|f(s,\varphi(s))\| ds + \sum_j \|G(t,\tau_j)\| \|g(\varphi(\tau_j))\|$$

$$\leq \int_{-\infty}^{\infty} Me^{-\beta|t-s|} H ds + \sum_j Me^{-\beta|t-\tau_j|} H$$

$$\leq 2MH \left(\frac{1}{\beta} + \frac{1}{1-e^{-\beta\theta}} \right) \leq h.$$

By Lemma 37 [2, p. 214], if $\varphi(t)$ is an W-almost periodic function and $\inf_i \tau_i^1 = \theta > 0$, then $\{\varphi(\tau_i)\}$ is an almost periodic sequence. By using the method of finding common almost periods, it is possible to show that the sequence $\{g_i(\varphi(\tau_i))\}$ is almost periodic.

Let r be an ε-almost period of the function $\varphi(t)$. By analogy with the proof of Lemma 5, we can show that, for $t \in \mathbb{R}$, $|t - \tau_j| > \varepsilon$, $j \in \mathbb{Z}$, the following inequality holds:

$$\|(T\varphi)(t+r) - (T\varphi)(t)\| = \left\| \int_{-\infty}^{+\infty} G(t+r,s) f(s) ds + \sum_j G(t+r,\tau_j) g_j \right.$$

$$\left. - \int_{-\infty}^{+\infty} G(t,s) f(s) ds - \sum_j G(t,\tau_j) g_j \right\| \leq \Gamma(\varepsilon)\varepsilon,$$

where $\Gamma(\varepsilon)$ is a bounded function of ε. Hence, we have proved that $T(D_h) \subseteq D_h$.

If $\varphi, \psi \in D_h$, then

$$\|(T\varphi)(t) - (T\psi)(t)\| \leq \int_{-\infty}^{\infty} \|G(t,s)\| \|f(s,\varphi(s)) - f(s,\phi(s))\| ds$$

$$+ \sum_k \|G(t,\tau_k)\| \|g_k(\varphi(\tau_k)) - g_k(\phi(\tau_k))\|$$

$$\leq 2MH \left(\frac{1}{\beta} + \frac{1}{1-e^{-\beta\theta}} \right) \|\varphi - \psi\|_0.$$

For sufficiently small $N > 0$, the operator T is a contraction in the domain D_h and, hence, there exists a unique W-almost periodic solution of system (1), (2).

REFERENCES

1. A. Halanay and D. Wexler *Qualitative Theory of Systems with Impulses* [Russian translation], Nauka, Moscow (1971).
2. A. M. Samoilenko and N. A. Perestyuk *Impulsive Differential Equations,* World Scientific, Singapore (1995).
3. G. T. Stamov, "Almost periodic solutions of impulsive differential equations," *Lect. Notes Math.,* **2047** (2012).
4. Y. M. Myslo and V. I. Tkachenko, "Almost periodic solutions of Mackey–Glass equations with pulse action," *Nonlin. Oscillat.,* **15**, No. 4, 537–546 (2012).
5. A. M. Samoilenko and S. I. Trofimchuk, "Almost periodic impulse systems," *Different. Equat.,* **29**, No. 5, 684–691 (1993).
6. M. U. Akhmetov and N. A. Perestyuk, "Periodic and almost periodic solutions of strongly nonlinear impulse systems," *J. Appl. Math. Mech.,* **56**, No. 6, 829–837 (1992).
7. D. Henry, "Geometric theory of semilinear parabolic equations," *Lect. Notes Math.,* **840** (1981).
8. S.-N. Chow and H. Leiva, "Existence and roughness of the exponential dichotomy for skew-product semiflow in Banach spaces," *J. Different. Equat.,* **120**, 429–477 (1995).
9. V. A. Pliss and G. R. Sell, "Robustness of exponential dichotomies in infinite-dimensional dynamical systems," *J. Dynam. Different. Equat.,* **11**, 471–513 (1999).
10. I. Kmit, L. Recke, and V. I. Tkachenko, "Robustness of exponential dichotomies of boundary-value problems for the general first-order hyperbolic systems," *Ukr. Math. J.,* **65**, No. 2, 236–251 (2013).
11. V. I. Tkachenko, "On multi-frequency systems with impulses," *Nelin. Kolyv.,* No. 1, 107–116 (1998).
12. V. I. Tkachenko, "On the exponential dichotomy of pulse evolution systems," *Ukr. Math. J.,* **46**, No. 4, 441–448 (1994).
13. L. Barreira and C. Valls, "Robustness for impulsive equations," *Nonlin. Anal.,* **72**, 2542–2563 (2010).
14. R. Naulin and M. Pinto, "Splitting of linear systems with impulses," *Rocky Mountain J. Math.,* **29**, No. 3, 1067–1084 (1999).