DOI 10.1007/s10958-015-2668-y
Journal of Mathematical Sciences, Vol. 212, No. 3, January, 2016

ALMOST PERIODIC SOLUTIONS OF NONLINEAR DISCRETE SYSTEMS THAT CAN BE NOT ALMOST PERIODIC IN BOCHNER'S SENSE

V. Yu. Slyusarchuk

UDC 517.929

We introduce a new class of almost periodic operators and establish the conditions of existence of almost periodic solutions of nonlinear discrete equations. These solutions can be not almost periodic in Bochner's sense.

1. Basic Definitions and Notation

Let $\mathbb{Z}$ be the set of all integers, let G be an arbitrary additive countable group, let $\mathbb{K}$ be a scalar field (this is either a real field $\mathbb{R}$ or a complex field $\mathbb{C}$), let E be an arbitrary Banach space over the field $\mathbb{K}$ with a norm $\|\cdot\|_E$, and let $L(X,X)$ be a Banach space of linear continuous operators $A : X \to X$ (X is an arbitrary Banach space) with the norm

$$\|A\|_{L(X,X)} = \sup_{\|x\|_X=1} \|Ax\|_X.$$

By $\mathfrak{M} = \mathfrak{M}(G,E)$, we denote the Banach space of representations $\mathbf{x} = \mathbf{x}(g)$ defined on G and taking values from E with the norm

$$\|\mathbf{x}\|_{\mathfrak{M}} = \sup_{g\in G} \|\mathbf{x}(g)\|_E$$

and the zero element $\mathbf{0}$. For each of these representations,

$$\sup_{g\in G} \|\mathbf{x}(g)\|_E < \infty.$$

Moreover, by $R(\mathbf{x})$, we denote the set of values of the representation $\mathbf{x} = \mathbf{x}(g)$, i.e., the set $\{\mathbf{x}(g) : g \in G\}$.

In the space $\mathfrak{M}$, we define the operator of shift S_h , $h \in G$, by the formula

$$(S_h\mathbf{x})(g) = \mathbf{x}(g+h), \quad g \in G. \tag{1}$$

Definition 1. *An element $\mathbf{y} \in \mathfrak{M}$ is called almost periodic (in Bochner's sense) (see [1, 2]) if the closure of the set $\{S_h\mathbf{y} : h \in G\}$ in the space $\mathfrak{M}$ is a compact subset of this space.*

The set $\mathfrak{B}$ of almost periodic elements of the space $\mathfrak{M}$ is a subspace of this space with the norm

$$\|\mathbf{x}\|_{\mathfrak{B}} = \|\mathbf{x}\|_{\mathfrak{M}}.$$

National University of Water Management and Nature Resources, Soborna Str., 11, Rivne, 33000, Ukraine;
e-mail: V.E.Slyusarchuk@gmail.com.

Translated from Neliniini Kolyvannya, Vol. 17, No. 3, pp. 407–418, July–September, 2014. Original article submitted November 5, 2013.

Definition 2. *An operator* $\mathbf{A} \in L(\mathfrak{M}, \mathfrak{M})$ *is called almost periodic (in Bochner's sense) if the closure of the set* $\{S_h \mathbf{A} S_{-h} : h \in G\}$ *in the space* $L(\mathfrak{M}, \mathfrak{M})$ *is compact in* $L(\mathfrak{M}, \mathfrak{M})$.

In what follows, to study discrete systems, we use a new class of almost periodic operators that can be not Bochner almost periodic.

We fix an arbitrary open set $D \subset E$ that may coincide with E. By $\mathcal{K}_D$, we denote the set of all nonempty compact subsets $K \subset D$. For the set $D_1 \subset D$, we denote the set of all elements $\mathbf{x} \in \mathfrak{M}$ for each of which $R(\mathbf{x}) \subset D_1$ by $\mathfrak{D}_{D_1}$.

Definition 3. *A mapping* $\mathbf{H} : \mathfrak{D}_D \to \mathfrak{M}$ *is called almost periodic if, for each set* $K \in \mathcal{K}_D$ *and each sequence* $(h_k)_{k \geq 1}$ *of elements of the group* G, *there exists a subsequence* $(h_{k_l})_{l \geq 1}$ *such that*

$$\lim_{l_1 \to \infty,\, l_2 \to \infty} \sup_{\mathbf{x} \in \mathfrak{D}_K} \left\| \mathbf{S}_{h_{l_1}} \mathbf{H} \mathbf{S}_{-h_{l_1}} \mathbf{x} - \mathbf{S}_{h_{l_2}} \mathbf{H} \mathbf{S}_{-h_{l_2}} \mathbf{x} \right\|_{\mathfrak{M}} = 0.$$

It is clear that every operator almost periodic in Bochner's sense $\mathbf{A} \in L(\mathfrak{M}, \mathfrak{M})$ is also almost periodic in the sense of Definition 3. It is also clear that, in the case where $D = E$, the space E is finite-dimensional, and the operator $\mathbf{H} : \mathfrak{D}_D \to \mathfrak{M}$ is linear, Definitions 2 and 3 are equivalent. However, in the infinite-dimensional space E, the operator $\mathbf{H}$ almost periodic in the sense of Definition 3 can be not almost periodic in the sense of Definition 2 (an example of operator of this kind is presented in the next section).

2. Example of an Operator Almost Periodic According to Definition 3 but Not Bochner Almost Periodic

Assume that $\mathbb{K} = \mathbb{R}$, $G = \mathbb{Z}$, $D = E$, and $\dim E = \infty$. By $\mathfrak{S} = \mathfrak{S}(\mathbb{Z}, E)$, we denote the set of elements of the space $\mathfrak{M}(\mathbb{Z}, E)$. For each of these elements, the closure of the set of values in the space E is a compact set. It is clear that $\mathfrak{B} \subset \mathfrak{S}$ and $\mathbf{x} + \mathbf{y}, \alpha \mathbf{x} \in \mathfrak{S}$ if $\mathbf{x}, \mathbf{y} \in \mathfrak{S}$ and $\alpha \in \mathbb{R}$. Therefore, $\mathfrak{S}$ is a vector space.

We now show that

$$\overline{\mathfrak{S}} = \mathfrak{S}. \tag{2}$$

This implies that the vector space $\mathfrak{S}$ is a subspace of the space $\mathfrak{M}$.

Let $\mathbf{x}$ be an arbitrary element of the set $\overline{\mathfrak{S}}$. There exists a sequence $(\mathbf{x}_m)_{m \geq 1}$ of elements of the set $\mathfrak{S}$ such that

$$\lim_{m \to \infty} \|\mathbf{x}_m - \mathbf{x}\|_{\mathfrak{M}(\mathbb{Z}, E)} = 0. \tag{3}$$

We fix an arbitrary number $\varepsilon > 0$. In view of relation (3), for a certain number $m_0 \in \mathbb{N}$, we get

$$\|\mathbf{x}_{m_0} - \mathbf{x}\|_{\mathfrak{M}(\mathbb{Z}, E)} < \varepsilon. \tag{4}$$

Since $\mathbf{x}_{m_0} \in \mathfrak{S}$, the set $\overline{R(\mathbf{x}_{m_0})}$ is compact in E. Hence, for this set, there exists a finite ε-grid M. According to relation (4), the set M is a (2ε)-grid for $\overline{R(\mathbf{x})}$. Then the set $\overline{R(\mathbf{x})}$ is compact in view of the arbitrariness of the choice of $\varepsilon > 0$ and the Hausdorff theorem (see [3]).

Thus, equality (2) holds and the vector space $\mathfrak{S}$ is a subspace of the space $\mathfrak{M}$.

Further, we consider the set $X = \{x_1, x_2, \ldots, x_k, \ldots\} \subset E$ whose elements satisfy the relation

$$\left\| \sum_{l=1}^{p} \beta_l x_{k_l} \right\|_E = \sum_{l=1}^{p} |\beta_l| \tag{5}$$

for any $p \in \mathbb{N}$, real numbers $\beta_1, \ldots, \beta_p$ and natural numbers $k_1, \ldots, k_p$ that do not coincide with each other. A set with the indicated property exists if, e.g., $E = C^0$, where C^0 is a Banach space of functions $x = x(t)$ bounded and continuous on $\mathbb{R}$ with values in $\mathbb{R}$. The norm in this space is introduced as follows:

$$\|x\|_{C^0} = \sup_{t \in \mathbb{R}} |x(t)|.$$

As the elements $x_1, x_2, \ldots, x_k, \ldots,$ we can take the functions $\sin \lambda_1 t, \sin \lambda_2 t, \ldots, \sin \lambda_k t, \ldots,$ respectively, where the numbers $\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots$ are linearly independent. In other words, the equality

$$n_1 \lambda_1 + n_2 \lambda_2 + \ldots + n_k \lambda_k = 0,$$

where $n_1, n_2, \ldots, n_k$ are integers, yields $n_1 = n_2 = \ldots = n_k = 0$ [2]. It is clear that the closure of the set X in the space E is not compact in E.

We define an element $\mathbf{y} = \mathbf{y}(n)$ of the space $\mathfrak{M}(\mathbb{Z}, E)$ by the formula

$$\mathbf{y}(n) = \begin{cases} x_1 & \text{for} \quad n \le 1, \\ x_n & \text{for} \quad n \ge 2. \end{cases} \tag{6}$$

Consider a set

$$Y = \{S_m \mathbf{y} : m \in \mathbb{Z}\},$$

where S_m is the operator of shift given by relation (1) in the case where $G = \mathbb{Z}$. We also consider the linear span span (Y) of this set, i.e., the minimal vector subspace of the space $\mathfrak{M}(\mathbb{Z}, E)$ that contains the set Y. Note that every element $\mathbf{u} \in \operatorname{span}(Y)$ is a linear combination of elements $S_{m_1}\mathbf{y}, \ldots, S_{m_p}\mathbf{y} \in Y$, i.e.,

$$\mathbf{u} = \sum_{k=1}^{p} \beta_k S_{m_k} \mathbf{y}$$

(here, $\beta_1, \ldots, \beta_p$ are real numbers and p is a natural number). Since, by virtue of relations (5) and (6), the following relations are satisfied for all sufficiently large natural numbers n and ω:

$$\begin{aligned} \|\mathbf{u}(n) - \mathbf{u}(n+\omega)\|_E &= \left\| \sum_{k=1}^{p} \beta_k \mathbf{y}(n + m_k) - \sum_{k=1}^{p} \beta_k \mathbf{y}(n + \omega + m_k) \right\|_E \\ &= \left\| \sum_{k=1}^{p} \beta_k x_{n+m_k} - \sum_{k=1}^{p} \beta_k x_{n+\omega+m_k} \right\|_E = 2 \sum_{k=1}^{p} |\beta_k|, \end{aligned}$$

the closure of the set of values of the nonzero element

$$\mathbf{u} = \mathbf{u}(n) = \left(\sum_{k=1}^{p} \beta_k S_{m_k} \mathbf{y}\right)(n)$$

of the vector subspace $\operatorname{span}(Y)$ is not a compact set. In other words, $\mathbf{u} \notin \mathfrak{S}$ if $\mathbf{u} \neq \mathbf{0}$. It is clear that the following limit exists:

$$\lim_{n\to-\infty} \mathbf{u}(n) = \left(\sum_{k=1}^{p} \beta_k\right) x_1.$$

We now show that the nonzero elements of the closure $\overline{\operatorname{span}(Y)}$ of the vector subspace $\operatorname{span}(Y)$ in the space $\mathfrak{M}(\mathbb{Z}, E)$ have the same properties. Let $\mathbf{z}$ be any element from $\overline{\operatorname{span}(Y)} \setminus \operatorname{span}(Y)$ and let $(\mathbf{z}_k)_{k\geq 1}$ be a sequence of elements from $\operatorname{span}(Y)$ such that

$$\lim_{k\to\infty} \|\mathbf{z}_k - \mathbf{z}\|_{\mathfrak{M}(\mathbb{Z},E)} = 0. \tag{7}$$

Further, we show that the following relation is true for some $\alpha \in \mathbb{R}$:

$$\lim_{n\to-\infty} \mathbf{z}(n) = \alpha x_1. \tag{8}$$

Indeed, let

$$\lim_{n\to-\infty} \mathbf{z}_k(n) = \alpha_k x_1, \quad \alpha_k \in \mathbb{R}, \quad k \geq 1, \tag{9}$$

where the sequence $(\alpha_k)_{k\geq 1}$ is convergent (this requirement does not decrease the generality of our considerations). In other words, for a certain number $\alpha \in \mathbb{R}$, we get

$$\lim_{k\to\infty} \alpha_k = \alpha. \tag{10}$$

It is clear that, for all $n \in \mathbb{Z}$ and $k \geq 1$,

$$\mathbf{z}(n) = (\mathbf{z}(n) - \mathbf{z}_k(n)) + (\mathbf{z}_k(n) - \alpha_k x_1) + (\alpha_k a - \alpha x_1) + \alpha x_1.$$

Therefore,

$$\|\mathbf{z}(n) - \alpha x_1\|_E \leq \|\mathbf{z}(n) - \mathbf{z}_k(n)\|_E + \|\mathbf{z}_k(n) - \alpha_k x_1\|_E + \|\alpha_k x_1 - \alpha x_1\|_E, \quad n \in \mathbb{Z}, \quad k \geq 1.$$

In view of (9), this yields that, for any $k \geq 1$,

$$\begin{aligned} 0 &\leq \overline{\lim_{n\to-\infty}} \|\mathbf{z}(n) - \alpha x_1\|_E \\ &\leq \overline{\lim_{n\to-\infty}} \|\mathbf{z}(n) - \mathbf{z}_k(n)\|_E + \overline{\lim_{n\to-\infty}} \|\mathbf{z}_k(n) - \alpha_k x_1\|_E + \overline{\lim_{n\to-\infty}} \|\alpha_k x_1 - \alpha x_1\|_E \end{aligned}$$

$$\leq \|\mathbf{z}-\mathbf{z}_k\|_{\mathfrak{M}(\mathbb{Z},E)} + \|(\alpha_k-\alpha)x_1\|_E.$$

Since these relations hold for all $k \geq 1$, relation (8) is true in view of (7) and (10).

We now show that, for an element $\mathbf{z} \in \overline{\text{span}\,(Y)} \setminus \text{span}\,(Y)$, the set $\overline{R(\mathbf{z})}$ is not compact in E. Let $(\mathbf{z}_k)_{k\geq 1}$ be a sequence of elements from $\text{span}\,(Y)$ for which relation (7) is satisfied. Note that, for every $k \geq 1$, there exist numbers $p_k \in \mathbb{N}$, $\delta_{1,k},\ldots,\delta_{p_k,k} \in \mathbb{R}$, and $m_{1,k},\ldots,m_{p_k,k} \in \mathbb{Z}$ such that the element $\mathbf{z}_k = \mathbf{z}_k(n)$ can be represented in the form

$$\mathbf{z}_k(n) = \sum_{l=1}^{p_k} \delta_{l,k}\mathbf{y}(n+m_{l,k}) = \sum_{l=1}^{p_k} \delta_{l,k}x_{n+m_{l,k}}$$

[here, we take into account relation (6)]. Thus, in view of (5), we can write

$$\|\mathbf{z}_k\|_{\mathfrak{M}(\mathbb{Z},E)} = \sum_{l=1}^{p_k} |\delta_{l,k}|$$

and, for all sufficiently large numbers $n,\omega \in \mathbb{N}$,

$$\|\mathbf{z}_k(n)-\mathbf{z}_k(n+\omega)\|_E = \left\|\sum_{l=1}^{p_k} \delta_{l,k}x_{n+m_{l,k}} - \sum_{l=1}^{p_k} \delta_{l,k}x_{n+\omega+m_{l,k}}\right\|_E = 2\sum_{l=1}^{p_k} |\delta_{l,k}| = 2\|\mathbf{z}_k\|_{\mathfrak{M}(\mathbb{Z},E)}.$$

These equalities imply that, for all sufficiently large numbers $n,\omega \in \mathbb{N}$,

$$\begin{aligned}\|\mathbf{z}(n)-\mathbf{z}(n+\omega)\|_E &\geq \|\mathbf{z}_k(n)-\mathbf{z}_k(n+\omega)\|_E - \|(\mathbf{z}(n)-\mathbf{z}_k(n))-(\mathbf{z}(n+\omega)-\mathbf{z}_k(n+\omega))\|_E \\ &\geq \|\mathbf{z}_k(n)-\mathbf{z}_k(n+\omega)\|_E - \|\mathbf{z}(n)-\mathbf{z}_k(n)\|_E - \|\mathbf{z}(n+\omega)-\mathbf{z}_k(n+\omega)\|_E \\ &\geq 2\|\mathbf{z}_k\|_{\mathbb{M}(\mathbb{Z},E)} - 2\|\mathbf{z}-\mathbf{z}_k\|_{\mathbb{M}(\mathbb{Z},E)}, \quad k \geq 1.\end{aligned}$$

This fact, the inclusion $\mathbf{z} \in \overline{\text{span}\,(Y)} \setminus \text{span}\,(Y)$, and relation (7) imply that, for some number $\gamma > 0$ and all sufficiently large numbers $n,\omega \in \mathbb{N}$,

$$\|\mathbf{z}(n)-\mathbf{z}(n+\omega)\|_E \geq \gamma,$$

which means that the set $\overline{R(\mathbf{z})}$ is not compact in the space E.

Hence, $\overline{\text{span}\,(Y)}$ is a subspace of the space $\mathfrak{M}(\mathbb{Z},E)$.

Consider a subspace

$$L = \mathfrak{S} \oplus \overline{\text{span}\,(Y)}$$

of the space $\mathfrak{M}(\mathbb{Z},E)$. Note that every vector $\mathbf{x} \in L$ admits a unique representation in the form $\mathbf{x} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u} \in \mathfrak{S}$ and $\mathbf{v} \in \overline{\text{span}\,(Y)}$. Indeed, if there exist two representations of this kind, namely,

$$\mathbf{x} = \mathbf{u}_1 + \mathbf{v}_1,$$

$$\mathbf{x} = \mathbf{u}_2 + \mathbf{v}_2 \left(\mathbf{u}_1,\mathbf{u}_2 \in \mathfrak{S},\ \mathbf{v}_1,\mathbf{v}_2 \in \overline{\text{span}\,(Y)}\right),$$

then $\mathbf{u}_1 + \mathbf{v}_1 = \mathbf{u}_2 + \mathbf{v}_2$. Hence, we get the equality $\mathbf{v}_1 = \mathbf{v}_2$ for $\mathbf{u}_1 = \mathbf{u}_2$ and the equality $\mathbf{u}_1 - \mathbf{u}_2 = \mathbf{v}_2 - \mathbf{v}_1$ for $\mathbf{u}_1 \neq \mathbf{u}_2$. This contradicts the inclusion $\mathbf{u}_2 - \mathbf{u}_1 \in \mathfrak{S}$ because

$$\mathbf{v}_2 - \mathbf{v}_1 \in \overline{\operatorname{span}(Y)} \setminus \{\mathbf{0}\}$$

and the set $\mathfrak{S} \cap \left(\overline{\operatorname{span}(Y)} \setminus \{\mathbf{0}\}\right)$ is empty.

Consider a linear continuous functional

$$\psi : \overline{\operatorname{span}\ (\{x_k : k \in \mathbb{N}\})} \to \mathbb{R}$$

such that $\psi(x_1) = 1$ and $\|\psi\| = 1$. This functional exists (see, e.g., [4]).

We define a linear functional $\varphi : L \to \mathbb{R}$ as follows: Every vector $\mathbf{x} = \mathbf{u} + \mathbf{v} \in L$, where $\mathbf{u} \in \mathfrak{S}$ and $\mathbf{v} \in \overline{\operatorname{span}(Y)}$, is associated with a number

$$\varphi(\mathbf{x}) = \varphi(\mathbf{u}) + \varphi(\mathbf{v}),$$

where

$$\varphi(\mathbf{u}) = 0$$

and

$$\varphi(\mathbf{u}) = \psi\left(\lim_{n \to -\infty} \mathbf{u}(n)\right).$$

This functional is continuous due to the continuity of the functional ψ.

By the Hahn–Banach theorem on continuation of linear continuous functionals [4], there exists a linear continuous functional $l : \mathfrak{M}(\mathbb{Z}, E) \to \mathbb{R}$, such that $l(\mathbf{x}) = \varphi(\mathbf{x})$ for all $\mathbf{x} \in L$ and $\|l\| = \|\varphi\|$.

We fix an arbitrary element $\mathbf{s} \in \mathfrak{M} \setminus \mathfrak{B}$ and define a linear continuous operator $C : \mathfrak{M} \to \mathfrak{M}$ by the formula

$$C\mathbf{x} = l(\mathbf{x})\mathbf{s}, \quad \mathbf{x} \in \mathfrak{M}. \tag{11}$$

We now show that this operator is almost periodic in the sense of Definition 3 and is not almost periodic in the sense of Definition 2.

Note that, in view of (11),

$$S_m C S_{-m} \mathbf{x} = l(S_{-m}\mathbf{x}) S_m \mathbf{s} \tag{12}$$

for any $\mathbf{x} \in \mathfrak{M}(\mathbb{Z}, E)$ and

$$l(S_{-m}\mathbf{x}) = 0$$

for any $\mathbf{x} \in \mathfrak{S}$. Thus, for any compact set $K \subset E$, the closure of the set $\{S_m C S_{-m}\mathbf{x} : m \in \mathbb{Z}, \mathbf{x} \in \mathfrak{D}_K\}$ in the space $\mathfrak{M}(\mathbb{Z}, E)$ is compact in $\mathfrak{M}(\mathbb{Z}, E)$ because this set coincides with $\{\mathbf{0}\}$. This means that the operator C is almost periodic in the sense of Definition 3. However, the closure of the set $\{S_m C S_{-m} : m \in \mathbb{Z}\}$ in the space $L(\mathfrak{M}(\mathbb{Z}, E), \mathfrak{M}(\mathbb{Z}, E))$ is not compact in $L(\mathfrak{M}(\mathbb{Z}, E), \mathfrak{M}(\mathbb{Z}, E))$. Indeed, by virtue of (11) and (12), the element $\mathbf{y}$ specified with the help of (6) satisfies the relation

$$S_m C S_{-m} \mathbf{y} = S_m \mathbf{s}, \quad m \in \mathbb{Z},$$

and, hence,

$$\{S_m C S_{-m} : m \in \mathbb{Z}\} \mathbf{y} = \{S_m \mathbf{s} : m \in \mathbb{Z}\}. \tag{13}$$

If the operator C is almost periodic in the sense of Definition 2, i.e., $\{S_m C S_{-m} : m \in \mathbb{Z}\}$ is a precompact set in the space $L(\mathfrak{M}(\mathbb{Z}, E), \mathfrak{M}(\mathbb{Z}, E))$, then the set $\{S_m C S_{-m} : m \in \mathbb{Z}\}\mathbf{y}$ is precompact in the space $\mathfrak{M}(\mathbb{Z}, E)$. In view of equality (13), the set $\{S_m \mathbf{s} : m \in \mathbb{Z}\}$ should also be precompact in the space $\mathfrak{M}(\mathbb{Z}, E)$. However, for $\{S_m \mathbf{s} : m \in \mathbb{Z}\}$, this property does not hold because the element $\mathbf{s}$ is not almost periodic (see Definition 1).

Hence, the construction of the operator with required properties is completed.

Remark 1. Assume that the Banach space E coincides with the space $l_1 = l_1(\mathbb{N}, \mathbb{R})$ of sequences $a = (a_1, a_2, \ldots, a_k, \ldots)$ each of which satisfies the inequality

$$\sum_{k=1}^{\infty} |a_k| < \infty$$

with the norm

$$\|a\|_{l_1} = \sum_{k=1}^{\infty} |a_k|. \tag{14}$$

As the set

$$X = \{x_1, x_2, \ldots, x_k, \ldots\} \subset E$$

used in the construction of the presented example, we can take a set $\tilde{X}$ of sequences

$$x_k = (\delta_{k1}, \delta_{k2}, \delta_{k3}, \ldots), \quad k \in \mathbb{N},$$

where δ_{kl} is the Kronecker symbol: $\delta_{kl} = 1$ for $k = l$ and $\delta_{kl} = 0$ for $k \neq l$.

Thus, it is clear that, in view of (14), the elements of the set $\tilde{X}$ satisfy relation (5).

3. Main Object of Investigations

Let Ω be an arbitrary domain from the space E. Consider a mapping $\mathbf{F} : \mathfrak{D}_\Omega \to \mathfrak{M}$ such that, *for every* $K \in \mathcal{K}_\Omega$, *the closure of the set* $\{S_h \mathbf{F} S_{-h} \mathbf{x} : h \in G, \ \mathbf{x} \in \mathfrak{D}_K\}$ *in the space* $\mathfrak{M}$ *is compact in* $\mathfrak{M}$, i.e., the mapping $\mathbf{F}$ is almost periodic in the sense of Definition 3.

It is clear that, for any $K \in \mathcal{K}$ and a sequence $(h_k)_{k\geq 1}$ of elements of the group G, there exists a subsequence $(h_{k_l})_{l\geq 1}$ such that the sequence $\left(S_{h_{k_l}} \mathbf{F} S_{-h_{k_l}} \mathbf{x}\right)_{l\geq 1}$ converges uniformly on $\mathfrak{D}_K$.

The aim of the present work is to establish the conditions of existence of almost periodic solutions of the equation

$$\mathbf{F}\mathbf{x} = \mathbf{0}. \tag{15}$$

In analyzing this equation, we use a functional defined on the set of solutions of the equation with precompact ranges of values.

Note that difference equations are special cases of Eq. (15). Thus, in particular, this is true for the equation

$$x_{n+1} = f_n(x_n), \quad n \in \mathbb{Z}.$$

The existence of almost periodic solutions of this equation was studied by the author in [5].

4. Functional δ. Separated and Strongly Separated Solutions of Eq. (15)

We fix an arbitrary set $K \in \mathcal{K}$. By $\mathcal{N}(\mathbf{F}, K)$, we denote the set of all solutions $\mathbf{x}$ of Eq. (15) for each of which $R(\mathbf{x}) \subset K$ and $\overline{R(\mathbf{x})} \neq K$.

We now fix an arbitrary element $\mathbf{x}^* \in \mathcal{N}(\mathbf{F}, K)$ (it is assumed that $\mathcal{N}(\mathbf{F}, K) \neq \varnothing$) and set

$$r(\mathbf{x}^*, K) = \sup\left\{\|x - y\|_E : x \in \overline{R(\mathbf{x}^*)},\ y \in K\right\}. \tag{16}$$

In view of the inequality $\overline{R(\mathbf{x})} \neq K$, we get

$$r(\mathbf{x}^*, K) > 0.$$

We also fix an arbitrary number $\varepsilon \in [0, r(\mathbf{x}^*, K)]$. By $\Omega(\mathbf{x}^*, K, \varepsilon)$, we denote the set of all elements $\mathbf{y} \in \mathfrak{M}$ such that

$$R(\mathbf{x}^* + \mathbf{y}) \subset K \tag{17}$$

and

$$\|\mathbf{y}\|_{\mathfrak{M}} \geq \varepsilon. \tag{18}$$

Similarly, we can define the set $\Omega(\mathbf{z}, K, \varepsilon)$ for any other element $\mathbf{z} \in \mathfrak{M}$ such that $R(\mathbf{z}) \subset K$.

Consider a functional

$$\delta(\mathbf{x}^*, K, \varepsilon) = \inf_{\mathbf{y} \in \Omega(\mathbf{x}^*, K, \varepsilon)} \left\|\mathbf{F}(\mathbf{x}^* + \mathbf{y})\right\|_{\mathfrak{M}}. \tag{19}$$

Definition 4. *A solution $\mathbf{z} \in \mathcal{N}(\mathbf{F}, K)$ of Eq. (15) is called separated on the set $G \times K$ if this solution is unique on the set $G \times K$ or the following inequality is satisfied for any other solution $\mathbf{u} = \mathbf{u}(g)$ with values in K:*

$$\inf_{g \in G} \|\mathbf{z}(g) - \mathbf{u}(g)\|_E \geq \rho.$$

Here, ρ is a positive constant depending only on $\mathbf{z}$.

Definition 5. *A solution* $\mathbf{z} \in \mathcal{N}(\mathbf{F}, K)$ *of Eq. (15) is called strongly separated on the set* $G \times K$ *if*

$$\delta(\mathbf{z}, K, \varepsilon) > 0$$

for every $\varepsilon \in (0, r(\mathbf{z}, K))$.

It is clear that any solution $\mathbf{z} \in \mathcal{N}(\mathbf{F}, K)$ of Eq. (15) strongly separated on the set $G \times K$ is a solution of this equation separated on the set $G \times K$. At the same time, the solution $\mathbf{z} \in \mathcal{N}(\mathbf{F}, K)$ of Eq. (15) separated on the set $G \times K$ can be a solution of this equation, which is not strongly separated on the set $G \times K$ (the corresponding example is constructed in [5] for the case where $G = \mathbb{Z}$).

The applications of the functional δ to the investigation of the nonlinear equation (15) and a similar linear equation are discussed in the next sections.

Similar functionals were used by the author in [6–8] to study nonlinear almost periodic equations,

$$x(t+1) = f(t, x(t)),$$

$$\frac{dx(t)}{dt} = f(t, x(t)), \quad t \in \mathbb{R},$$

and

$$f(t, x(t)) = 0, \quad t \in \mathbb{R}, \tag{20}$$

with a continuous mapping $f : \mathbb{R} \times \Omega \to E$, where Ω is an arbitrary domain of the space E.

5. Main Result

We now present conditions for the existence of almost periodic solutions of Eq. (15) in which, unlike the well-known Amerio theorem on almost periodic solutions of nonlinear differential equations, the $\mathcal{H}$-class of Eq. (15) is not used (see [9, 10]).

Let Λ be a bounded subset of the space E. We define the diameter $\operatorname{diam}\Lambda$ of the set Λ by the equality

$$\operatorname{diam}\Lambda = \sup\{\|x - y\|_E : x, y \in \Lambda\}.$$

Theorem 1. *If the solution* $\mathbf{z} \in \mathcal{N}(\mathbf{F}, K)$ *of Eq. (15), where* $K \in \mathcal{K}$, *is such that* $\operatorname{diam} R(\mathbf{z}) \neq 0$ *and*

$$\delta(\mathbf{z}, K, \varepsilon) > 0 \tag{21}$$

for any $\varepsilon \in (0, r(\mathbf{z}, K))$, *then this solution is almost periodic.*

Remark 2. The solution $\mathbf{z} \in \mathcal{N}(\mathbf{F}, K)$ of Eq. (15) for which $\operatorname{diam} R(\mathbf{z}) = 0$ is stationary and, hence, almost periodic.

Proof. We assume that the solution $\mathbf{z} \in \mathcal{N}(\mathbf{F}, K)$ of Eq. (15) is not an element of the space $\mathfrak{B}$. Then there exists a sequence $\left(S_{h_p}\mathbf{z}\right)_{p\ge1}$ (here, $h_p \in \mathbb{Z}$, $p \ge 1$) for which every subsequence $\left(S_{k_p}\mathbf{z}\right)_{p\ge1}$ is divergent. Hence, for some sequences of natural numbers $(p_r)_{r\ge1}$ and $(q_r)_{r\ge1}$ and a number $\gamma \in (0, \operatorname{diam} R(\mathbf{z}))$, we get

$$\left\| S_{k_{p_r}}\mathbf{z} - S_{k_{q_r}}\mathbf{z} \right\|_{\mathfrak{M}} \ge \gamma, \quad r \ge 1. \tag{22}$$

Note that

$$\operatorname{diam} R(\mathbf{z}) \le r(\mathbf{z}, K).$$

Without loss of generality, we can assume that the sequence

$$\left(S_{k_p}\mathbf{F}S_{-k_p}\mathbf{x}\right)_{p\ge 1}$$

is uniformly convergent on $\mathfrak{D}_K$. Thus,

$$\lim_{p,q\to\infty} \sup_{\mathbf{x}\in\mathfrak{D}_K} \left\| S_{k_p}\mathbf{F}S_{-k_p}\mathbf{x} - S_{k_q}\mathbf{F}S_{-k_q}\mathbf{x} \right\|_{\mathfrak{M}} = 0. \tag{23}$$

Consider the elements

$$\mathbf{y}_r = S_{k_{p_r}}\mathbf{z} - S_{k_{q_r}}\mathbf{z}, \quad r \ge 1,$$

of the space $\mathfrak{M}$. It is clear that

$$\mathbf{y}_r \in \Omega(S_{k_{q_r}}\mathbf{z}, K, \gamma), \quad r \ge 1. \tag{24}$$

We now show that

$$\delta(\mathbf{z}, K, \gamma) = 0. \tag{25}$$

In view of relations (19) and (24) and the fact that

$$S_{k_{p_r}}\mathbf{F}\mathbf{z} = \mathbf{0}, \quad r \ge 1,$$

the following relations hold for every $r \ge 1$:

$$\begin{aligned}
\delta(\mathbf{z}, K, \gamma) &= \inf_{\mathbf{y}\in\Omega(\mathbf{z},K,\gamma)} \|\mathbf{F}(\mathbf{z}+\mathbf{y})\|_{\mathfrak{M}} = \inf_{\mathbf{y}\in\Omega(S_{k_{q_r}}\mathbf{z},K,\gamma)} \|S_{k_{q_r}}\mathbf{F}(\mathbf{z}+S_{-k_{q_r}}\mathbf{y})\|_{\mathfrak{M}} \\
&= \inf_{\mathbf{y}\in\Omega(S_{k_{q_r}}\mathbf{z},K,\gamma)} \|S_{k_{q_r}}\mathbf{F}S_{-k_{q_r}}(S_{k_{q_r}}\mathbf{z}+\mathbf{y})\|_{\mathfrak{M}} \le \|S_{k_{q_r}}\mathbf{F}S_{-k_{q_r}}(S_{k_{q_r}}\mathbf{z}+\mathbf{y}_r)\|_{\mathfrak{M}} \\
&= \|S_{k_{q_r}}\mathbf{F}S_{-k_{q_r}}(S_{k_{q_r}}\mathbf{z}+(S_{k_{p_r}}\mathbf{z}-S_{k_{q_r}}\mathbf{z}))\|_{\mathfrak{M}} = \|S_{k_{q_r}}\mathbf{F}S_{-k_{q_r}}S_{k_{p_r}}\mathbf{z}\|_{\mathfrak{M}} \\
&\le \|S_{k_{p_r}}\mathbf{F}S_{-k_{p_r}}S_{k_{p_r}}\mathbf{z}\|_{\mathfrak{M}} + \|S_{k_{q_r}}\mathbf{F}S_{-k_{q_r}}S_{k_{p_r}}\mathbf{z} - S_{k_{p_r}}\mathbf{F}S_{-k_{p_r}}S_{k_{p_r}}\mathbf{z}\|_{\mathfrak{M}} \\
&= \|S_{k_{p_r}}\mathbf{F}\mathbf{z}\|_{\mathfrak{M}} + \|S_{k_{q_r}}\mathbf{F}S_{-k_{q_r}}S_{k_{p_r}}\mathbf{z} - S_{k_{p_r}}\mathbf{F}S_{-k_{p_r}}S_{k_{p_r}}\mathbf{z}\|_{\mathfrak{M}} \\
&= \|S_{k_{q_r}}\mathbf{F}S_{-k_{q_r}}S_{k_{p_r}}\mathbf{z} - S_{k_{p_r}}\mathbf{F}S_{-k_{p_r}}S_{k_{p_r}}\mathbf{z}\|_{\mathfrak{M}}
\end{aligned}$$

$$\leq \sup_{\mathbf{x}\in\mathfrak{D}_K} \left\| S_{k_{qr}} \mathbf{F} S_{-k_{qr}} \mathbf{x} - S_{k_{pr}} \mathbf{F} S_{-k_{pr}} \mathbf{x} \right\|_{\mathfrak{M}} .$$

By virtue of (23), this yields relation (25), which contradicts (21).

Hence, the assumption that the solution $\mathbf{z} \in \mathcal{N}(\mathbf{F}, K)$ of Eq. (15) is not almost periodic is wrong.

Theorem 1 is proved.

Note that the validity of relation (21) means that the solution $\mathbf{z} \in \mathcal{N}(\mathbf{F}, K)$ of Eq. (15) is strongly separated on the set $G \times K$. Hence, this theorem can be formulated in the following form:

Theorem 2. *Let K belong to $\mathcal{K}$. If the solution $\mathbf{z} \in \mathcal{N}(\mathbf{F}, K)$ of Eq. (15) is strongly separated on the set $G \times K$, then this solution is almost periodic.*

Remark 3. The condition of strong separability of a bounded solution of Eq. (15) is not a condition necessary for this solution to belong to the space $\mathfrak{B}$ (this is a sufficient condition). The solution of Eq. (15) can be almost periodic but not separated on the set $G \times K$, which is confirmed by the difference equation

$$x_{n+1} = x_n, \quad n \in \mathbb{Z}.$$

6. The Case of Linear Equation (15)

Consider an equation

$$\mathbf{A}\mathbf{x} = \mathbf{h}, \tag{26}$$

where

$$\mathbf{A} : \mathfrak{M} \to \mathfrak{M}$$

is a linear operator continuous and almost periodic in the sense of Definition 3 (this operator can be not Bochner almost periodic), and $\mathbf{h} \in \mathfrak{B}$.

Since Eq. (26) is a special case of Eq. (15) (the operator $\mathbf{F}$ is defined by the formula

$$\mathbf{F}\mathbf{x} = \mathbf{A}\mathbf{x} - \mathbf{h}, \quad \mathbf{x} \in \mathfrak{M}\,),$$

the following assertion is true by virtue of Theorem 2:

Theorem 3. *Let K belong to $\mathcal{K}$. The solution $\mathbf{z}$ of Eq. (26) strongly separated on the set $G \times K$ is almost periodic.*

We now present the conditions of strong separability of the solution $\mathbf{z}$ of Eq. (26) on $G \times K$.

Consider the linear homogeneous equation

$$\mathbf{A}\mathbf{x} = \mathbf{0} \tag{27}$$

corresponding to Eq. (26).

Theorem 4. *Let K belong to $\mathcal{K}$. The solution $\mathbf{z}$ of Eq. (26) with values in K is strongly separated on $G \times K$ iff the trivial solution $\mathbf{0}$ of Eq. (27) is strongly separated on $G \times K$.*

Proof. Since $\mathbf{z}$ is a solution of Eq. (26), every element $\mathbf{u}$ of the space $\mathfrak{M}$ for which

$$\mathbf{A}(\mathbf{z}+\mathbf{u})=\mathbf{h},$$

is a solution of Eq. (27), i.e.,

$$\mathbf{A}\mathbf{u}=\mathbf{0},$$

and vice versa. If we now use the definitions of the set $\Omega(\mathbf{x}^*, K, \varepsilon)$ [see (17) and (18)] and the functional $\delta(\mathbf{x}^*, K, \varepsilon)$ [see (19)], then, in the case of linear equations, we conclude that, for any $\varepsilon \in (0, r(\mathbf{z}, K))$ [see (16)],

$$\inf_{\mathbf{y}\in\Omega(\mathbf{z},K,\varepsilon)} \|\mathbf{A}(\mathbf{z}+\mathbf{y})-\mathbf{h}\|_{\mathfrak{M}} = \inf_{\mathbf{y}\in\Omega(\mathbf{0},K,\varepsilon)} \|\mathbf{A}\mathbf{y}\|_{\mathfrak{M}} > 0.$$

In other words, for all $\varepsilon \in (0, r(\mathbf{z}, K))$, we find

$$\delta(\mathbf{z}, K, \varepsilon) = \delta(\mathbf{0}, K, \varepsilon) > 0.$$

This yields the assertion of the theorem.

Theorem 4 is proved.

Theorem 5. *If*

$$\inf_{\mathbf{x}\in\mathfrak{S},\ \|\mathbf{x}\|_{\mathfrak{M}}=1} \|\mathbf{A}\mathbf{x}\|_{\mathfrak{M}} > 0, \tag{28}$$

then every solution $\mathbf{z}\in\mathfrak{S}$ *of Eq. (26) is almost periodic.*

Proof. Since $\mathbf{z}\in\mathfrak{S}$, we get

$$\overline{R(\mathbf{z})} \subset K$$

for some $K \in \mathcal{K}$. Thus, in view of (28) and the linearity of the operator $\mathbf{A}$, we find

$$\inf_{\mathbf{x}\in\mathfrak{S},\ \overline{R(\mathbf{x})}\subset K,\ \|\mathbf{x}\|_{\mathfrak{M}}=\varepsilon} \|\mathbf{A}\mathbf{x}\|_{\mathfrak{M}} > 0$$

for every $\varepsilon \in (\mu(\mathbf{z}, K), r(\mathbf{z}, K)]$, where

$$\mu(\mathbf{z}, K) = \inf\left\{\|x-y\|_E : x \in \overline{R(\mathbf{x}^*)},\ y \in K\right\}$$

and, hence,

$$\inf_{\mathbf{x}\in\mathfrak{S},\ \overline{R(\mathbf{x})}\subset K,\ \|\mathbf{x}\|_{\mathfrak{M}}\geq\varepsilon} \|\mathbf{A}\mathbf{x}\|_{\mathfrak{M}} > 0$$

for any $\varepsilon \in (0, r(\mathbf{z}, K)]$. In view of the last relation, the trivial solution $\mathbf{0}$ of Eq. (27) is strongly separated on $G\times K$. Therefore, by virtue of Theorem 4, the solution $\mathbf{z}\in\mathfrak{S}$ of Eq. (26) is also strongly separated on $G\times K$.

Hence, by Theorem 3, the solution $\mathbf{z}\in\mathfrak{S}$ of Eq. (26) is almost periodic.

Theorem 5 is proved.

Remark 4. The set of equations almost periodic in the sense of Definition 3 (for which it is possible to apply the theorems from Sections 5 and 6) is nonempty. An element of this set is, e.g., the equation

$$\mathbf{x} + C\mathbf{x} = \mathbf{h}, \tag{29}$$

where

$$C : \mathfrak{M}(\mathbb{Z}, E) \to \mathfrak{M}(\mathbb{Z}, E)$$

is a linear continuous operator defined by relation (11) and $\mathbf{h}$ is an almost periodic element of the space $\mathfrak{M}(\mathbb{Z}, E)$.

It is clear that the operator $I + C$, where

$$I : \mathfrak{M}(\mathbb{Z}, E) \to \mathfrak{M}(\mathbb{Z}, E)$$

is the identity operator, is almost periodic in the sense of Definition 3 but is not Bochner almost periodic.

Since $\mathbf{h} \in \mathfrak{S}$ and $C\mathbf{y} = \mathbf{0}$ for any $\mathbf{y} \in \mathfrak{S}$, Eq. (29) has a unique solution $\mathbf{x}$ in the space $\mathfrak{S}$ that coincides with $\mathbf{h}$ and is strongly separated [according to Definition 5 and the definition of the functional Δ (see (19))] on each set $\mathbb{Z} \times K$, where K is an arbitrary compact set in E for which $\overline{R(\mathbf{h})} \subset K$.

In conclusion, we note that the presented conditions for the existence of almost periodic solutions of Eqs. (15) and (26) are new even in the case $G = \mathbb{Z}$. Unlike the Amerio theorem, the $\mathcal{H}$-class of Eq. (15) is not used in Theorems 1 and 2 and the Banach space E can be infinite-dimensional. Similarly, in Theorems 3 and 5, the $\mathcal{H}$-class of Eq. (26) is also not used and the operator $\mathbf{A}$ can be not Bochner almost periodic.

We also note that there are numerous works devoted to the investigation of almost periodicity of the solutions of equations. We now mention only a part of these publications. For ordinary linear differential equations, the first theorems on almost periodic solutions were proved by Favard [11]. For the nonlinear differential equations, theorems of this sort were obtained by Amerio [9]. In the cited works, the $\mathcal{H}$-classes of the analyzed equations were essentially used. Moreover, the requirement of separability of bounded solutions of the equations was also used in [9]. Favard's results were improved by Mukhamadiev [12, 13]. The Mukhamadiev theorems were later generalized in [14–16]. In this direction, significant results were also obtained by Levitan [2], Amerio [17], and Zhikov [18]. The conditions of almost periodicity of bounded solutions of nonlinear difference and differential equations, as well as of Eq. (20), were established by the author (without using the $\mathcal{H}$-classes of these equations) in [5–8].

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