DOI 10.1007/s10958-015-2662-4
Journal of Mathematical Sciences, Vol. 212, No. 3, January, 2016

PROPERTIES OF CONTINUOUS PERIODIC SOLUTIONS OF SYSTEMS OF FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH SMALL PARAMETER

N. L. Denysenko UDC 517.9

We establish sufficient conditions for the existence of periodic solutions of a system of nonlinear functional-differential equations with small parameter and linear deviations of the argument and study the properties of these solutions.

Various special cases of systems of functional-differential equations of the form

$$\dot{x}(t) = f(t, x(t), x(\lambda_1 t), \ldots, x(\lambda_k t)),$$

where $\lambda_i > 0$, $i = \overline{1,k}$, and f is a vector function of dimension n, were investigated by numerous mathematicians. Thus, at present, numerous problems from the theory of these systems are fairly well studied (see [1–7] and the references therein). Indeed, the asymptotic properties of the solutions of a linear scalar equation ($n = 1$) were comprehensively investigated in [1]. Sufficient conditions for the existence and uniqueness of a solution bounded on the entire real axis were established for a system of nonlinear functional-differential equations of neutral type in [3]. In [4], the problem of existence of periodic solutions of systems of functional-differential equations with linear deviations of the argument were investigated and properties of these solutions were studied. In the present paper, we continue our investigation of the existence of periodic solutions for systems of functional-differential equations with small parameter and study the properties of these solutions.

Consider a system of nonlinear functional-differential equations of the form

$$\dot{x}(t) = Ax(t) + f(t, x(t), x(\lambda_1 t), \ldots, x(\lambda_k t)) + \varepsilon F(t, x(t), x(\lambda_1 t), \ldots, x(\lambda_k t), \varepsilon) \tag{1}$$

in the case where $\lambda_i \in \mathbb{N}$, $i = \overline{1,k}$, ε is a sufficiently small nonnegative scalar parameter, $t \in \mathbb{R} = (-\infty, +\infty)$, A is a real constant $(n \times n)$-matrix, and $f : \mathbb{R} \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}^n$ and $F : \mathbb{R} \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ are vector functions continuous in all variables and T-periodic in t, i.e.,

$$f(t+T, x(t), x(\lambda_1 t), \ldots, x(\lambda_k t)) \equiv f(t, x(t), x(\lambda_1 t), \ldots, x(\lambda_k t)),$$

$$F(t+T, x(t), x(\lambda_1 t), \ldots, x(\lambda_k t), \varepsilon) \equiv F(t, x(t), x(\lambda_1 t), \ldots, x(\lambda_k t), \varepsilon).$$

Assume that the eigenvalues a_j, $j = \overline{1,n}$, of the matrix A satisfy the condition

$$\operatorname{Re} a_j(A) \neq 0, \quad j = \overline{1,n}.$$

It is known that, in this case, there exists a nonsingular matrix C that reduces the matrix A to the form

$$A = C^{-1} \operatorname{diag}(A_1, A_2) C,$$

"Kyiv Polytechnic Institute" Ukrainian National Technical University, Ukraine, 03056, Kyiv, Peremoha Avenue, 37; e-mail: natalia_den@bigmir.net.

Translated from Neliniini Kolyvannya, Vol. 17, No. 3, pp. 332–340, July–September, 2014. Original article submitted December 20, 2013.

 1072-3374/16/2123–0254

where A_1 and A_2 are constant matrices of dimensions $p \times p$ and $(n-p)\times(n-p)$, respectively, whose eigenvalues satisfy the conditions

$$\begin{aligned} &\operatorname{Re} a_j(A_1) < 0, \quad j = 1,\ldots,p, \\ &\operatorname{Re} a_j(A_2) > 0, \quad j = p+1,\ldots,n \quad (0 < p \le n). \end{aligned} \tag{2}$$

1. In [5], we studied the problem of existence of T-periodic solutions of the system of equations (1) for $\varepsilon = 0$, i.e., of the system of equations

$$\dot{x}(t) = Ax(t) + f(t, x(t), x(\lambda_1 t), \ldots, x(\lambda_k t)). \tag{3}$$

To this end, we use the transformation

$$\dot{x}(t) = Ax(t) + y(t), \tag{4}$$

where $y(t) \in C^0$ and C^0 is a space of continuous T-periodic vector functions with the norm

$$\|y(t)\| = \max_t |y(t)|.$$

Thus, it directly follows from (3) that $x(t)$ is uniquely defined by the relation

$$x(t) = \int_{-\infty}^{+\infty} G(t-\tau)\, y(\tau)\, d\tau, \tag{5}$$

where

$$G(t) = \begin{cases} C^{-1}\operatorname{diag}(e^{A_1 t}, 0)\, C & \text{for} \quad t > 0, \\ -C^{-1}\operatorname{diag}(0, e^{A_2 t})\, C & \text{for} \quad t < 0. \end{cases} \tag{6}$$

It is easy to see that the matrix function $G(t) = (g_{ij}(t))$ satisfies the following conditions:

(a) $G(+0) - G(-0) = E$, where E is the identity matrix of dimension $n \times n$;

(b) $|G(t)| \le K e^{-\alpha|t|}$ for all $t \ne 0$, where $K > 0$, $\alpha > 0$, and

$$|G| = \max_{1 \le i \le n} \sum_{j=1}^{n} |g_{ij}|;$$

(c) $\dot{G} = AG$ for $t \ne 0$.

As a result of transformation (4), the system of equations (3) takes the form

$$y(t) = f\Bigg(t, \int\limits_{-\infty}^{+\infty} G(t-\tau)y(\tau)d\tau, \lambda_1 \int\limits_{-\infty}^{+\infty} G(\lambda_1(t-\tau))y(\lambda_1\tau)d\tau,$$

$$\ldots, \lambda_k \int\limits_{-\infty}^{+\infty} G(\lambda_k(t-\tau))y(\lambda_k\tau)d\tau\Bigg), \qquad (7)$$

where $G(t)$ is given by relation (6).

The following theorem is proved for the system of equations (7):

Theorem 1. *Suppose that the following conditions are satisfied:*

(i) all eigenvalues a_j, $j = \overline{1,n}$, *of the matrix* A *are such that (2) is true, i.e.,*

$$\exists\, K > 0,\ \alpha > 0:\ |G(t)| \le Ke^{-\alpha|t|}$$

for all $t \neq 0$;

(ii) all components of the vector function $f(t, y_0, y_1, \ldots, y_k)$ *are functions continuous in all variables and* T *-periodic in* t *and, in addition,*

$$\max_{t\in\mathbb{R}} |f(t,0,\ldots,0)| \le f^* < \infty;$$

(iii)

$$|f(t,\tilde{y}_0,\tilde{y}_1,\ldots,\tilde{y}_k) - f(t,\tilde{\tilde{y}}_0,\tilde{\tilde{y}}_1,\ldots,\tilde{\tilde{y}}_k)| \le l\sum_{i=0}^{k}|\tilde{y}_i - \tilde{\tilde{y}}_i|,$$

where $t \in \mathbb{R}$, $\tilde{y}_i, \tilde{\tilde{y}}_i \in \mathbb{R}^n$, $i = \overline{0,k}$, *and* $l = \text{const} > 0$;

(iv) $\dfrac{2Kl(k+1)}{\alpha} < 1.$

Then there exists a unique continuous T *-periodic solution* $\gamma = \gamma(t)$ *of the system of equations (7).*

By using Theorem 1 and relation (5), we conclude that the vector function

$$\bar{x}(t) = \int\limits_{-\infty}^{+\infty} G(t-\tau)\gamma(\tau)\,d\tau$$

is a unique continuous T -periodic solution of system (3), i.e., of the system of equations (1) for $\varepsilon = 0$.

2. We now consider T -periodic solutions of the system of equations (1) for $\varepsilon \neq 0$.

In the system of equations (1), we perform the following bijective change of variables

$$x(t) = y(t) + \overline{x}(t), \tag{8}$$

where $\overline{x}(t)$ is a T-periodic solution of the system of equations (1) for $\varepsilon = 0$. Then the investigation of the problem of existence of a T-periodic solution of the system of equations (1) is reduced to the investigation of a T-periodic solution of the system of equations

$$\dot{y}(t) = Ay(t) + \varphi(t, y(t), y(\lambda_1 t), \ldots, y(\lambda_k t)) + \varepsilon\Phi(t, y(t), y(\lambda_1 t), \ldots, y(\lambda_k t), \varepsilon), \tag{9}$$

where

$$\varphi\,(t, y(t), y(\lambda_1 t), \ldots, y(\lambda_k t)) = f\,(t, y(t) + \overline{x}(t), y(\lambda_1 t) + \overline{x}(\lambda_1 t), \ldots, y(\lambda_k t) + \overline{x}(\lambda_k t))$$
$$- f\,(t, \overline{x}(t), \overline{x}(\lambda_1 t), \ldots, \overline{x}(\lambda_k t)),$$

$$\Phi\,(t, y(t), y(\lambda_1 t), \ldots, y(\lambda_k t), \varepsilon) = F\,(t, y(t) + \overline{x}(t), y(\lambda_1 t) + \overline{x}(\lambda_1 t), \ldots, y(\lambda_k t) + \overline{x}(\lambda_k t), \varepsilon).$$

In view of conditions (ii) and (iii) in Theorem 1, one can easily show that the vector function $\varphi(t, y(t), y(\lambda_1 t), \ldots, y(\lambda_k t))$ is continuous in all variables, T-periodic in t, $\varphi(t, 0, \ldots, 0) \equiv 0$, and satisfies the Lipschitz condition

$$|\varphi(t, \tilde{y}_0, \tilde{y}_1, \ldots, \tilde{y}_k) - \varphi(t, \tilde{\tilde{y}}_0, \tilde{\tilde{y}}_1, \ldots, \tilde{\tilde{y}}_k)| \le l_1 \sum_{i=0}^{k} |\tilde{y}_i - \tilde{\tilde{y}}_i|,$$

where $t \in \mathbb{R}$, $\tilde{y}_i, \tilde{\tilde{y}}_i \in \mathbb{R}^n$, $i = \overline{0, k}$, and $l_1 = \text{const} > 0$.

The vector function $\Phi\,(t, y(t), y(\lambda_1 t), \ldots, y(\lambda_k t), \varepsilon)$ is also continuous in all variables and T-periodic in t.

To study the problem of existence of T-periodic solutions of the system of equations (9), we perform the transformation

$$\dot{y}(t) = Ay(t) + z(t), \tag{10}$$

where $z(t) \in C^0$ and C^0 is a space of T-periodic vector functions continuous on $\mathbb{R}$ with the norm $\|z(t)\| = \max_t |z(t)|$. Thus, in view of (10), system (9) immediately implies that $y(t)$ is uniquely determined by the equality

$$y(t) = \int_{-\infty}^{+\infty} G(t - \tau) z(\tau) d\tau, \tag{11}$$

where $G(t)$ is given by relation (6). As a result of transformation (10), the system of equations (9) takes the form

$$z(t) = \varphi\left(t, \int_{-\infty}^{+\infty} G(t-\tau)z(\tau)d\tau, \int_{-\infty}^{+\infty} G(\lambda_1 t-\tau)z(\tau)d\tau, \ldots, \int_{-\infty}^{+\infty} G(\lambda_k t-\tau)z(\tau)d\tau\right)$$

$$+\varepsilon\Phi\left(t,\int_{-\infty}^{+\infty}G(t-\tau)z(\tau)d\tau,\int_{-\infty}^{+\infty}G(\lambda_1 t-\tau)z(\tau)d\tau,\ldots,\int_{-\infty}^{+\infty}G(\lambda_k t-\tau)z(\tau)d\tau,\varepsilon\right)$$

or

$$z(t)=\varphi\left(t,\int_{-\infty}^{+\infty}G(t-\tau)z(\tau)d\tau,\lambda_1\int_{-\infty}^{+\infty}G(\lambda_1(t-\tau))z(\lambda_1\tau)d\tau,\right.$$

$$\left.\ldots,\lambda_k\int_{-\infty}^{+\infty}G(\lambda_k(t-\tau))z(\lambda_k\tau)d\tau\right)$$

$$+\varepsilon\Phi\left(t,\int_{-\infty}^{+\infty}G(t-\tau)z(\tau)d\tau,\lambda_1\int_{-\infty}^{+\infty}G(\lambda_1(t-\tau))z(\lambda_1\tau)d\tau,\right.$$

$$\left.\ldots,\lambda_k\int_{-\infty}^{+\infty}G(\lambda_k(t-\tau))z(\lambda_k\tau)d\tau,\varepsilon\right), \tag{12}$$

where $G(t)$ is determined from relation (6).

The following theorem is true for the system of equations (12):

Theorem 2. *Suppose that the following conditions are satisfied:*

(i) *all eigenvalues* a_j, $j=\overline{1,n}$, *of the matrix* A *are such that conditions (2) are satisfied, i.e.,*

$$\exists K>0,\ \alpha>0:\ |G(t)|\le Ke^{-\alpha|t|}$$

for all $t\neq 0$;

(ii) *all components of the vector functions* $\varphi(t,y_0,y_1,\ldots,y_k)$ *and* $\Phi(t,y_0,y_1,\ldots,y_k,\varepsilon)$ *are functions continuous in all variables and* T *-periodic in* t *;*

(iii) $\varphi(t,0,\ldots,0)\equiv 0$;

(iv) $\sup_{t,\varepsilon}|\Phi(t,0,\ldots,0,\varepsilon)|\le\Delta$;

(v) $|\varphi(t,\tilde{y}_0,\tilde{y}_1,\ldots,\tilde{y}_k)-\varphi(t,\tilde{\tilde{y}}_0,\tilde{\tilde{y}}_1,\ldots,\tilde{\tilde{y}}_k)|\le l_1\sum_{i=0}^{k}|\tilde{y}_i-\tilde{\tilde{y}}_i|$,

$$|\Phi(t,\tilde{y}_0,\tilde{y}_1,\ldots,\tilde{y}_k,\varepsilon)-\Phi(t,\tilde{\tilde{y}}_0,\tilde{\tilde{y}}_1,\ldots,\tilde{\tilde{y}}_k,\varepsilon)|\le l_2\sum_{i=0}^{k}|\tilde{y}_i-\tilde{\tilde{y}}_i|,$$

where $t\in\mathbb{R}$, $\tilde{y}_i,\tilde{\tilde{y}}_i\in\mathbb{R}^n$, $i=\overline{0,k}$, $l_1=\text{const}>0$, *and* $l_2=\text{const}>0$;

(vi) $\dfrac{2K(k+1)(l_1+\varepsilon l_2)}{\alpha} < 1.$

Then there exists a unique continuous T-periodic solution $\widetilde{\gamma} = \widetilde{\gamma}(t,\varepsilon)$ of the system of equations (12) such that $\lim_{\varepsilon\to 0} \widetilde{\gamma}(t,\varepsilon) = 0.$

Proof. We construct a solution of the system of equations (12) by the method of successive approximations given by the formulas

$$z_0(t,\varepsilon) \equiv 0,$$

$$z_m(t,\varepsilon) = \varphi\left(t, \int_{-\infty}^{+\infty} G(t-\tau)z_{m-1}(\tau,\varepsilon)d\tau, \lambda_1 \int_{-\infty}^{+\infty} G(\lambda_1(t-\tau))z_{m-1}(\lambda_1\tau,\varepsilon)d\tau,\right.$$

$$\left. \ldots, \lambda_k \int_{-\infty}^{+\infty} G(\lambda_k(t-\tau))z_{m-1}(\lambda_k\tau,\varepsilon)d\tau\right)$$

$$+\,\varepsilon\Phi\left(t, \int_{-\infty}^{+\infty} G(t-\tau)z_{m-1}(\tau,\varepsilon)d\tau, \lambda_1 \int_{-\infty}^{+\infty} G(\lambda_1(t-\tau))z_{m-1}(\lambda_1\tau,\varepsilon)d\tau,\right.$$

$$\left. \ldots, \lambda_k \int_{-\infty}^{+\infty} G(\lambda_k(t-\tau))z_{m-1}(\lambda_k\tau,\varepsilon)d\tau, \varepsilon\right), \quad m = 1,2,\ldots. \tag{13}$$

We first show that, for all $m = 1,2,\ldots,$ $t \in \mathbb{R},$ the following relations are true:

$$|z_m(t,\varepsilon) - z_{m-1}(t,\varepsilon)| \le \varepsilon\Delta\theta^{m-1}, \tag{14}$$

where

$$\theta := \frac{2K(k+1)(l_1+\varepsilon l_2)}{\alpha}.$$

Indeed under the conditions of the theorem, we get

$$|z_1(t,\varepsilon) - z_0(t,\varepsilon)| \le |\varphi(t,0,\ldots,0) + \varepsilon\Phi(t,0,\ldots,0,\varepsilon)| \le \varepsilon|\Phi(t,0,\ldots,0,\varepsilon)| \le \varepsilon\Delta,$$

i.e., for $m = 1,$ estimate (14) is true. Reasoning by induction, we assume that estimate (14) is proved for some $m \ge 1$ and show that it remains true if we pass from m to $m+1.$ Indeed, under the conditions of the theorem, it follows from relations (13) that

$$|z_{m+1}(t,\varepsilon) - z_m(t,\varepsilon)| \le (l_1+\varepsilon l_2)\left(\int_{-\infty}^{+\infty} |G(t-\tau)||z_m(\tau,\varepsilon) - z_{m-1}(\tau,\varepsilon)|d\tau\right.$$

$$+ \sum_{i=1}^{k} \lambda_i \int_{-\infty}^{+\infty} |G(\lambda_i(t-\tau))||z_m(\lambda_i\tau,\varepsilon) - z_{m-1}(\lambda_i\tau,\varepsilon)|d\tau \Bigg)$$

$$\leq (l_1 + \varepsilon l_2) \Bigg(\int_{-\infty}^{+\infty} K e^{-\alpha|t-\tau|} \varepsilon \Delta \theta^{m-1} d\tau$$

$$+ \sum_{i=1}^{k} \lambda_i \int_{-\infty}^{+\infty} K e^{-\alpha\lambda_i|t-\tau|} \varepsilon \Delta \theta^{m-1} d\tau \Bigg)$$

$$\leq \varepsilon \Delta \theta^{m-1} (l_1 + \varepsilon l_2) K \left(\int_{-\infty}^{+\infty} e^{-\alpha|t-\tau|} d\tau + \sum_{i=1}^{k} \lambda_i \int_{-\infty}^{+\infty} e^{-\alpha\lambda_i|t-\tau|} d\tau \right)$$

$$\leq \varepsilon \Delta \theta^{m-1} (l_1 + \varepsilon l_2) K \left(\frac{2}{\alpha} + \sum_{i=1}^{k} \lambda_i \frac{2}{\alpha\lambda_i} \right)$$

$$= \varepsilon \Delta \theta^{m-1} (l_1 + \varepsilon l_2) K \frac{2(k+1)}{\alpha} = \varepsilon \Delta \theta^m.$$

This proves that estimate (14) is true for any $m \geq 1$.

By induction, we show that the approximations $z_m(t,\varepsilon)$, $m = 0, 1, 2, \ldots,$ are T-periodic vector functions of t, i.e.,

$$z_m(t+T,\varepsilon) = z_m(t,\varepsilon), \quad m = 0, 1, \ldots. \tag{15}$$

Indeed, according to the conditions of the theorem, we find

$$z_0(t+T,\varepsilon) = z_0(t,\varepsilon) \equiv 0,$$

i.e., relation (15) is true for some $m = 0$. We now assume that relation (15) holds for some $m \geq 0$ and show that it remains true if we pass from m to $m+1$. Thus, it follows from (13) that

$$z_{m+1}(t+T,\varepsilon) = \varphi \Bigg(t+T, \int_{-\infty}^{+\infty} G(t+T-\tau) z_m(\tau,\varepsilon) d\tau, \lambda_1 \int_{-\infty}^{+\infty} G(\lambda_1(t+T-\tau)) z_m(\lambda_1\tau,\varepsilon) d\tau,$$

$$\ldots, \lambda_k \int_{-\infty}^{+\infty} G(\lambda_k(t+T-\tau)) z_m(\lambda_k\tau,\varepsilon) d\tau \Bigg)$$

$$+ \varepsilon \Phi \Bigg(t+T, \int_{-\infty}^{+\infty} G(t+T-\tau) z_m(\tau,\varepsilon) d\tau, \lambda_1 \int_{-\infty}^{+\infty} G(\lambda_1(t+T-\tau)) z_m(\lambda_1\tau,\varepsilon) d\tau,$$

$$\left. \ldots, \lambda_k \int_{-\infty}^{+\infty} G(\lambda_k(t+T-\tau)) z_m(\lambda_k \tau, \varepsilon) d\tau, \varepsilon \right)$$

$$= \varphi\left(t, \int_{-\infty}^{+\infty} G(t-s) z_m(s,\varepsilon) ds, \lambda_1 \int_{-\infty}^{+\infty} G(\lambda_1(t-s)) z_m(\lambda_1 s, \varepsilon) ds, \right.$$

$$\left. \ldots, \lambda_k \int_{-\infty}^{+\infty} G(\lambda_k(t-s)) z_m(\lambda_k s, \varepsilon) ds \right)$$

$$+ \varepsilon \Phi\left(t, \int_{-\infty}^{+\infty} G(t-s) z_m(s,\varepsilon) ds, \lambda_1 \int_{-\infty}^{+\infty} G(\lambda_1(t-s)) z_m(\lambda_1 s, \varepsilon) ds, \right.$$

$$\left. \ldots, \lambda_k \int_{-\infty}^{+\infty} G(\lambda_k(t-s)) z_m(\lambda_k s, \varepsilon) ds, \varepsilon \right) = z_{m+1}(t,\varepsilon).$$

Hence, relations (15) are true for all $m \geq 0$.

Thus, all approximations $z_m(t,\varepsilon)$, $m = 0, 1, 2, \ldots,$ are meaningful. They are continuous T-periodic vector functions satisfying estimates (14). In view of condition (vi) of the theorem and estimates (14), we conclude that the series

$$\sum_{m=1}^{+\infty} (z_m(t,\varepsilon) - z_{m-1}(t,\varepsilon))$$

uniformly converges for any $t \in \mathbb{R}$ to a certain continuous T-periodic vector function $\widetilde{\gamma} = \widetilde{\gamma}(t,\varepsilon)$ which is a solution of the system of equations (12) [this can be easily proved by passing to the limit as $m \to +\infty$ in (13)]. Moreover,

$$|\widetilde{\gamma}(t,\varepsilon)| \leq \sum_{m=1}^{+\infty} |z_m(t,\varepsilon) - z_{m-1}(t,\varepsilon)| \leq \sum_{m=1}^{+\infty} \varepsilon \Delta \theta^{m-1} = \varepsilon \Delta \frac{1}{1-\theta},$$

and, hence,

$$\lim_{\varepsilon \to 0} \widetilde{\gamma}(t,\varepsilon) = 0.$$

We now show that system (12) does not have any other continuous T-periodic solutions. Indeed, assume that there exists one more continuous T-periodic solution $\eta(t,\varepsilon)$ of the system of equations (12) such that $\widetilde{\gamma}(t,\varepsilon) \neq \eta(t,\varepsilon)$. Then we find

$$|\widetilde{\gamma}(t,\varepsilon) - \eta(t,\varepsilon)| \leq (l_1 + \varepsilon l_2) \left(\int_{-\infty}^{+\infty} |G(t-\tau)| |\widetilde{\gamma}(\tau,\varepsilon) - \eta(\tau,\varepsilon)| d\tau \right.$$

$$+\sum_{i=1}^{k}\lambda_i\int_{-\infty}^{+\infty}|G(\lambda_i(t-\tau))||\widetilde{\gamma}(\lambda_i\tau,\varepsilon)-\eta(\lambda_i\tau,\varepsilon)|d\tau\Bigg)$$

$$\le(l_1+\varepsilon l_2)\Bigg(\int_{-\infty}^{+\infty}Ke^{-\alpha|t-\tau|}|\widetilde{\gamma}(\tau,\varepsilon)-\eta(\tau,\varepsilon)|d\tau$$

$$+\sum_{i=1}^{k}\lambda_i\int_{-\infty}^{+\infty}Ke^{-\alpha\lambda_i|t-\tau|}|\widetilde{\gamma}(\lambda_i\tau,\varepsilon)-\eta(\lambda_i\tau,\varepsilon)|d\tau\Bigg)$$

$$\le(l_1+\varepsilon l_2)\frac{2K(k+1)}{\alpha}\max_t|\widetilde{\gamma}(t,\varepsilon)-\eta(t,\varepsilon)|.$$

Hence,

$$\|\widetilde{\gamma}(t,\varepsilon)-\eta(t,\varepsilon)\|\le\theta\|\widetilde{\gamma}(t,\varepsilon)-\eta(t,\varepsilon)\|.$$

The obtained relation holds only for $\theta\ge 1,$ which contradicts the condition (vi) of the theorem. Thus, it is proved that the vector function $\widetilde{\gamma}(t,\varepsilon)$ is a unique continuous T-periodic solution of the system of equations (12).

The theorem is proved.

By using Theorem 2 and relation (11), we conclude that the vector function

$$\widetilde{y}(t,\varepsilon)=\int_{-\infty}^{+\infty}G(t-\tau)\,\widetilde{\gamma}(\tau,\varepsilon)d\tau$$

is a unique continuous T-periodic solution of the system of equations (1) for $\varepsilon\ne 0$ such that

$$\lim_{\varepsilon\to 0}\widetilde{y}(t,\varepsilon)=0.$$

Thus, by virtue of relation (8) and Theorems 1 and 2, we conclude that the system of equations (1) possesses a unique T-periodic solution

$$\widehat{x}(t,\varepsilon)=\widetilde{y}(t,\varepsilon)+\bar{x}(t)$$

such that

$$\lim_{\varepsilon\to 0}\widehat{x}(t,\varepsilon)=\bar{x}(t),$$

where $\bar{x}(t)$ is a T-periodic solution of the system of equations (1) for $\varepsilon=0$ and $\widetilde{y}(t,\varepsilon)$ is a T-periodic solution of the system of equations (1) for $\varepsilon\ne 0.$

REFERENCES

1. T. Kato and J. B. McLeod, "The functional-differential equation $y'(x) = ay(\lambda x) + by(x)$," *Bull. Amer. Math. Soc.,* **77**, 891–937 (1971).
2. M. Kwapisz, "On the existence and uniqueness of solutions of certain integral-differential equation," *Ann. Pol. Math.,* **31**, No. 1, 23–41 (1975).
3. A. M. Samoilenko and G. P. Pelyukh, "Solutions of systems of nonlinear functional-differential equations bounded in the entire real axis and their properties," *Ukr. Mat. Zh.,* **46**, No. 6, 737–747 (1994); ***English translation:*** *Ukr. Math. J.,* **46**, No. 6, 799–811 (1994).
4. N. L. Denysenko, "Periodic solutions of systems of functional differential equations with linear deviation of argument and their properties," *Nelin. Kolyvannya,* **12**, No. 2, 168–179 (2009); ***English translation:*** *Nonlin. Oscillations,* **12**, No. 2, 171–183 (2009).
5. N. L. Denysenko, *Investigation of the Properties of Solutions of Systems of Functional-Differential Equations with Linearly Transformed Argument* [in Ukrainian], Author's Abstract of the Candidate-Degree Thesis (Physics and Mathematics), Kyiv (2009).
6. Yu. A. Mitropol'skii, A. M. Samoilenko, and D. I. Martynyuk, *Systems of Evolutionary Equations with Periodic and Conditionally Periodic Coefficients* [in Russian], Naukova Dumka, Kiev (1985).
7. J. Hale, *Theory of Functional Differential Equations,* Springer, New York (1977).