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ON CONVERGENCE OF THE ACCELERATED NEWTON METHOD UNDER GENERALIZED LIPSCHITZ CONDITIONS

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We study the problem of local convergence of the accelerated Newton method for the solution of nonlinear functional equations under generalized Lipschitz conditions for the first- and second-order Fréchet derivatives. We show that the accelerated method is characterized by the quadratic order of convergence and compare it with the classical Newton method.

Introduction

Consider a nonlinear functional equation

$$F(x) = 0\,, \tag{1}$$

where $F(x)$ is a nonlinear operator acting from a Banach space X into a Banach space Y. The classical Newton method [3, 4] is the most popular and commonly used method for the solution of Eq. (1). A new method obtained as a modification of the Newton method was proposed in [1]. In the same paper, its semilocal convergence (under Kantorovich-type conditions) was investigated, and the class of problems for which this new method converges faster than the Newton method was outlined.

Assume that the following representation is true for a nonlinear operator φ in the Banach space X:

$$F(x) \equiv x - \varphi(x) = 0\,. \tag{2}$$

The iterative process for the solution of Eq. (2) has the form

$$x_{k+1} = x_k - \left[F'\left(\frac{x_k + \varphi(x_k)}{2}\right)\right]^{-1} F(x_k), \qquad k = 0,1,2,\ldots\,, \tag{3}$$

where x_0 is the initial value. Method (3) is a special case of the one-parameter class of methods

$$x_{k+1} = x_k - [F'((1-\mu)x_k + \mu\varphi(x_k))]^{-1} F(x_k), \qquad k = 0,1,2,\ldots, \tag{4}$$

studied in [2]. As shown in [2], method (4) for the value of the parameter $\mu = 0.5$, i.e., method (3), is especially efficient. Method (3) was studied fairly comprehensively in [7] under the classical conditions imposed on the

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first- and second-order derivatives. Method (4) was also investigated in [11] but the results were obtained under different (severer) conditions.

In the investigations of the Newton method, it is proposed [10] to use generalized Lipschitz conditions for the operator of derivative in which a positive integrable function is used instead of a constant L. In [9], we propose to use similar generalized Lipschitz conditions for the operator of divided difference of the first order and, under these conditions, study the convergence of the secant method. In [6], we study the convergence of the Steffensen method for the operator equations under generalized Lipschitz conditions imposed on the first divided differences of the nonlinear operator $F(x)$. The analysis of inexact methods was carried out in [5] under generalized Lipschitz conditions for the first-order divided differences. The problem of convergence of the inexact Newton method under a different generalized Lipschitz condition for the first-order derivative was investigated in [8].

In the present work, we consider method (3) under generalized Lipschitz conditions for the first- and second-order derivatives.

Definition and Auxiliary Lemmas

We denote by $B(x_0,r)=\{x:\|x-x_0\|<r\}$ an open ball and by $\overline{B(x_0,r)}=\{x:\|x-x_0\|\le r\}$ a closed ball of radius r centered at a point x_0.

The condition

$$\|F(x)-F(x^\tau)\| \le L\|x-x^\tau\| \qquad \forall\ x,x^\tau\in D$$

imposed on the operator F is called a Lipschitz condition in the domain D with constant L. If the domain D is a ball $B(x_0,r)$ of radius r centered at x_0 and $x\in B(x_0,r)$, then the segment $x^\tau=x_0+\tau(x-x_0)$, $0\le\tau\le1$, connects points x and x_0 of the ball $B(x_0,r)$. In this case, the condition

$$\|F(x)-F(x^\tau)\| \le L\|x-x^\tau\| \qquad \forall\ x\in B(x_0,r), \qquad 0\le\tau\le1, \tag{5}$$

is called the radius Lipschitz condition in the ball $B(x_0,r)$ with constant L.

At the same time, the condition

$$\|F(x)-F(x_0)\| \le L\|x-x_0\| \qquad \forall\ x\in B(x_0,r) \tag{6}$$

is called the center Lipschitz condition in a ball $B(x_0,r)$ with constant L.

However, the parameter L in the Lipschitz conditions is not necessarily constant but can be a positive integrable function. In this case, we replace (5) and (6) with

$$\|F(x)-F(x^\tau)\| \le \int_{\tau\rho(x)}^{\rho(x)} L(u)\,du \qquad \forall\ x\in B(x_0,r), \qquad 0\le\tau\le1, \tag{7}$$

and

$$\|F(x)-F(x_0)\| \le \int_0^{\rho(x)} L(u)du \qquad \forall\ x \in B(x_0,r)\,, \tag{8}$$

respectively. The Lipschitz conditions (7) and (8) are called the generalized Lipschitz conditions or the conditions with average L.

By using the Banach theorem [3], we arrive at the following result:

Lemma 1. *Suppose that F has a continuous derivative in the ball $B(x^*,r)$, there exists $F'(x^*)^{-1}$, and the derivative F' satisfies the center Lipschitz condition with average L:*

$$\left\|F'(x^*)^{-1}\left(F'\left(\frac{x+y}{2}\right)-F'(x^*)\right)\right\| \le \int_0^{\rho\left(\frac{x+y}{2}\right)} L(u)du \qquad \forall\ x,y \in B(x^*,r)\,, \tag{9}$$

where L is a positive integrable function and $\rho(x)=\|x-x^*\|$. Let r satisfy the condition

$$\int_0^r L(u)du \le 1\,. \tag{10}$$

Then $F'(x)$ is invertible in the ball $B(x^*,r)$ and, moreover,

$$\left\|F'\left(\frac{x+y}{2}\right)^{-1}F'(x^*)\right\| \le \left(1-\int_0^{\rho\left(\frac{x+y}{2}\right)} L(u)du\right)^{-1}.$$

Proof. Indeed, in view of the identity

$$F'\left(\frac{x+y}{2}\right)^{-1}F'(x^*) = \left[I-\left(I-F'(x^*)^{-1}F'\left(\frac{x+y}{2}\right)\right)\right]^{-1}$$

and inequalities (9) and (10), it follows from the Banach theorem that

$$\left\|F'\left(\frac{x+y}{2}\right)^{-1}F'(x^*)\right\| \le \left(1-\int_0^{\rho\left(\frac{x+y}{2}\right)} L(u)du\right)^{-1}.$$

Lemma 2 [5]. *Let*

$$h(t)=\frac{1}{t}\int_0^t L(u)du\,, \qquad 0 \le t \le r\,,$$

where $L(u)$ is a positive integrable function monotonically nondecreasing on $[0,r]$. Then $h(t)$ is a monotonically nondecreasing function of t.

Lemma 3. *Let*

$$g(t) = \frac{1}{t^3}\int_0^t N(u)(t-u)^2\,du, \qquad 0 \le t \le r,$$

where $N(u)$ is a positive integrable function monotonically nondecreasing on $[0,r]$. Then $g(t)$ is a monotonically nondecreasing function of t.

Proof. Indeed, it follows from the monotonicity of N for $0 < t_1 < t_2$ that

$$g(t_2) - g(t_1) = \frac{1}{t_2^3}\int_0^{t_2} N(u)(t_2-u)^2\,du - \frac{1}{t_1^3}\int_0^{t_1} N(u)(t_1-u)^2\,du$$

$$= \frac{1}{t_2^3}\int_{t_1}^{t_2} N(u)(t_2-u)^2\,du + \frac{1}{t_2^3}\int_0^{t_1} N(u)(t_1-u)^2\,du - \frac{1}{t_1^3}\int_0^{t_1} N(u)(t_1-u)^2\,du$$

$$= \frac{1}{t_2^3}\int_{t_1}^{t_2} N(u)(t_2-u)^2\,du + \left(\frac{1}{t_2^3} - \frac{1}{t_1^3}\right)\int_0^{t_1} N(u)(t_1-u)^2\,du$$

$$\ge N(t_1)\left[\frac{1}{t_2^3}\int_{t_1}^{t_2} (t_2-u)^2\,du + \left(\frac{1}{t_2^3} - \frac{1}{t_1^3}\right)\int_0^{t_1} (t_1-u)^2\,du\right]$$

$$= N(t_1)\left[\frac{1}{t_2^3}\int_0^{t_2} (t_2-u)^2\,du - \frac{1}{t_1^3}\int_0^{t_1} (t_1-u)^2\,du\right] = 0\,.$$

Hence,

$$g(t) = \frac{1}{t^3}\int_0^t N(u)(t-u)^2\,du, \qquad 0 \le t \le r,$$

is a monotonically nondecreasing function of t.

Convergence of the Accelerated Newton Method

We now study the local convergence of method (3). The radius of the domain of convergence and the order of convergence of this method are established by the following theorem:

Theorem 1. *Let F be a nonlinear operator defined in an open convex domain D of the space X with values in this space. Assume that*

(i) $F(x)=0$ *has a solution* $x^* \in D$; *at this point* x^*, *there exists a Fréchet derivative* $F'(x^*)$, *and it is invertible;*

(ii) there exist Fréchet derivatives F' *and* F'' *in* $B(x^*, Mr)$ *satisfy the Lipschitz conditions with average* L *and* M:

$$\left\|F'(x^*)^{-1}(F'(x)-F'(x^*))\right\| \leq \int_0^{\rho(x)} L(u)\,du\,, \tag{11}$$

$$\left\|F'(x^*)^{-1}(F''(x)-F''(y))\right\| \leq \int_0^{\|x-y\|} N(u)\,du\,, \tag{12}$$

where $x,y \in B(x^*,r)$, $\rho(x)=\left\|x-x^*\right\|$, *and* L *and* N *are positive nondecreasing functions ;*

(iii) $\|\varphi'(x)\| \leq \alpha,\ M=\max\{1,\alpha\}\ \ \forall\ x \in B(x^*,r)$;

(iv) $r>0$ *satisfies the inequality*

$$\frac{\frac{1}{8}\int_0^r N(u)(r-u)^2\,du + r\int_0^{\alpha r/2} L(u)\,du}{r\left(1-\int_0^{(1+\alpha)r/2} L(u)\,du\right)} \leq 1\,. \tag{13}$$

Then the accelerated Newton method converges for all $x_0 \in B(x^*,r)$, *and*

$$\rho(x_{n+1}) = \left\|x_{n+1}-x^*\right\| \leq (A\rho(x_n)+C\alpha)\rho(x_n)^2 \leq (A\rho(x_0)+C\alpha)\rho(x_n)^2$$

$$\leq \frac{q}{\rho(x_0)}\rho(x_n)^2 \leq \ldots \leq q^{2^{n+1}-1}\rho(x_0), \qquad n=0,1,2,\ldots, \tag{14}$$

where

$$A = \frac{\int_0^{\rho(x_0)} N(u)(\rho(x_0)-u)^2\,du}{8\left(1-\int_0^{(1+\alpha)\rho(x_0)/2} L(u)\,du\right)\rho(x_0)^3}\,,$$

$$C = \frac{\int_0^{\alpha\rho(x_0)/2} L(u)\,du}{\left(1 - \int_0^{(1+\alpha)\rho(x_0)/2} L(u)\,du\right)\alpha\rho(x_0)},$$

$$q = A\rho(x_0)^2 + C\alpha\rho(x_0) < 1. \tag{15}$$

Proof. We choose arbitrary $x_0 \in B(x^*, r)$, where r satisfies inequality (13). Then q defined by (15) is smaller than 1. Indeed, since L and N are monotone, it follows from Lemmas 2 and 3 that

$$\frac{1}{t}\int_0^t L(u)\,du \qquad \text{and} \qquad \frac{1}{t^3}\int_0^t N(u)(t-u)^2\,du$$

are nondecreasing as functions of t. Hence,

$$q = \frac{\int_0^{\rho(x_0)} N(u)(\rho(x_0)-u)^2\,du\,\rho(x_0)^2}{8\rho(x_0)^3\left(1 - \int_0^{(1+\alpha)\rho(x_0)/2} L(u)\,du\right)} + \frac{\int_0^{\alpha\rho(x_0)/2} L(u)\,du\,\frac{\alpha\rho(x_0)}{2}}{\frac{\alpha\rho(x_0)}{2}\left(1 - \int_0^{(1+\alpha)\rho(x_0)/2} L(u)\,du\right)}$$

$$\le \frac{\int_0^{r} N(u)(r-u)^2\,du\,\rho(x_0)^2}{8r^3\left(1 - \int_0^{(1+\alpha)r/2} L(u)\,du\right)} + \frac{\int_0^{\alpha r/2} L(u)\,du\,\frac{\alpha\rho(x_0)}{2}}{\frac{\alpha r}{2}\left(1 - \int_0^{(1+\alpha)r/2} L(u)\,du\right)}$$

$$\le \frac{1}{r}\left(\frac{\int_0^{r} N(u)(r-u)^2\,du}{8r\left(1 - \int_0^{(1+\alpha)r/2} L(u)\,du\right)} + \frac{\int_0^{\alpha r/2} L(u)\,du}{\left(1 - \int_0^{(1+\alpha)r/2} L(u)\,du\right)}\right)\rho(x_0)$$

$$\le \frac{\rho(x_0)}{r} < 1. \tag{16}$$

If $x_k \in B(x^*, r)$, then, according to (5), we find

$$x_{k+1} - x^* = x_k - x^* - F'\left(\frac{x_k + \varphi(x_k)}{2}\right)^{-1} F(x_k)$$

$$= F'\left(\frac{x_k + \varphi(x_k)}{2}\right)^{-1}\left[F'\left(\frac{x_k + \varphi(x_k)}{2}\right)(x_k - x^*) - F(x_k) + F(x^*)\right]$$

$$= F'\left(\frac{x_k+\varphi(x_k)}{2}\right)^{-1} F'(x^*)\left(F'(x^*)^{-1}\left[F'\left(\frac{x_k+\varphi(x_k)}{2}\right)(x_k-x^*)-F(x_k)+F(x^*)\right]\right)$$

$$= F'\left(\frac{x_k+\varphi(x_k)}{2}\right)^{-1} F'(x^*)\left(F'(x^*)^{-1}\left[F'\left(\frac{x_k+x^*}{2}\right)(x_k-x^*)-F(x_k)+F(x^*)\right]\right.$$

$$\left.+F'(x^*)^{-1}\left[F'\left(\frac{x_k+\varphi(x_k)}{2}\right)-F'\left(\frac{x_k+x^*}{2}\right)\right](x_k-x^*)\right).$$

We now write the identity from Lemma 1 in [12, p. 336] for $\omega = \frac{1}{2}$:

$$F(x)-F(y)-F'\left(\frac{x+y}{2}\right)(x-y)$$

$$= \frac{1}{4}\int_0^1 (1-t)\left[F''\left(\frac{x+y}{2}+\frac{t}{2}(x-y)\right)-F''\left(\frac{x+y}{2}+\frac{t}{2}(y-x)\right)\right](x-y)(x-y)\,dt\,.$$

Taking $x = x^*$, $y = x$ in this equality, we obtain

$$\left\|F'(x^*)^{-1}\left[F(x^*)-F(x_k)-F'\left(\frac{x_k+x^*}{2}\right)(x^*-x_k)\right]\right\|$$

$$= \frac{1}{4}\left\|\int_0^1 (1-t)F'(x^*)^{-1}\left[F''\left(\frac{x_k+x^*}{2}+\frac{t}{2}(x^*-x_k)\right)\right.\right.$$

$$\left.\left.-F''\left(\frac{x_k+x^*}{2}+\frac{t}{2}(x_k-x^*)\right)(x^*-x_k)(x^*-x_k)\right]dt\right\|$$

$$\le \frac{1}{4}\int_0^1 (1-t)\int_0^{t\|x_k-x^*\|} N(u)\,du\,\|x_k-x^*\|^2\,dt$$

$$= \frac{1}{8}\int_0^{\|x_k-x^*\|}\left(1-\frac{u}{\|x_k-x^*\|}\right)^2 N(u)\,du\,\|x_k-x^*\|^2$$

$$= \frac{1}{8}\int_0^{\rho(x_k)} N(u)(\rho(x_k)-u)^2\,du\,.$$

Thus, according to Lemmas 1–3 and conditions (11) and (12), in view of the last inequality, we get

$$\begin{aligned}
\|x_{k+1}-x^*\| &\le \left\| F'\left(\frac{x_k+\varphi(x_k)}{2}\right)^{-1} F'(x^*)\right\| \\
&\quad \times \left(\left\| F'(x^*)^{-1}\left[F'\left(\frac{x_k+x^*}{2}\right)(x_k-x^*) - F(x_k) + F(x^*)\right]\right\| \right. \\
&\quad \left. + \left\| F'(x^*)^{-1}\left[F'\left(\frac{x_k+\varphi(x_k)}{2}\right) - F'\left(\frac{x_k+x^*}{2}\right)\right](x_k-x^*)\right\| \right) \\
&\le \left\| F'\left(\frac{x_k+\varphi(x_k)}{2}\right)^{-1} F'(x^*)\right\| \\
&\quad \times \left(\frac{1}{4}\int_0^1 (1-t)\left\| F'(x^*)^{-1}\left[F''\left(\frac{x_k+x^*}{2}+\frac{t}{2}(x_k-x^*)\right)\right.\right.\right. \\
&\quad \left.\left. - F''\left(\frac{x_k+x^*}{2}+\frac{t}{2}(x^*-x_k)\right)\right](x_k-x^*)(x_k-x^*)\right\| dt \\
&\quad \left. + \left\| F'(x^*)^{-1}\left[F'\left(\frac{x_k+\varphi(x_k)}{2}\right) - F'\left(\frac{x_k+x^*}{2}\right)\right](x_k-x^*)\right\| \right) \\
&\le \frac{\int_0^{\rho(x_k)} N(u)(\rho(x_k)-u)^2\,du\,\rho(x_k)^3}{8\left(1-\int_0^{\rho\left(\frac{x_k+\varphi(x_k)}{2}\right)} L(u)\,du\right)\rho(x_k)^3} + \frac{\int_0^{\rho(\varphi(x_k))/2} L(u)\,du\,\rho(x_k)\frac{\rho(\varphi(x_k))}{2}}{\left(1-\int_0^{\rho\left(\frac{x_k+\varphi(x_k)}{2}\right)} L(u)\,du\right)\frac{\rho(\varphi(x_k))}{2}} \\
&\le \frac{\int_0^{\rho(x_0)} N(u)(\rho(x_0)-u)^2\,du\,\rho(x_k)^3}{8\left(1-\int_0^{(1+\alpha)\rho(x_0)/2} L(u)\,du\right)\rho(x_0)^3} + \frac{\int_0^{\alpha\rho(x_0)/2} L(u)\,du\,\rho(x_k)\frac{\alpha\rho(x_k)}{2}}{\left(1-\int_0^{(1+\alpha)\rho(x_0)/2} L(u)\,du\right)\frac{\alpha\rho(x_0)}{2}} \\
&\le (A\rho(x_k)+C\alpha)\rho(x_k)^2 \le (A\rho(x_0)+C\alpha)\rho(x_k)^2 \le \frac{q}{\rho(x_0)}\rho(x_k)^2\,. \qquad (17)
\end{aligned}$$

We also write the following estimate:

$$\|\varphi(x_{k+1}) - x^*\| = \|\varphi(x_{k+1}) - \varphi(x^*)\|$$

$$\le \|\varphi'(x_{k+1} + \theta(x_{k+1} - x^*))\| \|x_{k+1} - x^*\|$$

$$\le \alpha \|x_{k+1} - x^*\|, \qquad 0 \le \theta \le 1 . \tag{18}$$

Setting $k = 0$ in estimates (17) and (18), we obtain

$$\|x_1 - x^*\| \le q\|x_0 - x^*\| < \|x_0 - x^*\|,$$

$$\|\varphi(x_1) - x^*\| \le \alpha\|x_1 - x^*\| < M\|x_1 - x^*\| \le M\|x_0 - x^*\| .$$

Hence, x_1 and $\varphi(x_1)$ belong to the ball $B(x^*, \alpha r)$. This means that we can repeat (17) infinitely many times. Therefore, by induction, all x_k, $\varphi(x_k) \in B(x^*, \alpha r)$, whereas

$$\rho(x_k) = \|x_k - x^*\| \qquad \text{and} \qquad \rho(\varphi(x_k)) = \|\varphi(x_k) - x^*\|$$

are monotonically decreasing.

Further, for all $k = 0, 1, \ldots$, we have

$$\|x_{k+1} - x^*\| \le \frac{q}{\rho(x_0)} \rho(x_k)^2 = \frac{q}{\rho(x_0)} \|x_k - x^*\|^2 \le \ldots \le q^{2^{k+1} - 1} \|x_0 - x^*\| .$$

Thus, estimate (14) is proved.

The domain of uniqueness of the solution can be found by analogy with [10].

In studying the Newton method, it is usually assumed that derivatives satisfy the Lipschitz conditions. If we suppose that L and N are constant, then we obtain the following corollary of Theorem 1:

Corollary 1. *Assume that* $F(x^*) = 0$ *and* F *has the first- and second-order continuous derivatives in* $B(x^*, Mr)$. *Suppose that* $F'(x^*)^{-1}$ *exists and that* $F'(x^*)^{-1} F'(x)$ *and* $F'(x^*)^{-1} F''(x)$ *satisfy the Lipschitz conditions*

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le L\|x - x^*\|,$$

$$\left\| F'(x^*)^{-1}(F''(x) - F''(y)) \right\| \le N\|x - y\|,$$

where $x, y \in B(x^*, Mr)$, $\|\varphi'(x)\| \le \alpha$, and $M = \max\{1, \alpha\}$ $\forall x \in B(x^*, r)$.

Let $r > 0$ *satisfy the equation*

$$\frac{Nr^2}{12} + L(1 + 2\alpha)r - 2 = 0 .$$

Then the accelerated Newton method (3) converges for all $x_0 \in B(x^*, r)$ *and inequality (14) is true for*

$$q = \frac{L\alpha + \frac{N}{12}\|x_0 - x^*\|}{2 - L(1+\alpha)\|x_0 - x^*\|}\|x_0 - x^*\|.$$

Comparing the obtained value of q with the quantity

$$q_N = \frac{L\|x_0 - x^*\|}{2\left(1 - L\|x_0 - x^*\|\right)}$$

introduced for the Newton method in [10], we conclude that q (for $\alpha < 1$ and a sufficiently close initial approximation) is lower than q_N and, hence, the order of convergence of the accelerated Newton method is greater in this case.

CONCLUSIONS

In [9], we have studied the local convergence of the Newton method in the case where the generalized Lipschitz conditions (in which a certain positive integrable function is used instead of the Lipschitz constant) are satisfied for the first-order derivatives. In the present work, we study the local convergence of the accelerated version of this method under generalized Lipschitz conditions for the first- and second-order derivatives. It is shown that, under certain conditions, the proposed method converges faster than the Newton method.

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