DOI 10.1007/s10958-015-2632-x
Journal of Mathematical Sciences, Vol. 211, No. 6, December, 2015

THREE-DIMENSIONAL VORTEX FLOWS OF TWO–VELOCITY INCOMPRESSIBLE MEDIA IN THE CASE OF CONSTANT VOLUME SATURATION

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UDC 532.5

Using scalar functions, we describe flows of incompressible viscous two-velocity fluids in the case of the pressure balance and substances of constant volume saturation. We derive a system of differential equations for these scalar functions an illustrate the method by an example. Bibliography: 5 *titles. Illustrations*: 1 *figure.*

1 Equations of Two-Velocity Hydrodynamics with One Pressure

In this paper, we use the method of [1]. A nonlinear two-velocity model of the fluid flow through a deformable porous medium was constructed in [2, 3]. The two-velocity two-fluid hydrodynamic theory with the pressure balance was developed in [4]. In the isothermic case, the equations of motion of a two-velocity medium in a dissipative system with one pressure has the form (cf. [4]):

$$\frac{\partial\overline{\rho}}{\partial t}+\operatorname{div}(\widetilde{\rho}\widetilde{\mathbf{v}}+\rho\mathbf{v})=0,\quad \frac{\partial\widetilde{\rho}}{\partial t}+\operatorname{div}(\widetilde{\rho}\widetilde{\mathbf{v}})=0, \tag{1}$$

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v},\nabla)\mathbf{v}=-\frac{\nabla p}{\overline{\rho}}+\nu\Delta\mathbf{v}+\frac{\widetilde{\rho}}{2\overline{\rho}}\nabla(\widetilde{\mathbf{v}}-\mathbf{v})^2+\mathbf{f}, \tag{2}$$

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Translated from *Vestnik Novosibirskogo Gosudarstvennogo Universiteta: Seriya Matematika, Mekhanika, Informatika* **14**, No. 2, 2014, pp. 15-23.

$$\frac{\partial \widetilde{\mathbf{v}}}{\partial t} + (\widetilde{\mathbf{v}}, \nabla)\widetilde{\mathbf{v}} = -\frac{\nabla p}{\overline{\rho}} + \widetilde{\nu}\Delta\widetilde{\mathbf{v}} - \frac{\rho}{2\overline{\rho}}\nabla(\widetilde{\mathbf{v}} - \mathbf{v})^2 + \mathbf{f}, \tag{3}$$

where $\widetilde{\mathbf{v}}$ and $\mathbf{v}$ are the velocity vectors of subsystems forming a two-velocity continuum with densities $\widetilde{\rho}$ and ρ respectively, ν and $\widetilde{\nu}$ are the corresponding kinematic viscosities, $\overline{\rho} = \widetilde{\rho} + \rho$ is the common density and $p = p(\overline{\rho}, (\widetilde{\mathbf{v}} - \mathbf{v})^2)$ is the equation of state for the two-velocity continuum, $\mathbf{f}$ is the vector of mass force per unit mass. We write Equations (2) and (3) in the equivalent form

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}\nabla(v^2) - \mathbf{v} \times \mathrm{rot}\mathbf{v} = -\frac{\nabla p}{\overline{\rho}} + \nu\Delta\mathbf{v} + \frac{\widetilde{\rho}}{2\overline{\rho}}\nabla(\widetilde{\mathbf{v}} - \mathbf{v})^2 + \mathbf{f}, \tag{4}$$

$$\frac{\partial \widetilde{\mathbf{v}}}{\partial t} + \frac{1}{2}\nabla(\widetilde{v}^2) - \widetilde{\mathbf{v}} \times \mathrm{rot}\widetilde{\mathbf{v}} = -\frac{\nabla p}{\overline{\rho}} + \widetilde{\nu}\Delta\widetilde{\mathbf{v}} - \frac{\rho}{2\overline{\rho}}\nabla(\widetilde{\mathbf{v}} - \mathbf{v})^2 + \mathbf{f}. \tag{5}$$

From these equations we can derive other equations describing change of vortex in time. For this purpose we apply the operator rot to both sides of Equations (4) and (5):

$$\frac{\partial\, \mathbf{\Omega}}{\partial t} - \mathrm{rot}\,(\mathbf{v} \times\, \mathbf{\Omega}) = -\mathrm{rot}\left(\frac{\nabla p}{\overline{\rho}}\right) + \nu\Delta\, \mathbf{\Omega} + \mathrm{rot}\left(\frac{\widetilde{\rho}}{2\overline{\rho}}\nabla(\widetilde{\mathbf{v}} - \mathbf{v})^2\right) + \mathrm{rot}\mathbf{f},$$

$$\frac{\partial\, \widetilde{\mathbf{\Omega}}}{\partial t} - \mathrm{rot}\left(\widetilde{\mathbf{v}} \times\, \widetilde{\mathbf{\Omega}}\right) = -\mathrm{rot}\left(\frac{\nabla p}{\overline{\rho}}\right) + \widetilde{\nu}\Delta\, \widetilde{\mathbf{\Omega}} - \mathrm{rot}\left(\frac{\rho}{2\overline{\rho}}\nabla(\widetilde{\mathbf{v}} - \mathbf{v})^2\right) + \mathrm{rot}\mathbf{f}.$$

Using formulas of vector analysis, we find

$$\frac{\partial\, \mathbf{\Omega}}{\partial t} - \mathrm{rot}\,(\mathbf{v} \times\, \mathbf{\Omega}) = \frac{1}{\overline{\rho}^2}\left(\nabla\overline{\rho} \times \nabla p\right) + \nu\Delta\, \mathbf{\Omega} + \frac{1}{2}\left(\nabla\left(\frac{\widetilde{\rho}}{\overline{\rho}}\right) \times \nabla(\widetilde{\mathbf{v}} - \mathbf{v})^2\right) + \mathrm{rot}\mathbf{f}, \tag{6}$$

$$\frac{\partial\, \widetilde{\mathbf{\Omega}}}{\partial t} - \mathrm{rot}\left(\widetilde{\mathbf{v}} \times\, \widetilde{\mathbf{\Omega}}\right) = \frac{1}{\overline{\rho}^2}\left(\nabla\overline{\rho} \times \nabla p\right) + \widetilde{\nu}\Delta\, \widetilde{\mathbf{\Omega}} - \frac{1}{2}\left(\nabla\left(\frac{\rho}{\overline{\rho}}\right) \times \nabla(\widetilde{\mathbf{v}} - \mathbf{v})^2\right) + \mathrm{rot}\mathbf{f}. \tag{7}$$

2 Scalar Description of Three-Dimensional Vortex Flows

In this section, we present a scalar description of three-dimensional vortex flows of two-velocity hydrodynamics in an incompressible medium under the condition that the volume saturation is constant. If the mass forces are absent ($\mathbf{f} = 0$), then Equations (1)–(3) have the solution $\mathbf{v} = 0$, $\widetilde{\mathbf{v}} = 0$, $\rho = \rho^0$, $\widetilde{\rho} = \widetilde{\rho}^0$, $p = p^0$ for a mixture of fluids at rest with the uniform pressure $p = p^0$, partial densities ρ^0, $\widetilde{\rho}^0$, and temperature T. In the case of homogeneous media, i.e., $\rho^f = \mathrm{const}$ and $\widetilde{\rho}^f = \mathrm{const}$, where ρ^f, $\widetilde{\rho}^f$ are the phase physical densities provided that the volume saturation is constant for substances forming the two-phase continuum, we have $\rho = \mathrm{const}$ and $\widetilde{\rho} = \mathrm{const}$ which implies div $\mathbf{v} = 0$ and div$\widetilde{\mathbf{v}} = 0$ which is equivalent to the relations $\mathbf{v} = \mathrm{rot}\mathbf{A}$ and $\widetilde{\mathbf{v}} = \mathrm{rot}\widetilde{\mathbf{A}}$, where $\mathbf{A}$ and $\widetilde{\mathbf{A}}$ are the corresponding vector potentials of velocities $\mathbf{v}$ and $\widetilde{\mathbf{v}}$. In other words, $\mathbf{v}$ and $\widetilde{\mathbf{v}}$ are solenoidal vectors. Because of the gradient invariance of the vector potential, one of its components can be made zero without loss of generality. Following [1], we assume that div$\mathbf{A} = 0$ and div$\widetilde{\mathbf{A}} = 0$. This assumption restricts the class of flows under consideration. Then two-component vector potentials are written in terms of scalar functions $\sigma(x, y, z, t)$ and $\widetilde{\sigma}(x, y, z, t)$ as

$$\mathbf{A} = \frac{\partial \sigma}{\partial y}\mathbf{i} - \frac{\partial \sigma}{\partial x}\mathbf{j},$$

$$\widetilde{\mathbf{A}} = \frac{\partial \widetilde{\sigma}}{\partial y}\mathbf{i} - \frac{\partial \widetilde{\sigma}}{\partial x}\mathbf{j}.$$

Hence the vorticity fields are also two-component ones. Indeed, since

$$\mathbf{\Omega} = \mathrm{rot}\mathbf{v} = \mathrm{rotrot}\mathbf{A} = -\Delta\mathbf{A} + \nabla\mathrm{div}\mathbf{A},$$

$$\widetilde{\mathbf{\Omega}} = \mathrm{rot}\widetilde{\mathbf{v}} = \mathrm{rotrot}\widetilde{\mathbf{A}} = -\Delta\widetilde{\mathbf{A}} + \nabla\mathrm{div}\widetilde{\mathbf{A}},$$

we have

$$\mathbf{\Omega} = -\frac{\partial\Delta\sigma}{\partial y}\mathbf{i} + \frac{\partial\Delta\sigma}{\partial x}\mathbf{j},$$

$$\widetilde{\mathbf{\Omega}} = -\frac{\partial\Delta\widetilde{\sigma}}{\partial y}\mathbf{i} + \frac{\partial\Delta\widetilde{\sigma}}{\partial x}\mathbf{j}.$$

Moreover, the velocity fields remain to be three-dimensional

$$\mathbf{v} = \frac{\partial^2\sigma}{\partial x\partial z}\mathbf{i} + \frac{\partial^2\sigma}{\partial y\partial z}\mathbf{j} - \left(\frac{\partial^2\sigma}{\partial x^2} + \frac{\partial^2\sigma}{\partial y^2}\right)\mathbf{k} = \frac{\partial^2\sigma}{\partial x\partial z}\mathbf{i} + \frac{\partial^2\sigma}{\partial y\partial z}\mathbf{j} + \left(\frac{\partial^2\sigma}{\partial z^2} - \Delta\sigma\right)\mathbf{k},$$

$$\widetilde{\mathbf{v}} = \frac{\partial^2\widetilde{\sigma}}{\partial x\partial z}\mathbf{i} + \frac{\partial^2\widetilde{\sigma}}{\partial y\partial z}\mathbf{j} - \left(\frac{\partial^2\widetilde{\sigma}}{\partial x^2} + \frac{\partial^2\widetilde{\sigma}}{\partial y^2}\right)\mathbf{k} = \frac{\partial^2\widetilde{\sigma}}{\partial x\partial z}\mathbf{i} + \frac{\partial^2\widetilde{\sigma}}{\partial y\partial z}\mathbf{j} + \left(\frac{\partial^2\widetilde{\sigma}}{\partial z^2} - \Delta\widetilde{\sigma}\right)\mathbf{k}.$$

Since the third component of the vorticity field is absent, the projection of (4) and (5) on the z-axis yields $\mathrm{rot}(\mathbf{v}\times\ \mathbf{\Omega}) = 0$ and $\mathrm{rot}(\widetilde{\mathbf{v}}\times\ \widetilde{\mathbf{\Omega}}) = 0$ which implies

$$J\left(\Delta\sigma, \frac{\partial^2\sigma}{\partial z^2}\right) = 0, \quad \widetilde{J}\left(\Delta\widetilde{\sigma}, \frac{\partial^2\widetilde{\sigma}}{\partial z^2}\right) = 0,$$

where $J(f,g) \equiv f_x g_y - f_y g_x$. From these relations we find

$$\Delta\sigma = -H\left(\frac{\partial^2\sigma}{\partial z^2}\right), \quad \Delta\widetilde{\sigma} = -\widetilde{H}\left(\frac{\partial^2\widetilde{\sigma}}{\partial z^2}\right),$$

where H and $\widetilde{H}$ are arbitrary functions. It is convenient to introduce the functions

$$\Phi(x,y,z,t) = \frac{\partial\sigma}{\partial z}, \quad \widetilde{\Phi}(x,y,z,t) = \frac{\partial\widetilde{\sigma}}{\partial z}.$$

Then the velocity fields are represented as

$$\mathbf{v} = \frac{\partial\Phi}{\partial x}\mathbf{i} + \frac{\partial\Phi}{\partial y}\mathbf{j} + \left[\frac{\partial\Phi}{\partial z} + H\left(\frac{\partial\Phi}{\partial z}\right)\right]\mathbf{k} = \nabla\Phi + H\left(\frac{\partial\Phi}{\partial z}\right)\mathbf{k}, \tag{8}$$

$$\widetilde{\mathbf{v}} = \frac{\partial\widetilde{\Phi}}{\partial x}\mathbf{i} + \frac{\partial\widetilde{\Phi}}{\partial y}\mathbf{j} + \left[\frac{\partial\widetilde{\Phi}}{\partial z} + \widetilde{H}\left(\frac{\partial\widetilde{\Phi}}{\partial z}\right)\right]\mathbf{k} = \nabla\widetilde{\Phi} + \widetilde{H}\left(\frac{\partial\widetilde{\Phi}}{\partial z}\right)\mathbf{k}. \tag{9}$$

In the special case

$$H\left(\frac{\partial\Phi}{\partial z}\right) \equiv 0, \quad \widetilde{H}\left(\frac{\partial\widetilde{\Phi}}{\partial z}\right) \equiv 0,$$

the velocity fields are potentials and the functions $\Phi(x,y,z,t)$, $\widetilde{\Phi}(x,y,z,t)$ are hydrodynamic potentials. Following [1], we call such functions *quasipotentials*.

The vorticity fields are expressed in terms of quasipotentials as follows:

$$\mathbf{\Omega} = \frac{\partial H}{\partial x}\mathbf{i} - \frac{\partial H}{\partial y}\mathbf{j} = H' \frac{\partial}{\partial z}\left(\frac{\partial \Phi}{\partial x}\mathbf{i} - \frac{\partial \Phi}{\partial y}\mathbf{j}\right), \tag{10}$$

$$\widetilde{\mathbf{\Omega}} = \frac{\partial \widetilde{H}}{\partial x}\mathbf{i} - \frac{\partial \widetilde{H}}{\partial y}\mathbf{j} = \widetilde{H}' \frac{\partial}{\partial z}\left(\frac{\partial \widetilde{\Phi}}{\partial x}\mathbf{i} - \frac{\partial \widetilde{\Phi}}{\partial y}\mathbf{j}\right), \tag{11}$$

where the prime means the differentiation of H and $\widetilde{H}$ with respect to the corresponding variables. The equations of continuity are written in the form

$$\Delta\Phi + \frac{\partial H}{\partial z} = 0, \tag{12}$$

$$\Delta\widetilde{\Phi} + \frac{\partial \widetilde{H}}{\partial z} = 0 \tag{13}$$

or

$$\Delta\Phi + H' \frac{\partial^2 \Phi}{\partial z^2} = 0, \tag{14}$$

$$\Delta\widetilde{\Phi} + \widetilde{H}' \frac{\partial^2 \widetilde{\Phi}}{\partial z^2} = 0. \tag{15}$$

In the case of homogeneous media, substituting (6), (7) into (4), (5), for the first two components we find the motion integrals

$$\frac{\partial \Phi}{\partial t} + \frac{(\nabla\Phi)^2}{2} + \int H d\Phi_z = -\frac{p}{\overline{\rho}} + \nu\Delta\Phi - F + R(z,t) + \frac{\widetilde{\rho}}{2\overline{\rho}}[(\Phi_x - \widetilde{\Phi}_x)^2 + (\Phi_y - \widetilde{\Phi}_y)^2 + (\Phi_z - \widetilde{\Phi}_z + H - \widetilde{H})^2], \tag{16}$$

$$\frac{\partial \widetilde{\Phi}}{\partial t} + \frac{(\nabla\widetilde{\Phi})^2}{2} + \int \widetilde{H} d\widetilde{\Phi}_z = -\frac{p}{\overline{\rho}} + \widetilde{\nu}\Delta\widetilde{\Phi} - F + \widetilde{R}(z,t) - \frac{\rho}{2\overline{\rho}}[(\Phi_x - \widetilde{\Phi}_x)^2 + (\Phi_y - \widetilde{\Phi}_y)^2 + (\Phi_z - \widetilde{\Phi}_z + H - \widetilde{H})^2], \tag{17}$$

where $F(x,y,z,t)$ is the potential of mass forces and $R(z,t)$, $\widetilde{R}(z,t)$ are arbitrary functions defined by the boundary conditions. From these equations we find

$$\frac{\partial(\rho\Phi + \widetilde{\rho}\widetilde{\Phi})}{\partial t} + \frac{\rho(\nabla\Phi)^2 + \widetilde{\rho}(\nabla\widetilde{\Phi})^2}{2} + \rho\int H d\Phi_z + \widetilde{\rho}\int \widetilde{H} d\widetilde{\Phi}_z + p + \overline{\rho}F = \nu\rho\Delta\Phi + \widetilde{\nu}\widetilde{\rho}\Delta\widetilde{\Phi} + \rho R(z,t) + \widetilde{\rho}\widetilde{R}(z,t). \tag{18}$$

From the third components of the velocities (5), (6) for quasipotentials we have

$$H' \frac{\partial}{\partial z}\left(\frac{\partial \Phi}{\partial t} + \frac{(\nabla\Phi)^2}{2} + \int H d\Phi_z\right) = \nu\Delta H - \frac{\partial R}{\partial z}, \tag{19}$$

$$\widetilde{H}' \frac{\partial}{\partial z}\left(\frac{\partial \widetilde{\Phi}}{\partial t} + \frac{(\nabla\widetilde{\Phi})^2}{2} + \int \widetilde{H} d\widetilde{\Phi}_z\right) = \widetilde{\nu}\Delta\widetilde{H} - \frac{\partial \widetilde{R}}{\partial z}, \tag{20}$$

The system (14), (15) is a generalization of the Bernoulli equation for two-velocity hydrodynamics. It becomes the known Bernoulli equation for potential flows [1] if the velocities and densities of the phases coincide, R, $\widetilde{R}$ depend only on time, and

$$H\left(\frac{\partial\Phi}{\partial z}\right) \equiv 0, \quad \widetilde{H}\left(\frac{\partial\widetilde{\Phi}}{\partial z}\right) \equiv 0.$$

Using (16), we can find the pressure field provided that the quasipotentials are known for given functions $H\Big(\frac{\partial\Phi}{\partial z}\Big)$ and $\widetilde{H}\Big(\frac{\partial\widetilde{\Phi}}{\partial z}\Big)$. Thus, to find the velocity fields and the corresponding vorticity fields and pressure, we need to solve the system (10), (11), (17), (18) for quasipotentials and then apply Equations (6)–(9) and (16).

To illustrate the above approach, we consider the case of linear functions $H(\Phi_z)$ and $\widetilde{H}(\widetilde{\Phi}_z)$, i.e., $H(\Phi_z) = \lambda\Phi_z$ and $\widetilde{H}(\widetilde{\Phi}_z) = \widetilde{\lambda}\widetilde{\Phi}_z$. Then Equations (12), (13), (17), (18) take the form

$$\Delta\Phi + \lambda\Phi_{zz} = 0, \tag{21}$$

$$\frac{\partial\Phi}{\partial t} + \frac{(\nabla\Phi)^2}{2} + \frac{\lambda}{2}(\Phi_z)^2 + \lambda\nu\frac{\partial^2\Phi}{\partial z^2} = -\frac{R(z,t)}{\lambda} + Q(x,y,t), \tag{22}$$

$$\Delta\widetilde{\Phi} + \widetilde{\lambda}\widetilde{\Phi}_{zz} = 0, \tag{23}$$

$$\frac{\partial\widetilde{\Phi}}{\partial t} + \frac{(\nabla\widetilde{\Phi})^2}{2} + \frac{\widetilde{\lambda}}{2}(\widetilde{\Phi}_z)^2 + \widetilde{\lambda}\widetilde{\nu}\frac{\partial^2\widetilde{\Phi}}{\partial z^2} = -\frac{\widetilde{R}(z,t)}{\widetilde{\lambda}} + \widetilde{Q}(x,y,t), \tag{24}$$

where $Q(x,y,t)$ and $\widetilde{Q}(x,y,t)$ are arbitrary functions.

In [1], for the system (19)–(22) solutions of three type were constructed:

I. Solutions of the form

$$\Phi(x,y,z,t) = e^{-\nu\lambda k^2 t}(Ae^{-kz} + Be^{kz})\sin(\alpha x + \beta y),$$

$$\widetilde{\Phi}(x,y,z,t) = e^{-\widetilde{\nu}\widetilde{\lambda}\widetilde{k}^2 t}(\widetilde{A}e^{-\widetilde{k}z} + \widetilde{B}e^{\widetilde{k}z})\sin(\widetilde{\alpha}x + \widetilde{\beta}y)$$

with $\lambda > 0$, $\widetilde{\lambda} > 0$, $\alpha^2 + \beta^2 = (1+\mu)k^2$, $\widetilde{\alpha}^2 + \widetilde{\beta}^2 = (1+\widetilde{\lambda})\widetilde{k}^2$, and

$$R(z,t) = -\frac{\lambda}{2}(\alpha^2+\beta^2)e^{-2\nu\lambda k^2 t}(Ae^{-kz} + Be^{kz})^2,$$

$$Q(x,y,t) = -2AB(\alpha^2+\beta^2)e^{-2\nu\lambda k^2 t}\sin^2(\alpha x + \beta y),$$

$$\widetilde{R}(z,t) = -\frac{\widetilde{\lambda}}{2}(\widetilde{\alpha}^2+\widetilde{\beta}^2)e^{-2\widetilde{\nu}\widetilde{\lambda}\widetilde{k}^2 t}(\widetilde{A}e^{-\widetilde{k}z} + \widetilde{B}e^{\widetilde{k}z})^2,$$

$$\widetilde{Q}(x,y,t) = -2\widetilde{A}\widetilde{B}(\widetilde{\alpha}^2+\widetilde{\beta}^2)e^{-2\widetilde{\nu}\widetilde{\lambda}\widetilde{k}^2 t}\sin^2(\widetilde{\alpha}x + \widetilde{\beta}y).$$

II. Solutions of the form

$$\Phi(x,y,z,t) = e^{\nu\lambda k^2 t}(Ae^{-(\alpha x+\beta y)} + Be^{\alpha x+\beta y})\sin kz,$$

$$\widetilde{\Phi}(x,y,z,t) = e^{\widetilde{\nu}\widetilde{\lambda}\widetilde{k}^2 t}(\widetilde{A}e^{-(\widetilde{\alpha}x+\widetilde{\beta}y)} + \widetilde{B}e^{\widetilde{\alpha}x+\widetilde{\beta}y})\sin\widetilde{k}z$$

with $-1 < \lambda < 0$, $-1 < \widetilde{\lambda} < 0$, $\alpha^2 + \beta^2 = (1+\lambda)k^2$, $\widetilde{\alpha}^2 + \widetilde{\beta}^2 = (1+\widetilde{\lambda})\widetilde{k}^2$, and

$$R(z,t) = -2\lambda AB(\alpha^2+\beta^2)e^{2\nu\lambda k^2 t}\cos^2 kz,$$

$$Q(x,y,t) = \frac{1}{2}(\alpha^2+\beta^2)e^{2\nu\lambda k^2 t}(Ae^{-(\alpha x+\beta y)} + Be^{\alpha x+\beta y})^2,$$

$$\widetilde{R}(z,t) = -2\widetilde{\lambda}\widetilde{A}\widetilde{B}(\widetilde{\alpha}^2+\widetilde{\beta}^2)e^{2\widetilde{\nu}\widetilde{\lambda}\widetilde{k}^2 t}\cos^2 \widetilde{k}z,$$

$$\widetilde{Q}(x,y,t) = \frac{1}{2}(\widetilde{\alpha}^2+\widetilde{\beta}^2)e^{2\widetilde{\nu}\widetilde{\lambda}\widetilde{k}^2 t}(\widetilde{A}e^{-(\widetilde{\alpha} x+\widetilde{\beta} y)} + \widetilde{B}e^{\widetilde{\alpha} x+\widetilde{\beta} y})^2.$$

III. Solutions of the form

$$\Phi(x,y,z,t) = Ae^{\nu\lambda k^2 t}\sin(\alpha x+\beta y)\sin kz,$$

$$\widetilde{\Phi}(x,y,z,t) = \widetilde{A}e^{\widetilde{\nu}\widetilde{\lambda}\widetilde{k}^2 t}\sin(\widetilde{\alpha} x+\widetilde{\beta} y)\sin \widetilde{k}z$$

with $\lambda < -1$, $\widetilde{\lambda} < -1$, $\alpha^2 + \beta^2 = -(1+\lambda)k^2$, $\widetilde{\alpha}^2 + \widetilde{\beta}^2 = -(1+\widetilde{\lambda})\widetilde{k}^2$, and

$$R(z,t) = -\frac{\lambda}{2}A^2(\alpha^2+\beta^2)e^{2\nu\lambda k^2 t}\sin^2 kz,$$

$$Q(x,y,t) = -\frac{A^2}{2}(\alpha^2+\beta^2)e^{2\nu\lambda k^2 t}\sin^2(\alpha x+\beta y),$$

$$\widetilde{R}(z,t) = -\frac{\widetilde{\lambda}}{2}\widetilde{A}^2(\widetilde{\alpha}^2+\widetilde{\beta}^2)e^{2\widetilde{\nu}\widetilde{\lambda}\widetilde{k}^2 t}\sin^2 \widetilde{k}z,$$

$$\widetilde{Q}(x,y,t) = -\frac{\widetilde{A}^2}{2}(\widetilde{\alpha}^2+\widetilde{\beta}^2)e^{2\widetilde{\nu}\widetilde{\lambda}\widetilde{k}^2 t}\sin^2(\widetilde{\alpha} x+\widetilde{\beta} y).$$

A solution of type III can be regarded as a doubly periodic Kolmogorov flow [5]. Rotating the coordinates about the z-axis, we can assume that the solution depends only on x in the x, y–plane. Therefore, without loss of generality we set $\beta = 0$. Moreover, the velocity fields contain two components x and z:

$$\mathbf{v} = \alpha Ae^{2\nu\lambda k^2 t}\left[\cos\alpha x\sin kz\mathbf{i} + \frac{1}{\sqrt{-(1+\lambda)}}\sin\alpha x\cos kz\mathbf{k}\right],$$

$$\widetilde{\mathbf{v}} = \widetilde{\alpha}\widetilde{A}e^{2\widetilde{\nu}\widetilde{\lambda}\widetilde{k}^2 t}\left[\cos\widetilde{\alpha} x\sin \widetilde{k}z\mathbf{i} + \frac{1}{\sqrt{-(1+\widetilde{\lambda})}}\sin\widetilde{\alpha} x\cos \widetilde{k}z\mathbf{k}\right],$$

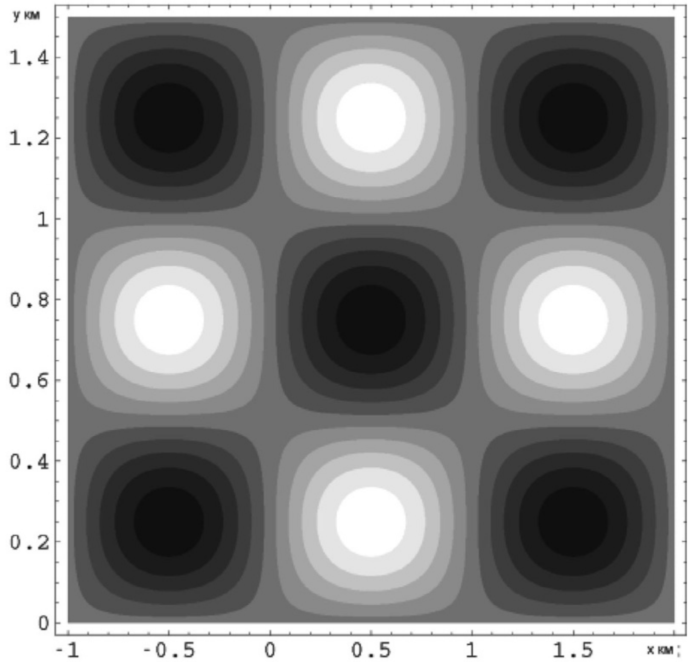
whereas the vorticity fields have only one y-component:

$$\mathbf{\Omega} = -\lambda\alpha Ae^{2\nu\lambda k^2 t}\cos\alpha x\sin kz\mathbf{j},$$

$$\widetilde{\mathbf{\Omega}} = -\widetilde{\lambda}\widetilde{\alpha}\widetilde{A}e^{2\widetilde{\nu}\widetilde{\lambda}\widetilde{k}^2 t}\cos\widetilde{\alpha} x\sin \widetilde{k}z\mathbf{j}.$$

Ignoring the external forces, setting $F(x,y,z,t) \equiv 0$, and using (16), we find the pressure

$$p = \rho\frac{(\alpha A)^2}{2}e^{2\nu\lambda k^2 t}[\sin^2\alpha x - (1+\lambda)\sin^2 kz] + \widetilde{\rho}\frac{(\widetilde{\alpha}\widetilde{A})^2}{2}e^{2\widetilde{\nu}\widetilde{\lambda}\widetilde{k}^2 t}[\sin^2\widetilde{\alpha} x - (1+\widetilde{\lambda})\sin^2 \widetilde{k}z].$$



The figure presents the flow fields in the viscous case for $\nu = 0$ and $\lambda = -1{,}25$. The opposite flow fields are colored by black and white. Since the solution is periodic in the x, z-plane, we can extract an elementary cell such that its lateral boundary can be assumed to be solid. In the case of the absence of viscosity, it is required only that the normal velocity components vanish on the walls, which is valid for solutions of type III.

More complicated examples of flows can be constructed by matching solutions of type I–III. Other classes of flows can be constructed by choosing suitable nonlinear $H(\Phi_z)$ and $\widetilde{H}(\widetilde{\Phi}_z)$.

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Submitted on May 28, 2012