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FORMAL MATRICES AND THEIR DETERMINANTS

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Abstract. In the present paper, we study formal matrix rings over a given ring and determinants of such matrices.

In Secs. 1–4, some general properties of formal matrix rings are presented. In Secs. 5–9, formal matrix rings over a given ring are considered. We formulate and study the realization problem (or the characterization problem), the classification problem, and the isomorphism problem of such rings. Then we introduce the notion of the determinant of a formal matrix over a commutative ring. To such determinants, we extend the main properties of the ordinary determinant of matrices over commutative rings.

All rings are assumed to be associative and with nonzero identity element. If R is a ring, then $J(R)$ is the Jacobson radical of R and $Z(R)$ is the center of R. In this paper, we use standard notions and the notation of ring theory (see, e.g., [24, 25]).

1. Construction of Formal Matrix Rings of Order 2

Given two rings R and S, an R-S-bimodule M, and an S-R-bimodule N, we denote by K the set of all matrices of the form

$$\begin{pmatrix} r & m \\ n & s \end{pmatrix}, \quad \text{where} \quad r \in R, \quad s \in S, \quad m \in M, \quad n \in N.$$

With respect to the matrix addition, K is an Abelian group. To turn K into a ring, we need to know how to calculate "the product" $mn \in R$ and "the product" $nm \in S$. This can be done correctly as follows. We assume that there are bimodule homomorphisms $\varphi\colon M \otimes_S N \to R$ and $\psi\colon N \otimes_R M \to S$. We set $\varphi(m \otimes n) = mn$ and $\psi(n \otimes m) = nm$ for all $m \in M$ and $n \in N$. Now we can multiply matrices in K as matrices in ordinary matrix rings:

$$\begin{pmatrix} r & m \\ n & s \end{pmatrix}\begin{pmatrix} r_1 & m_1 \\ n_1 & s_1 \end{pmatrix} = \begin{pmatrix} rr_1 + mn_1 & rm_1 + ms_1 \\ nr_1 + sn_1 & nm_1 + ss_1 \end{pmatrix}, \quad r, r_1 \in R, \quad s, s_1 \in S, \quad m, m_1 \in M, \quad n, n_1 \in N.$$

Note that rm_1, ms_1, nr_1, and sn_1 are the corresponding module products. We also assume that for all $m, m' \in M$ and $n, n' \in N$, the associativity relations $(mn)m' = m(nm')$ and $(nm)n' = n(mn')$ hold. Then the set K is a ring with respect to the mentioned addition and multiplication operations. When verifying ring axioms, we also need to take into account the main properties of tensor products and the property that φ and ψ are bimodule homomorphisms. The converse is also true: if K is a ring, then the mentioned associativity relations are true. The ring K is called a *formal matrix ring* (of order 2); it is denoted by

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}.$$

The term "a ring of generalized matrices" is also used. Sometimes, we simply write "the matrix ring."

If $N = 0$ or $M = 0$, then K is a ring of formal upper or lower triangular matrices

$$\begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} R & 0 \\ N & S \end{pmatrix},$$

respectively. To define such rings, the homomorphisms φ and ψ are not required.

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The images I and J of the homomorphisms φ and ψ are ideals of rings R and S, respectively. They are called *trace ideals* of the ring K. We say that K is a *ring with zero trace ideals* or a *trivial* ring provided $\varphi = 0 = \psi$, i.e., $I = 0 = J$. Obviously, the ring of formal triangular matrices is a ring with zero trace ideals.

We denote by MN (NM) the set of all finite sums of elements of the form mn (respectively, nm). We have the relations

$$I = MN, \quad J = NM, \quad IM = MJ, \quad NI = JN.$$

How shall the problem of the study of formal matrix rings be correctly formulated? It is natural to consider the study of the ring

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

as the study of the dependence of properties of this ring on properties of the rings R and S, the bimodules M and N, and the homomorphisms φ and ψ.

Sometimes, it is convenient to identify matrices with corresponding elements. For example, we can identify the matrix

$$\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$$

with the element $r \in R$, and so on. We use similar conventions for sets of matrices. For example, the set of matrices

$$\begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix}$$

is presented in the form (X, Y) (or simply X provided $Y = 0$). Similar rules act for matrices with zero upper row.

If $M = 0 = N$, then the ring K can be identified with the direct product $R \times S$. Usually, we assume that the product $R \times S$ is a matrix ring.

Let K be some formal matrix ring

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}.$$

Using the convention on representations of matrices, we have the relation

$$K = \begin{pmatrix} eKe & eK(1-e) \\ (1-e)Ke & (1-e)K(1-e) \end{pmatrix}, \tag{1.1}$$

where

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Under such approach, the action of the corresponding homomorphisms φ and ψ coincides with the multiplication in the ring K.

In a certain sense, the converse is true. Namely, let an abstract ring T contain an idempotent e that is not equal to 0 or 1. We can form the formal matrix ring

$$K = \begin{pmatrix} eTe & eT(1-e) \\ (1-e)Te & (1-e)T(1-e) \end{pmatrix}.$$

The rings T and K are isomorphic to each other. The correspondence

$$t \to \begin{pmatrix} ete & et(1-e) \\ (1-e)te & (1-e)t(1-e) \end{pmatrix}, \quad t \in T,$$

defines the corresponding isomorphism.

Let K be some formal matrix ring presented in the form (1.1). It is easy to describe the structure of ideals and the factor rings of the ring K; also, see the end of Sec. 4 and Propositions 6.3 and 6.4.

If L is an ideal of the ring K, then it is directly verified that L coincides with the set of matrices

$$\begin{pmatrix} eLe & eL(1-e) \\ (1-e)Le & (1-e)L(1-e) \end{pmatrix},$$

where eLe and $(1-e)L(1-e)$ are ideals in the rings R and S, respectively, and $eL(1-e)$ and $(1-e)Le$ are subbimodules in M and N, respectively. Subgroups that are placed in one of four positions in L coincide with sets of corresponding components of elements in L.

We form the group of matrices $\bar{K}$:

$$\begin{pmatrix} (eKe)/(eLe) & \big(eK(1-e)\big)/\big(eL(1-e)\big) \\ \big((1-e)Ke\big)/\big((1-e)Le\big) & \big((1-e)K(1-e)\big)/\big((1-e)L(1-e)\big) \end{pmatrix}.$$

In fact, we have the formal matrix ring $\bar{K}$ (we consider formal matrix rings in the agreed general sense). The multiplication of matrices in $\bar{K}$ is induced by the multiplication in K. It is directly verified that the mapping

$$K/L \to \bar{K}, \quad \begin{pmatrix} r & m \\ n & s \end{pmatrix} + L \to \begin{pmatrix} \bar{r} & \bar{m} \\ \bar{n} & \bar{s} \end{pmatrix}$$

is a ring isomorphism, where the overline denotes the corresponding residue class.

A concrete formal matrix ring is defined with the use of two bimodule homomorphisms φ and ψ. The choice of another pair of homomorphisms generally leads to another ring. We can formulate the problem on classification of formal matrix rings depending on corresponding pairs of bimodule homomorphisms. To this problem, the following *Isomorphism Problem* is related.

Let K and K_1 be two formal matrix rings with bimodule homomorphisms φ, ψ and φ_1, ψ_1, respectively. Which interrelations between homomorphisms φ, ψ and φ_1, ψ_1 are necessary and sufficient for an isomorphism $K \cong K_1$ to exist?

How many formal matrix rings exist? It follows from the above that the class of formal matrix rings coincides with the class of rings that have nontrivial idempotents (if we consider direct products of rings as matrix rings).

The class of formal matrix rings also coincides with the class of endomorphism rings of modules that are decomposable into direct sums. Indeed, let $G = A \oplus B$ be a right module over some ring T. The endomorphism ring of G is isomorphic to the matrix ring

$$\begin{pmatrix} \mathrm{End}_T(A) & \mathrm{Hom}_T(B,A) \\ \mathrm{Hom}_T(A,B) & \mathrm{End}_T(B) \end{pmatrix}$$

with ordinary operations of addition and multiplication of matrices (we consider the composition of homomorphisms as the product of the homomorphisms). Conversely, for the ring

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

we can present the decomposition $K_K = (R, M) \oplus (N, S)$ into the direct sum of right ideals and verify that the ring $\mathrm{End}_K(K)$ is isomorphic to the ring

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}.$$

There are many classes of rings that do not necessarily have a matrix origin but are close to formal matrix rings. In particular, there are various triangular matrix rings. In [5], the authors study so-called *trivial extensions* of rings defined as follows. If R is a ring and M is an R-R-bimodule, then we denote by T the direct sum of Abelian groups R and M:

$$T = \{(r, m) \mid r \in R,\ m \in M\}.$$

The group T turns into a ring if the multiplication is defined by the relation $(r, m)(r_1, m_1) = (rr_1, rm_1 + mr_1)$. This ring is a trivial extension of the ring R by the bimodule M.

Now we consider the triangular matrix ring

$$\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$$

and its subring

$$\Gamma = \left\{ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \middle| r \in R,\ m \in M \right\}.$$

The rings T and Γ are isomorphic to each other under the correspondence

$$(r, m) \to \begin{pmatrix} r & m \\ 0 & r \end{pmatrix}.$$

Thus, trivial extensions consist of triangular matrices.

Each ring of formal triangular matrices

$$\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$$

is a trivial extension. Indeed, we can consider M as an $(R \times S)$-$(R \times S)$-bimodule, by assuming that $(r, s)m = rm$ and $m(r, s) = ms$. We take the trivial extension

$$T = \{((r, s), m) \mid r \in R,\ s \in S,\ m \in M\}$$

of the ring $R \times S$. The correspondence

$$\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \to ((r, s), m)$$

defines an isomorphism of the rings K and T. However, there exists a class of triangular matrix rings containing all trivial extensions. Let $f \colon R \to S$ be a ring homomorphism. In the ring

$$\begin{pmatrix} R & M \\ 0 & S \end{pmatrix},$$

all matrices of the form

$$\begin{pmatrix} r & m \\ 0 & f(r) \end{pmatrix}$$

form a subring.

We present a more general construction of ring extensions (see [19]). Let M be an R-R-bimodule and let $\Phi \colon M \otimes_R M \to R$ be some R-R-bimodule homomorphism. We define a multiplication in $R \oplus M$ by the relation

$$(r, m)(r_1, m_1) = (rr_1 + \Phi(m \otimes m_1), rm_1 + mr_1).$$

This multiplication is associative if and only if

$$\Phi(m \otimes m_1)m_2 = m\Phi(m_1 \otimes m_2) \tag{1.2}$$

for all $m, m_1, m_2 \in M$. In this case, $R \oplus M$ is a ring. This ring is denoted by $R \times_\Phi M$; it is called a *semitrivial extension* of the ring R by M and Φ.

The formal matrix rings are semitrivial extensions. Let

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

be a formal matrix ring with bimodule homomorphisms φ and ψ. We set $T = R \times S$ and $V = M \times N$. We consider V as a natural T-T-bimodule. We denote by Φ the T-T-bimodule homomorphism

$$(\varphi, \psi) \colon V \otimes_T V \to T.$$

It satisfies the corresponding relation (1.2). Consequently, we have a semitrivial extension $T \times_\Phi V$. The rings $T \times_\Phi V$ and

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

are isomorphic to each other under the correspondence

$$(r, s) + (m, n) \to \begin{pmatrix} r & m \\ n & s \end{pmatrix}.$$

On the other hand, every semitrivial extension can be embedded in a suitable formal matrix ring. Indeed, let $T \times_\Phi V$ be a semitrivial extension. The relation that corresponds to (1.2) means that there exists a formal matrix ring

$$\begin{pmatrix} T & V \\ V & T \end{pmatrix}.$$

The bimodule homomorphisms of the ring coincide with Φ. The mapping

$$T \times_\Phi V \to \begin{pmatrix} T & V \\ V & T \end{pmatrix}, \quad (t, v) \to \begin{pmatrix} t & v \\ v & t \end{pmatrix}$$

is a ring embedding. Thus, we can identify $T \times_\Phi V$ with the matrix ring of the form

$$\begin{pmatrix} t & v \\ v & t \end{pmatrix}.$$

Let T be some commutative ring. If the rings R and S are T-algebras, then the ring

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

also is a T-algebra. In this case, we say that K is a *formal matrix algebra.*

2. Examples of Formal Matrix Rings of Order 2

We present some examples of formal matrix rings.

(1) Let M be a right module over some ring S, $R = \mathrm{End}(M_S)$, and let $M^* = \mathrm{Hom}(M_S, S_S)$. Then M is an R-S-bimodule and M^* is an S-R-bimodule, where

$$(s\alpha)m = s\alpha(m), \ (\alpha r)m = \alpha\big(r(m)\big), \quad \alpha \in M^*, \quad s \in S, \quad r \in R, \quad m \in M.$$

There exist an R-R-bimodule homomorphism $\varphi\colon M \otimes_S M^* \to R$ and an S-S-bimodule homomorphism $\psi\colon M^* \otimes_R M \to S$ that are defined by the relations

$$\Big(\varphi\Big(\sum m_i \otimes \alpha_i\Big)\Big)(m) = \sum m_i\alpha_i(m), \quad \psi\Big(\sum \alpha_i \otimes m_i\Big) = \sum \alpha_i(m_i),$$

where $m_i, m \in M$ and $\alpha_i \in M^*$. For φ and ψ, two associativity laws are true. Consequently, we obtain the matrix ring

$$\begin{pmatrix} R & M \\ M^* & S \end{pmatrix}.$$

(2) Let R be a ring, X be a left ideal in R, Y be a right ideal in R, and let S be any subring in R such that $YX \subseteq S \subseteq X \cap Y$. Then

$$\begin{pmatrix} R & X \\ Y & S \end{pmatrix}$$

is a formal matrix ring such that the actions of mappings φ and ψ are reduced to the multiplication in R. As a special case, we obtain the ring

$$\begin{pmatrix} R & Re \\ eR & eRe \end{pmatrix},$$

where e is some idempotent.

(3) Let R be a ring, Y be a right ideal in R, and let S be any subring in R containing Y as an ideal. Then S is called a *subidealizator* of the ideal Y in R, and

$$\begin{pmatrix} R & R \\ Y & S \end{pmatrix}$$

is a matrix ring.

(4) Endomorphism rings of Abelian groups. Let G be an Abelian group that is a direct sum of groups A and B. Then the endomorphism ring $\mathrm{End}(G)$ of G is a formal matrix ring (see Sec. 1). Abelian groups provide many interesting and useful examples of formal matrix rings. First of all, these are triangular matrix rings. Thus, endomorphism rings of the groups $\mathbb{Q} \oplus \mathbb{Z}$, $\mathbb{Z}(p^n) \oplus \mathbb{Z}$, and $\mathbb{Z}(p^\infty) \oplus \mathbb{Q}$ are isomorphic to the rings

$$\begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix}, \quad \begin{pmatrix} \mathbb{Z}_{p^n} & \mathbb{Z}_{p^n} \\ 0 & \mathbb{Z} \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \hat{\mathbb{Z}}_p & A_p \\ 0 & \mathbb{Q} \end{pmatrix},$$

respectively, where $\hat{\mathbb{Z}}_p$ is the ring of p-adic integers and A_p is the field of p-adic numbers.

The endomorphism ring of the p-group $\mathbb{Z}(p^n) \oplus \mathbb{Z}(p^m)$, $n < m$, is an informative illustration to the notion of a formal matrix ring. It can be identified with the formal matrix ring

$$\begin{pmatrix} \mathbb{Z}_{p^n} & \mathbb{Z}_{p^n} \\ \mathbb{Z}_{p^n} & \mathbb{Z}_{p^m} \end{pmatrix}.$$

We denote this ring by K or

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}.$$

How are matrices multiplied in the ring K? First of all, we remark that $\mathbb{Z}_{p^n} = \mathbb{Z}_{p^m}/(p^{m-n} \cdot 1)$. Therefore, the rings R and S act on M and N by an ordinary, uniquely possible method. Further, we pass to homomorphisms $\varphi\colon M \otimes_S N \to R$ and $\psi\colon N \otimes_R M \to S$. If we consider K as the original endomorphism ring, then the action of φ and ψ coincides with the composition of corresponding homomorphisms. Taking this into account, we obtain the following property. If $\bar{a} \in M$ and $\bar{b} \in N$, where the over-line denotes the residue class, then

$$\varphi(\bar{a} \otimes \bar{b}) = \bar{a} \circ \bar{b} = p^{m-n}\bar{a}\bar{b}.$$

Further, we have

$$\psi(\bar{b} \otimes \bar{a}) = \bar{b} \circ \bar{a} = p^{m-n}\bar{b}\bar{a},$$

where the last symbols $\bar{b}$ and $\bar{a}$ denote the residue classes in $\mathbb{Z}_{p^m}$ with representatives b and a.

We obtain that the trace ideals I and J of the ring K are equal to $(p^{m-n} \cdot 1)$ and $(p^{m-n} \cdot 1)$, respectively. Therefore, we have $I \subseteq J(R)$ and $J \subseteq J(S)$. There exists a surjective homomorphism $e\colon S \to R$ with $e(\bar{y}) = \bar{y}$, $\bar{y} \in S$. In addition, $\mathrm{Ker}(e) \subseteq J(S)$ and the relation $e(\bar{b} \circ \bar{a}) = \bar{a} \circ \bar{b}$ holds.

In [3], the case $n = 1$, $m = 2$ is considered in detail. In the ring K, all invertible matrices are described. This is used for constructing a cryptosystem.

(5) Complete matrix rings. Let R be some ring. The complete matrix ring $\mathrm{M}(n, R)$ can be represented in the form of a formal matrix ring of order 2

$$\mathrm{M}(n, R) = \begin{pmatrix} R & \mathrm{M}(1 \times (n-1), R) \\ \mathrm{M}\big((n-1) \times 1, R\big) & \mathrm{M}(n-1, R) \end{pmatrix}.$$

This ring is an example of a block matrix ring. A more general situation will appear in the proof of Proposition 3.3 and in the first paragraph after this proof.

(6) See [4]. Let R be a ring, G be a finite subgroup of the automorphism group of the ring R, and let R^G be the ring of invariants of the ring R, i.e., R^G is the subring

$$\{x \in R \mid x^g = x \text{ for all } g \in G\}.$$

We consider the skew group ring $R * G$ consisting of all formal sums of the form

$$\sum_{g \in G} r_g g, \quad r_g \in R.$$

The sums are added componentwise; for multiplication, we use the distributivity law and the relation $rg \cdot sh = rs^{g^{-1}}gh$ for all $r, s \in R$ and $g, h \in G$. It is clear that R is a left R^G-module and a right R^G-module. We can also consider R as a left and right $R * G$-module as follows: for any two elements $x = \sum_{g \in G} r_g g \in R * G$ and $r \in R$, we set

$$x \cdot r = \sum_{g \in G} r_g r^{g^{-1}}, \quad r \cdot x = \sum_{g \in G} (r r_g)^g.$$

The mappings

$$\varphi \colon R \otimes_{R*G} R \to R^G, \quad \psi \colon R \otimes_{R^G} R \to R * G$$

are defined with the use of the relations

$$\varphi(x \otimes y) = \sum_{g \in G} (xy)^g, \quad \psi(y \otimes x) = \sum_{g \in G} y x^{g^{-1}} g,$$

respectively.

Two associativity conditions hold; as a result, we obtain the ring

$$\begin{pmatrix} R^G & R \\ R & R * G \end{pmatrix}.$$

3. Formal Matrix Rings of Order $n \geq 2$

We present several remarks about formal matrix rings of arbitrary order n. To understand how we must define such rings, the case $n = 2$, which was considered in Sec. 1, is sufficient.

We fix a positive integer $n \geq 2$. Let $R_1, \ldots, R_n$ be rings, M_{ij} be R_i-R_j-bimodules, and let $M_{ii} = R_i$, $i, j = 1, \ldots, n$. For all $i, j, k = 1, \ldots, n$ such that $i \neq j$ and $j \neq k$, we assume that an R_i-R_k-bimodule homomorphism

$$\varphi_{ijk} \colon M_{ij} \otimes_{R_j} M_{jk} \to M_{ik}$$

is defined. For subscripts $i = j$ and $j = k$, we assume that φ_{iik} and φ_{ikk} are canonical isomorphisms

$$R_i \otimes_{R_i} M_{ik} \to M_{ik}, \quad M_{ij} \otimes_{R_j} R_j \to M_{ij}.$$

Instead of $\varphi_{ijk}(a \otimes b)$, we write $a \circ b$ or ab. We also assume that, in this notation, $(ab)c = a(bc)$ for all elements $a \in M_{ij}$, $b \in M_{jk}$, $c \in M_{kl}$, and subscripts i, j, k, l.

We denote by K the set of all matrices (a_{ij}) of order n with values in bimodules M_{ij}. With respect to standard matrix operations of addition and multiplication, K is a ring. It can be presented in the form

$$\begin{pmatrix} R_1 & M_{12} & \ldots & M_{1n} \\ M_{21} & R_2 & \ldots & M_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ M_{n1} & M_{n2} & \ldots & R_n \end{pmatrix}. \tag{3.1}$$

We say that K is a *formal matrix ring of order* n. If $M_{ij} = 0$ for all i, j with $i < j$ ($j < i$), then we say that there exists a *ring of formal lower* (respectively, *upper*) *triangular matrices.*

To better understand the structure of formal matrix rings, we clarify their relations to idempotents and endomorphism rings.

Proposition 3.1. *A ring K is a formal matrix ring of order $n \geq 2$ if and only if K contains a complete orthogonal system consisting of n nonzero idempotents.*

Proof. If K is a formal matrix ring of order n, then the matrix units $E_{11}, \dots, E_{nn}$ (see Sec. 5) form the required system of idempotents of the ring K.

Conversely, if $\{e_1, \dots, e_n\}$ is a complete orthogonal system of nonzero idempotents of some ring T, then T is isomorphic to the formal matrix ring

$$\begin{pmatrix} e_1Te_1 & e_1Te_2 & \dots & e_1Te_n \\ e_2Te_1 & e_2Te_2 & \dots & e_2Te_n \\ \dots & \dots & \dots & \dots \\ e_nTe_1 & e_nTe_2 & \dots & e_nTe_n \end{pmatrix};$$

see Sec. 1 about such rings. □

The case of direct sums of two modules, which was considered in Sec. 1, can be generalized to direct sums of any finite number of summands.

Proposition 3.2. *The class of formal matrix rings of order n coincides with the class of endomorphism rings of modules that are decomposable into direct sums of n nonzero summands.*

Formal matrix rings of any order n can appear in concrete problems. In the general theory, matrix rings of order 2 are usually studied; the main reason is a technical convenience. In a certain sense, the case $n > 2$ can be sometimes reduced to the case of matrices of order 2.

Proposition 3.3. *Any formal matrix ring of order $n > 2$ is isomorphic to some formal matrix ring of order k for each $k = 2, \dots, n-1$.*

Proof. The assertion becomes quite understandable if we consider a representation of matrix rings with the use of idempotents or endomorphism rings (see Propositions 3.1 and 3.2). It is sufficient to "enlarge" idempotents or direct summands by a certain method. Of course, there also exists a direct proof. For example, we take $k = 2$. We introduce the following notation for sets of matrices. We set $R = R_1$, $M = (M_{12}, \dots, M_{1n})$,

$$N = \begin{pmatrix} M_{21} \\ \vdots \\ M_{n1} \end{pmatrix}, \quad S = \begin{pmatrix} R_2 & M_{23} & \dots & M_{2n} \\ \dots & \dots & \dots & \dots \\ M_{n2} & M_{n3} & \dots & R_n \end{pmatrix}.$$

Here S is a formal matrix ring of order $n-1$, M is an R-S-bimodule, N is an S-R-bimodule, and module multiplications are defined as products of rows and columns on matrices. The homomorphisms φ_{ijk}, defining the multiplication in K, induce bimodule homomorphisms $\varphi\colon M \otimes_S N \to R$ and $\psi\colon N \otimes_R M \to S$. In addition, two required associativity laws are true. As a result, we have the formal matrix ring

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

and the isomorphism

$$K \cong \begin{pmatrix} R & M \\ N & S \end{pmatrix}.$$

The isomorphism is obtained by decomposing each matrix into four blocks:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} (a_{11}) & (a_{12} \ \dots \ a_{1n}) \\ \begin{pmatrix} a_{21} \\ \dots \\ a_{n1} \end{pmatrix} & \begin{pmatrix} a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{n2} & \dots & a_{nn} \end{pmatrix} \end{pmatrix}.$$ □

In the proof of the proposition, we practically obtain that formal matrices can be decomposed into blocks, which is similar to the case of ordinary matrices; i.e., we can represent formal matrices in the form of block matrices. We perform actions over block matrices with the use of the rules that coincide with the rules for the case where we have single elements instead of the blocks. The multiplication of block matrices of the same order is always possible provided cofactors have the same block decompositions.

Thus, every formal matrix ring can be considered as a ring of (formal) block matrices. Rings of block upper (lower) triangular matrices naturally appear. The rings of (formal) block matrices are used in the theory of finite-dimensional algebras. In particular, rings of block triangular matrices over fields naturally appear in this theory.

Conversely, there exist several constructions that allow us to use a given formal matrix ring for constructing formal matrix rings of higher order. We consider the first method.

Given a formal matrix ring of the form (3.1), we fix some sequence of positive integers $s_1, \dots, s_n$. We denote by $\bar{M}_{ij}$ the set of $s_i \times s_j$ matrices with elements in M_{ij} (we recall that $M_{ii} = R$). Further, let $\bar{K}$ be the set of all block matrices $(\bar{M}_{ij})$, $i, j = 1, \dots, n$. We define operations of addition and multiplication of these matrices as usual. This means that the addition is componentwise; about the multiplication, we remark that $A_{ij} \cdot A_{jk} \in \bar{M}_{ik}$ for all matrices $A_{ij} \in \bar{M}_{ij}$ and $A_{jk} \in \bar{M}_{jk}$. Then $\bar{K}$ turns into a ring of formal block matrices; in addition, $\bar{K}$ is a formal matrix ring of order $s_1 + \dots + s_n$.

In what follows, we use the second easy method of constructing formal matrix rings of larger order. Given a formal matrix ring K of order 2,

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix},$$

we show that there exists a formal matrix ring

$$K_4 = \begin{pmatrix} K & \begin{pmatrix} M \\ S \end{pmatrix} \\ \begin{pmatrix} N & S \end{pmatrix} & S \end{pmatrix}.$$

First of all,

$$\begin{pmatrix} M \\ S \end{pmatrix}$$

is a natural K-S-bimodule, and

$$\begin{pmatrix} N & S \end{pmatrix}$$

is an S-K-bimodule. The mapping

$$\varphi\colon \begin{pmatrix} M \\ S \end{pmatrix} \otimes_S \begin{pmatrix} N & S \end{pmatrix} \to K, \quad \begin{pmatrix} m \\ x \end{pmatrix} \otimes (n, y) \to \begin{pmatrix} mn & my \\ xn & xy \end{pmatrix}$$

is a K-K-bimodule homomorphism, and the mapping

$$\psi\colon \begin{pmatrix} N & S \end{pmatrix} \otimes_K \begin{pmatrix} M \\ S \end{pmatrix} \to S, \quad (n, y) \otimes \begin{pmatrix} m \\ x \end{pmatrix} \to nm + yx$$

is an S-S-bimodule homomorphism. There are two familiar associativity relations for φ and ψ from Sec. 1. Consequently, the mentioned ring K_4 exists. The ring

$$K_2 = \begin{pmatrix} K & \begin{pmatrix} R \\ N \end{pmatrix} \\ \begin{pmatrix} R & M \end{pmatrix} & R \end{pmatrix}$$

is similarly defined. Now we remark that the ring

$$L = \begin{pmatrix} S & N \\ M & R \end{pmatrix}$$

always exists, along with the ring K. These rings are isomorphic to each other under the correspondence

$$\begin{pmatrix} r & m \\ n & s \end{pmatrix} \to \begin{pmatrix} s & n \\ m & r \end{pmatrix}.$$

Therefore, in addition to the rings K_4 and K_2, there exist rings K_3 and K_1 that are isomorphic to K_4 and K_2, respectively. However, we can also construct them directly.

4. Some Ideals of Formal Matrix Rings

For a formal matrix ring of order n, we find the Jacobson radical and the prime radical. First, we consider the case $n = 2$.

Given a ring

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix},$$

we define four subbimodules of the bimodules M and N. We set

$$J_l(M) = \{m \in M \mid Nm \subseteq J(S)\}, \quad J_r(M) = \{m \in M \mid mN \subseteq J(R)\},$$
$$J_l(N) = \{n \in N \mid Mn \subseteq J(R)\}, \quad J_r(N) = \{n \in N \mid nM \subseteq J(S)\}.$$

Now we form the following sets of matrices:

$$J_l(K) = \begin{pmatrix} J(R) & J_l(M) \\ J_l(N) & J(S) \end{pmatrix}, \quad J_r(K) = \begin{pmatrix} J(R) & J_r(M) \\ J_r(N) & J(S) \end{pmatrix}.$$

It is directly verified that we obtain a left ideal and a right ideal of the ring K.

Theorem 4.1 ([20])**.** *The following relations hold:*

$$J_l(K) = J(K) = J_r(K).$$

Proof. We have

$$J(K) = \begin{pmatrix} X & B \\ C & Y \end{pmatrix},$$

where X and Y are ideals of the rings R and S, respectively, and B and C are subbimodules in M and N, respectively (see Sec. 1). The following relations hold:

$$X = eJ(K)e = J(eKe) = J(R), \quad \text{where} \quad e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Similarly, we obtain that $Y = J(S)$. Further, we have

$$B \subseteq J_l(M) \cap J_r(M), \quad C \subseteq J_l(N) \cap J_r(N).$$

It is proved that $J(K) \subseteq J_l(K) \cap J_r(K)$.

Now we take an arbitrary matrix

$$\begin{pmatrix} r & m \\ n & s \end{pmatrix}$$

in $J_r(K)$ and the identity matrix E. The matrices

$$E - \begin{pmatrix} r & m \\ 0 & 0 \end{pmatrix}, \quad E - \begin{pmatrix} 0 & 0 \\ n & s \end{pmatrix}$$

are right invertible in K. Their right inverse matrices are matrices

$$\begin{pmatrix} x & xm \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ yn & y \end{pmatrix},$$

respectively, where x and y are right inverse elements for $1 - r$ and $1 - s$, respectively. Consequently, the matrices

$$\begin{pmatrix} r & m \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ n & s \end{pmatrix}, \quad \begin{pmatrix} r & m \\ n & s \end{pmatrix}$$

are contained in $J(K)$. Therefore, $J_r(K) \subseteq J(K)$. Similarly, we have that $J_l(K) \subseteq J(K)$. □

We obtain that $J_l(M) = J_r(M)$ and $J_l(N) = J_r(N)$. These ideals are denoted by $J(M)$ and $J(N)$, respectively. Thus, we have the relation

$$J(K) = \begin{pmatrix} J(R) & J(M) \\ J(N) & J(S) \end{pmatrix}.$$

For an arbitrary ring T, the intersection of all prime ideals of T is called the *prime radical*; it is denoted by $P(T)$.

It is well known that the prime radical of the ring T coincides with the set of all strongly nilpotent elements of T. We recall that an element $a \in T$ is said to be *strongly nilpotent* if all terms of each sequence $a_0, a_1, a_2, \dots$ such that

$$a_0 = a, \quad a_{n+1} \in a_n T a_n, \quad n \in \mathbb{N},$$

are equal to zero, beginning with some number.

We define ideals $P_l(M)$, $P_r(M)$, $P_l(N)$, and $P_r(N)$ that are similar to the ideals $J_l(M)$, $J_r(M)$, $J_l(N)$, and $J_r(N)$, respectively. We restrict ourselves by the "left-side" case. We set

$$P_l(M) = \{m \in M \mid Nm \subseteq P(S)\}, \quad P_r(M) = \{m \in M \mid mN \subseteq P(R)\}.$$

Then let

$$P_l(K) = \begin{pmatrix} P(R) & P_l(M) \\ P_l(N) & P(S) \end{pmatrix}.$$

Theorem 4.2 ([20])**.** *The following relations hold:*

$$P_l(K) = P(K) = P_r(K).$$

Proof. The proof of Theorem 4.2 consists of the verification of the above relations with the use of strongly nilpotent elements. □

The equal ideals $P_l(M)$ and $P_r(M)$ are denoted by $P(M)$, and the equal ideals $P_l(N)$ and $P_r(N)$ are denoted by $P(N)$.

Now we pass to a formal matrix ring K of any order n of the form (3.1) from Sec. 3. For all subscripts i and j, we define two subbimodules

$$J_l(M_{ij}) = \{x \in M_{ij} \mid M_{ji}x \subseteq J(R_j)\}, \quad J_r(M_{ij}) = \{x \in M_{ij} \mid xM_{ji} \subseteq J(R_i)\}.$$

For $i = j$, we obtain $J_l(R_i) = J_r(R_i) = J(R_i)$.

Theorem 4.3. *We have the relation*

$$J(K) = \begin{pmatrix} J(R_1) & J_l(M_{12}) & \dots & J_l(M_{1n}) \\ J_l(M_{21}) & J(R_2) & \dots & J_l(M_{2n}) \\ \dots & \dots & \dots & \dots \\ J_l(M_{n1}) & J_l(M_{n2}) & \dots & J(R_n) \end{pmatrix} \tag{4.1}$$

and a similar relation, in which the subscript l is replaced by r.

Proof. In the case $n = 2$, the assertion follows from Theorem 4.1. Let K be a formal matrix ring of order $n \geq 3$. We present K as the block matrix ring

$$\begin{pmatrix} R & M \\ N & R_n \end{pmatrix},$$

where R is a formal matrix ring of order $n - 1$ and M and N are corresponding bimodules (see the proof of Proposition 3.3). By Theorem 4.1, we have

$$J(K) = \begin{pmatrix} J(R) & J(M) \\ J(N) & J(R_n) \end{pmatrix}.$$

By the induction hypothesis, the radical $J(R)$ has the form mentioned in the theorem. We must show that the set in the right part of the relation (4.1) coincides with

$$\begin{pmatrix} J(R) & J(M) \\ J(N) & J(R_n) \end{pmatrix}.$$

It is sufficient to verify that

$$\begin{pmatrix} J_l(M_{1\,n}) \\ \dots \\ J_l(M_{n-1\,n}) \end{pmatrix} = J(M), \quad \big(J_l(M_{n1}), \dots, J_l(M_{n\,n-1})\big) = J(N),$$

where

$$J(M) = \{m \in M \mid Nm \subseteq J(R_n)\}, \quad J(N) = \{n \in N \mid Mn \subseteq J(R)\}.$$

The required assertion follows from the definition of subbimodules $J_l(M_{ij})$. We specialize the following fact. If $x \in J_l(M_{nj})$, then $M_{in}x \subseteq J_l(M_{ij})$ for all distinct $i, j = 1, \dots, n$. Indeed, we have

$$M_{ji}M_{in}x \subseteq M_{jn}x \subseteq J(R_j).$$

The proof of the analogue of the relation (4.1) for "the subscript r" is symmetric to the proof presented above. □

We have that $J_l(M_{ij}) = J_r(M_{ij})$ for distinct i and j. We denote this subbimodule by $J(M_{ij})$.

The prime radical $P(K)$ has a similar structure. Similarly to the subbimodules $J_l(M_{ij})$ and $J_r(M_{ij})$, we define two subbimodules $P_l(M_{ij})$ and $P_r(M_{ij})$. The following result is true.

Theorem 4.4. *We have the relation*

$$P(K) = \begin{pmatrix} P(R_1) & P_l(M_{12}) & \dots & P_l(M_{1n}) \\ P_l(M_{21}) & P(R_2) & \dots & P_l(M_{2n}) \\ \dots & \dots & \dots & \dots \\ P_l(M_{n1}) & P_l(M_{n2}) & \dots & P(R_n) \end{pmatrix}$$

and a similar relation, in which the subscript l is replaced by the subscript r.

We consider the structure of ideals of the ring K. All assertions from Sec. 1 that are related to ideals and the factor rings can be extended to formal matrix rings of any order n. The ideal L of the ring K is equal to

$$\begin{pmatrix} I_1 & A_{12} & \dots & A_{1n} \\ A_{21} & I_2 & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & I_n \end{pmatrix},$$

where I_i is an ideal of the ring R and A_{ij} is a subbimodule in M_{ij}. Between these ideals and subbimodules, there exist some interrelations, which can be easily found (in one particular case, they are mentioned in Sec. 6). The set of matrices

$$\begin{pmatrix} R_1/I_1 & M_{12}/A_{12} & \dots & M_{1n}/A_{1n} \\ M_{21}/A_{21} & R_2/I_2 & \dots & M_{2n}/A_{2n} \\ \dots & \dots & \dots & \dots \\ M_{n1}/A_{n1} & M_{n2}/A_{n2} & \dots & R_n/I_n \end{pmatrix}$$

naturally forms a formal matrix ring, which is canonically isomorphic to the factor ring K/L.

At the end of the section, we find the center of the formal matrix ring. We recall that the center of some ring T is denoted by $C(T)$.

Lemma 4.5. *The center of the formal matrix ring K consists of all diagonal matrices* $\operatorname{diag}(r_1, r_2, \dots, r_n)$ *such that $r_i \in C(R_i)$ and $r_i m = m r_j$ for all $m \in M_{ij}$ and distinct i and j.*

Proof. It is clear that diagonal matrices with mentioned construction are contained in $C(K)$.

We assume that the matrix $D = (d_{ij})$ is contained in $C(K)$. It follows from the relation $DE_{kk} = E_{kk}D$ that $d_{ik} = 0 = d_{kj}$ for $i \neq k$ and $k \neq j$. Therefore, $d_{ij} = 0$ for $i \neq j$, and D is a diagonal matrix.

Now we fix subscripts i and j and an element $m \in M_{ij}$. Let A_{ij} be the matrix that has m on the position (i, j) and has 0 on remaining positions. It follows from the relation $DA_{ij} = A_{ij}D$ that $d_i m = md_j$. In particular, for $i = j$, we obtain that $d_i \in C(R_i)$. □

5. Formal Matrix Rings over the Ring R

Let R be a ring. For any positive integer $n \geq 2$, there exists a matrix ring $\mathrm{M}(n, R)$ of order n. We can define other matrix multiplications to obtain formal matrix rings. For the ring $\mathrm{M}(n, R)$, the corresponding bimodule homomorphisms $R \otimes_R R \to R$ are equal to each other and act with the use of the relation $x \otimes y \to xy$. By taking other bimodule homomorphisms $R \otimes_R R \to R$, we can obtain matrix rings over R of order n (as formal matrix rings), which do not coincide with $\mathrm{M}(n, R)$.

We take some formal matrix ring K of order n, which is defined in Sec. 3, such that $R_1 = \cdots = R_n = R$ and $M_{ij} = R$ for all i and j. Such a ring is called a *formal matrix ring of order n over the ring R*. Let $\varphi_{ijk} \colon R \otimes_R R \to R$, $i, j, k = 1, \ldots, n$, be bimodule homomorphisms associated to the ring K. Similar to Sec. 3, for elements $x, y \in R$, we set $x \circ y = \varphi_{ijk}(x \otimes y)$. We also set $s_{ijk} = \varphi_{ijk}(1 \otimes 1)$ for every three subscripts i, j, k. For all elements $x, y \in R$ and subscripts i, j, k, we obtain

$$x \circ y = \varphi_{ijk}(x \otimes y) = x\varphi_{ijk}(1 \otimes 1)y = xs_{ijk}y.$$

Further, we obtain that

$$xs_{ijk} = \varphi_{ijk}(x \otimes 1) = \varphi_{ijk}(1 \otimes x) = s_{ijk}x.$$

Thus, s_{ijk} is a central element of the ring R and $x \circ y = s_{ijk}xy$.

For all i, j, k, l, the relations

$$s_{iik} = 1 = s_{ikk}, \quad s_{ijk} \cdot s_{ikl} = s_{ijl} \cdot s_{jkl} \tag{5.1}$$

hold. The first two relations follow from the property that φ_{iik} and φ_{ikk} coincide with the canonical isomorphism $R \otimes_R R \to R$, $x \otimes y \to xy$. The remaining relations follow from the property that $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in R$ (see the beginning of Sec. 3). In particular, the relation $(1 \circ 1) \circ 1 = 1 \circ (1 \circ 1)$ holds.

Now let us have an arbitrary set of central elements s_{ijk} of the ring R, $i, j, k = 1, \ldots, n$, which satisfy relations (5.1). For each of three subscripts i, j, and k, we define a bimodule homomorphism

$$\varphi_{ijk} \colon R \otimes_R R \to R, \quad \varphi_{ijk}(x \otimes y) = s_{ijk}xy, \quad x, y \in R.$$

These homomorphisms define a formal matrix ring of order n in the sense of Sec. 3. Indeed, φ_{iik} and φ_{ikk} are the canonical isomorphisms. Further, we set $x \circ y = \varphi_{ijk}(x \otimes y)$ and obtain that $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in R$ and corresponding subscripts i, j, k, and l.

We can conclude that there is a one-to-one correspondence between formal matrix rings of order n over a ring R and sets of central elements $\{s_{ijk} \mid i, j, k = 1, \ldots, n\}$ of the ring R, which satisfy relations (5.1). We denote such a concrete ring by $\mathrm{M}(n, R, \{s_{ijk}\})$ or $\mathrm{M}(n, R, \Sigma)$, where $\Sigma = \{s_{ijk} \mid i, j, k = 1, \ldots, n\}$; or we simply denote it by the symbol K. The set Σ is called a *system of factors*, and the elements of Σ are called *factors* of the ring K. If all s_{ijk} are equal to 1, then we obtain the ring $\mathrm{M}(n, R)$.

It is useful to present the formula for the multiplication of matrices in the ring $\mathrm{M}(n, R, \Sigma)$. Namely, if $A = (a_{ij})$, $B = (b_{ij})$, and $AB = C = (c_{ij})$, then

$$c_{ij} = \sum_{k=1}^{n} s_{ikj}a_{ik}b_{kj}.$$

We prove several relations for the factors s_{ijk} (we assume that we have a ring $\mathrm{M}(n, R, \Sigma)$). We again present relations (5.1); we will call them *main relations*:

$$s_{iik} = 1 = s_{ikk}, \quad s_{ijk} \cdot s_{ikl} = s_{ijl} \cdot s_{jkl}. \tag{5.2}$$

By setting $i = k$, we obtain $s_{iji} = s_{ijl} \cdot s_{jil}$. Consequently, $s_{jij} = s_{jil} \cdot s_{ijl}$. Therefore, the relation $s_{iji} = s_{jij}$ holds. For $j = l$, we have $s_{jkj} = s_{ijk} \cdot s_{ikj}$, whence $s_{iji} = s_{lij} \cdot s_{lji}$. Thus, we have

$$s_{iji} = s_{ijl} \cdot s_{jil} = s_{lij} \cdot s_{lji}. \tag{5.3}$$

From relations (5.3), we obtain the following relations, which follow from each other by a permutation of subscripts:

$$\begin{aligned} s_{iji} = s_{jij} = s_{ijk} \cdot s_{jik} = s_{kij} \cdot s_{kji}, \\ s_{iki} = s_{kik} = s_{ikj} \cdot s_{kij} = s_{jik} \cdot s_{jki}, \\ s_{jkj} = s_{kjk} = s_{jki} \cdot s_{kji} = s_{ijk} \cdot s_{ikj}. \end{aligned} \tag{5.4}$$

The following relations follow from relations (5.4):

$$\begin{aligned} s_{ijk} \cdot s_{iki} = s_{jki} \cdot s_{iji} = s_{kij} \cdot s_{jkj}, \\ s_{kji} \cdot s_{iki} = s_{ikj} \cdot s_{iji} = s_{jik} \cdot s_{jkj}. \end{aligned} \tag{5.5}$$

The relations (5.5) can be directly proved with the use of (5.2) if we put $l = i$ into (5.2) and interchange subscripts.

To a given ring $\mathrm{M}(n, R, \Sigma)$, we can relate several matrices. Namely, we set

$$S = (s_{iji}) = \begin{pmatrix} s_{111} & s_{121} & \dots & s_{1n1} \\ s_{212} & s_{222} & \dots & s_{2n2} \\ \dots & \dots & \dots & \dots \\ s_{n1n} & s_{n2n} & \dots & s_{nnn} \end{pmatrix}.$$

Then for every $k = 1, \dots, n$, we form the matrix

$$S_k = (s_{ikj}) = \begin{pmatrix} s_{1k1} & s_{1k2} & \dots & s_{1kn} \\ s_{2k1} & s_{2k2} & \dots & s_{2kn} \\ \dots & \dots & \dots & \dots \\ s_{nk1} & s_{nk2} & \dots & s_{nkn} \end{pmatrix}.$$

The matrices S, S_k are called *matrices of factors* of the ring $\mathrm{M}(n, R, \Sigma)$. The matrix S is symmetric. The main diagonal of the matrix S_k coincides with kth row (and kth column) of the matrix S; the kth row and the kth column of the matrix S_k consist of 1.

We formulate several interrelated problems on formal matrix rings $\mathrm{M}(n, R, \Sigma)$.

(I) The realization and characterization problem. Given matrices $T, T_1, \dots, T_n$ of order n with elements in the center $C(R)$, under which conditions are these matrices matrices of factors of some ring $\mathrm{M}(n, R, \Sigma)$? It is clear that the matrix T must be symmetric. In addition, we can assume that all matrices T_k are also symmetric.

(II) The classification problem. Describe formal matrix rings in relation to systems of factors or matrices of factors.

(III) The isomorphism problem. When do two systems of factors define isomorphic formal matrix rings? In a more general situation, the isomorphism problem is formulated in Sec. 1.

The above Problems (I)–(III) are considered in Secs. 7–9. Now we consider some easy methods of constructing systems of factors. We also consider standard situations, where we can state that formal matrix rings are isomorphic to each other.

(a) We can define the action of the symmetric group of order n on systems of factors; consequently, the group acts on formal matrix rings. The corresponding orbits consist of isomorphic rings.

Let τ be a permutation of order n. The action of τ on matrices is known. Namely, for the matrix $A = (a_{ij})$ of order n, we set $\tau A = (a_{\tau(i)\tau(j)})$. We mean that the matrix τA has the element $a_{\tau(i)\tau(j)}$ on the position (i, j), and the element a_{ij} passes to the position $\big(\tau^{-1}(i), \tau^{-1}(j)\big)$.

Now if $\Sigma = \{s_{ijk} \mid i,j,k = 1,\dots,n\}$ is some system of factors, then we set $t_{ijk} = s_{\tau(i)\tau(j)\tau(k)}$. Then $\{t_{ijk} \mid i,j,k = 1,\dots,n\}$ also is a system of factors, since it satisfies relations (5.2). We denote it by $\tau\Sigma$. Consequently, the formal matrix ring $\mathrm{M}(n,R,\tau\Sigma)$ exists. The rings $\mathrm{M}(n,R,\Sigma)$ and $\mathrm{M}(n,R,\tau\Sigma)$ are isomorphic to each other under the correspondence $A \to \tau A$.

We also remark that the action of permutations on matrices can be represented in another form (see [8]). Let $T = \big(\delta_{i\tau(j)}\big)$ be the permutation matrix τ, where $\delta_{i\tau(j)}$ is the Kronecker symbol. Then $\tau A = T^{-1}AT$, where $T^{-1} = \big(\delta_{i\tau^{-1}(j)}\big)$. The essence of the isomorphism $\mathrm{M}(n,R,\Sigma) \cong \mathrm{M}(n,R,\tau\Sigma)$ can be expressed by the relation

$$T^{-1}(A \circ B)T = (T^{-1}AT) \circ (T^{-1}BT),$$

where the left product is calculated in $\mathrm{M}(n,R,\Sigma)$, and the right product is calculated in $\mathrm{M}(n,R,\tau\Sigma)$.

(b) As above, Σ is a system of factors s_{ijk}, and α is an endomorphism of the ring R. We set $t_{ijk} = \alpha(s_{ijk})$. Then $\{t_{ijk}\}$ is a system of factors. We denote it by $\alpha\Sigma$. There exists a ring homomorphism

$$\mathrm{M}(n,R,\Sigma) \to \mathrm{M}(n,R,\alpha\Sigma), \quad (a_{ij}) \to (\alpha a_{ij}).$$

It is an isomorphism provided α is an automorphism of the ring R.

(c) If $\Sigma = \{s_{ijk}\}$ and $X = \{x_{ijk}\}$ are two systems of factors, then $\{x_{ijk}s_{ijk}\}$ also is a system of factors. We denote it by $X\Sigma$. For every $l = 1,\dots,n$, there exists a homomorphism

$$\zeta\colon \mathrm{M}(n,R,X\Sigma) \to \mathrm{M}(n,R,\Sigma), \quad (a_{ij}) \to (x_{ijl}a_{ij}).$$

If all factors x_{ijk} are invertible elements, then ζ is an isomorphism.

Proof. We take matrices $A = (a_{ij})$ and $B = (b_{ij})$ in the ring $\mathrm{M}(n,R,X\Sigma)$. We have $AB = (c_{ij})$, where

$$c_{ij} = \sum_{k=1}^{n} x_{ikj}s_{ikj}a_{ik}b_{kj}.$$

On the position (i,j) of the matrix $\zeta(AB)$, we have the element

$$\sum_{k=1}^{n} x_{ijl}x_{ikj}s_{ikj}a_{ik}b_{kj}.$$

Further, we have $\zeta(A) = (x_{ijl}a_{ij})$, $\zeta(B) = (x_{ijl}b_{ij})$, and the matrix $\zeta(A)\zeta(B)$ has the element

$$\sum_{k=1}^{n} s_{ikj}x_{ikl}x_{kjl}a_{ik}b_{kj}$$

on the position (i,j). However, it follows from relations (5.2) that $x_{ikj} \cdot x_{ijl} = x_{ikl} \cdot x_{kjl}$. Therefore, ζ preserves products. If all x_{ijk} are invertible, then ζ has the inverse homomorphism $\zeta^{-1}\colon (c_{ij}) \to (x_{ijl}^{-1}c_{ij})$. □

Let E_{ij} denote the matrix unit, i.e., the matrix E_{ij} has 1 on the position (i,j), and E_{ij} has 0 on remaining positions. We remark that $E_{ij} \cdot E_{jk} = s_{ijk}E_{ik}$.

(d) For every system of factors $\Sigma = \{s_{ijk}\}$, the set $\Sigma^t = \{t_{ijk} \mid t_{ijk} = s_{kji},\ i,j,k = 1,\dots,n\}$ also is a system of factors. Therefore, if the ring $\mathrm{M}(n,R,\Sigma)$ exists, then the ring $\mathrm{M}(n,R,\Sigma^t)$ also exists; it is denoted by K^t. In addition, if $S, S_1,\dots,S_n$ are matrices of factors of the ring K, then $S, S_1^t,\dots,S_n^t$ are matrices of factors of the ring K^t.

If R is a commutative ring, then the transposition $A \to A^t$ is an anti-isomorphism between the rings K and K^t. Thus, the relation $(AB)^t = B^tA^t$ holds, where the right product is calculated in K^t. This implies one implication of the following assertion.

Proposition 5.1. *Let R be a commutative ring. The relation $(AB)^t = B^tA^t$ holds for all matrices A and B if and only if $s_{ikj} = s_{jki}$ for all subscripts i, k, and j (i.e., the matrices $S_1,\dots,S_n$ are symmetric).*

Proof. The remaining implication follows from the relations

$$(E_{ik} \cdot E_{kj})^t = (s_{ikj} E_{ij})^t = s_{ikj} E_{ji}, \quad E_{kj}^t \cdot E_{ik}^t = E_{jk} E_{ki} = s_{jki} E_{ji}. \qquad \square$$

By the item (c), the ring $\mathrm{M}(n, R, \Sigma\Sigma^t)$ also exists. All matrices of factors of $\mathrm{M}(n, R, \Sigma\Sigma^t)$ are symmetric.

We consider one homomorphism that will be quite useful later. It is a particular case of the homomorphism ζ from the item (c).

Given a formal matrix ring $\mathrm{M}(n, R, \Sigma)$, $\Sigma = \{s_{ijk}\}$, we fix a subscript $l = 1, \ldots, n$ and set $t_{ij} = s_{ijl}$ for all $i, j = 1, \ldots, n$. We remark that $t_{ii} = 1$, and it follows from the main relations (5.2) that $s_{ijk} \cdot t_{ik} = t_{ij} \cdot t_{jk}$. In particular, $s_{iji} = t_{ij} \cdot t_{ji}$.

We define a mapping

$$\eta \colon \mathrm{M}(n, R, \Sigma) \to \mathrm{M}(n, R), \quad (a_{ij}) \to (t_{ij} a_{ij}).$$

Proposition 5.2.

(1) *η is a ring homomorphism;*

(2) *if for all i, j, and k, the factor s_{ikj} is divided by t_{ik} or t_{kj}, then* $\mathrm{Ker}(\eta)$ *is a nilpotent ideal of nilpotency index* 2;

(3) *the mapping η is injective if and only if all s_{ijk} are nonzero divisors;*

(4) *η is an isomorphism if and only if all elements s_{ijk} are invertible.*

Proof. (1) The assertion follows from (c) if we take the system consisting of 1 as Σ and take Σ as X.

(2) We take arbitrary matrices $A = (a_{ij})$ and $B = (b_{ij})$ in the ring $\mathrm{M}(n, R, \Sigma)$. If $\eta A = 0 = \eta B$, then $t_{ij} a_{ij} = 0 = t_{ij} b_{ij}$ for all i and j. It follows from the proof of (c) that $AB = 0$. $\square$

At the end of the section, we consider formal matrix rings of order 2 and 3 over a ring R. In the case $n = 2$, it is easy to obtain a complete answer. Indeed, we take an arbitrary ring $\mathrm{M}(2, R, \Sigma)$ and clarify the action of the multiplication in it.

By the use of the previous material, it is easy to prove the following property. There exists an element $s \in C(R)$ such that matrices in $\mathrm{M}(2, R, \Sigma)$ are multiplied by the relation

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + s a_{12} b_{21} & a_{11} b_{12} + a_{12} b_{22} \\ a_{21} b_{11} + a_{22} b_{21} & s a_{21} b_{12} + a_{22} b_{22} \end{pmatrix}. \tag{5.6}$$

The converse is also true. Any central element s of the ring R defines a ring of formal 2×2 matrices over R such that the multiplication in the ring satisfies the relation (5.6). Such rings are introduced in [12]; they are denoted by K_s in this paper. They are also studied and used in [6, 7, 10, 15, 17, 21, 22].

The situation with classification of rings $\mathrm{M}(3, R, \Sigma)$ is more complicated. The main relations (5.2) turn into the relations

$$s_{iik} = 1 = s_{ikk}, \quad i, k = 1, 2, 3,$$
$$s_{121} = s_{212} = s_{123} \cdot s_{213} = s_{321} \cdot s_{312},$$
$$s_{131} = s_{313} = s_{132} \cdot s_{312} = s_{231} \cdot s_{213},$$
$$s_{232} = s_{323} = s_{231} \cdot s_{321} = s_{132} \cdot s_{123}.$$

Every family of elements $\{s_{ijk} \mid i, j, k = 1, 2, 3\}$, which satisfy the previous relations, defines a formal matrix ring over R of order 3. If $s_{ikj} = s_{jki}$ for all $i, j, k = 1, 2, 3$, i.e., the matrices S_1, S_2, and S_3 are symmetric, then we can write these relations more compactly:

$$s_{iik} = 1 = s_{ikk}, \quad i, k = 1, 2, 3,$$
$$s_{121} = s_{212} = s_{123} \cdot s_{213},$$
$$s_{131} = s_{313} = s_{132} \cdot s_{312},$$
$$s_{232} = s_{323} = s_{231} \cdot s_{321}.$$

In connection with Problem (I), we can certainly say that not every symmetric matrix of order 3 can be the matrix S for some ring $\mathrm{M}(3, R, \Sigma)$.

6. Some Properties of Formal Matrix Rings over R

First of all, we remark that many results were proved by Tang and Zhou [23] for one class of formal matrix rings over R; see Section 7 on such rings.

Let $\mathrm{M}(n, R, \Sigma)$ be some formal matrix ring of order n over the ring R. We recall that $\Sigma = \{s_{ijk} \mid i, j, k = 1, \dots, n\}$ is a system of factors. The factors s_{ijk} satisfy relations (5.2) and their corollaries (5.4) and (5.5) from Sec. 5. Sometimes, we denote by R_{ij} the ring R that stands on the position (i, j).

We use Theorems 4.3 and 4.4 for calculation the Jacobson radical and the prime radical of the ring $\mathrm{M}(n, R, \Sigma)$. In the notation of Theorem 4.3, we have

$$J(R_{ij}) = \{x \in R_{ij} \mid R_{ji} \circ x \subseteq J(R)\}.$$

Further, $R_{ji} \circ x = s_{jij}Rx = s_{iji}Rx$. Therefore,

$$J(R_{ij}) = \{x \in R_{ij} \mid s_{iji}x \in J(R)\}.$$

We remark that the ideal $J(R_{ij})$ coincides with the intersection of all maximal left (right) ideals of the ring R that do not contain s_{iji}. We denote this ideal $J(R_{ij})$ by $J_{ij}(R)$. Now the following result follows from Theorem 4.3.

Corollary 6.1. *The Jacobson radical of the ring* $\mathrm{M}(n, R, \Sigma)$ *is equal to*

$$\begin{pmatrix} J(R) & J_{12}(R) & \dots & J_{1n}(R) \\ J_{21}(R) & J(R) & \dots & J_{2n}(R) \\ \dots & \dots & \dots & \dots \\ J_{n1}(R) & J_{n2}(R) & \dots & J(R) \end{pmatrix}. \tag{6.1}$$

The prime radical of the ring $\mathrm{M}(n, R, \Sigma)$ has a similar structure. We set

$$P(R_{ij}) = \{x \in R_{ij} \mid s_{iji}x \in P(R)\}.$$

Corollary 6.2. *The prime radical of the ring* $\mathrm{M}(n, R, \Sigma)$ *coincides with the ideal of matrices of the form* (6.1); *we only have to replace the symbol* J *by* P.

We present an internal description of ideals of the ring $\mathrm{M}(n, R, \Sigma)$ and determine the structure of its factor rings. For an arbitrary formal matrix ring, these questions were briefly considered in Secs. 1 and 4. The following proposition is directly verified.

Proposition 6.3. *Let* I *be an ideal of the ring* $\mathrm{M}(n, R, \Sigma)$. *Then*

$$I = \begin{pmatrix} I_{11} & I_{12} & \dots & I_{1n} \\ I_{21} & I_{22} & \dots & I_{2n} \\ \dots & \dots & \dots & \dots \\ I_{n1} & I_{n2} & \dots & I_{nn} \end{pmatrix},$$

where all I_{ij} *are ideals in* R. *In addition, the following relations hold*:

$$I_{ii} \subseteq \bigcap_{l=1}^{n} (I_{il} \cap I_{li}), \quad s_{iji}I_{ij} \subseteq I_{ii} \cap I_{jj}$$

for all $i, j = 1, \dots, n$, *and*

$$s_{ikj}I_{kj} \subseteq I_{ij}, \quad s_{jki}I_{jk} \subseteq I_{ji}$$

for all distinct i, j, *and* k.

The following proposition is also proved with the use of standard methods.

Proposition 6.4. *Let $I = (I_{ij})$ be an ideal of the ring $K = \mathrm{M}(n, R, \Sigma)$.*

(1) *The set of matrices*

$$\bar{K} = \begin{pmatrix} R/I_{11} & R/I_{12} & \dots & R/I_{1n} \\ R/I_{21} & R/I_{22} & \dots & R/I_{2n} \\ \dots & \dots & \dots & \dots \\ R/I_{n1} & R/I_{n2} & \dots & R/I_{nn} \end{pmatrix}$$

is a formal matrix ring with bimodule homomorphisms

$$\varphi_{ijk} \colon R/I_{ij} \otimes_{R/I_{jj}} R/I_{jk} \to R/I_{ik}, \quad \varphi_{ijk}(\bar{x} \otimes \bar{y}) = s_{ijk}xy + I_{ik}$$

for all $i, j, k = 1, \dots, n$.

(2) *There is an isomorphism*

$$K/I \cong \bar{K}, \quad (x_{ij}) + I \to (x_{ij} + I_{ij}).$$

For our matrix rings, we consider several usual ring properties. We recall that E_{ij} are matrix units.

Proposition 6.5. *Let R be a ring and let K be a formal matrix ring of order n over R with factors s_{ijk}, $i, j, k = 1, \dots, n$. The following assertions hold.*

(1) *K is a simple ring if and only if R is a simple ring and all factors s_{ijk} not are equal to zero.*
(2) *K is a prime ring if and only if R is a prime ring and all s_{ijk} not are equal to zero.*
(3) *K is a regular ring if and only if R is a regular ring and all elements s_{ijk} are invertible in R.*
(4) *K is a semiprimitive ring if and only if R is a semiprimitive ring and all s_{ijk} are nonzero divisors in R.*
(5) *K is a semiprime ring if and only if R is a semiprime ring and all s_{ijk} are nonzero divisors in R.*

Proof. (1) $\Longrightarrow$. If I is a nonzero ideal in R, then

$$\begin{pmatrix} I & \dots & I \\ \dots & \dots & \dots \\ I & \dots & I \end{pmatrix}$$

is an ideal in K. Therefore, since the ring K is simple, we have $I = R$. Consequently, R is a simple ring. The homomorphism η from Proposition 5.2 need to be injective. Therefore, each s_{ijk} is not equal to zero.

$\Longleftarrow$. The ring $\mathrm{M}(n, R)$ is simple. All central elements s_{ijk} are invertible, since the center of a simple ring is a field. By Proposition 5.2, $K \cong \mathrm{M}(n, R)$; consequently, K is a simple ring.

(2) $\Longrightarrow$. Since $R \cong E_{11}KE_{11}$, we have that R is a prime ring. We assume that $s_{ijk} = 0$ for some i, j, and k. Then $E_{ij}KE_{kl} = 0$. This contradicts the property that K is a prime ring.

$\Longleftarrow$. The center of a prime ring is a domain. Therefore, all s_{ijk} are nonzero divisors. We assume that $(a_{ij})K(b_{ij}) = 0$ for some nonzero matrices (a_{ij}) and (b_{ij}). For example, let $a_{kl} \neq 0$ and $b_{mp} \neq 0$. Then $s_{klm}s_{kmp}a_{kl}Rb_{mp} = 0$, whence $a_{kl}Rb_{mp} = 0$; this contradicts the primeness of the ring R. Thus, K is a prime ring.

(3) $\Longrightarrow$. Since the ring K is regular and $R \cong E_{11}KE_{11}$, the ring R is regular. Consequently, there exists a matrix (a_{ij}) with $E_{ij} = E_{ij}(a_{ij})E_{ij}$. Therefore, we obtain the relation $s_{iji}a_{ji} = 1$, and all elements s_{iji} are invertible. From relations (5.4), we obtain that all elements s_{ijk} also are invertible.

$\Longleftarrow$. The ring $\mathrm{M}(n, R)$ is regular and $K \cong \mathrm{M}(n, R)$ by Proposition 5.2.

(4) and (5). The assertions follow from relations (5.4) of Sec. 5 and Corollaries 6.1 and 6.2, respectively. □

7. Characterization of Matrices of Factors

In this section and in the following two sections, R denotes an arbitrary ring, $\mathrm{M}(n, R, \Sigma)$ is a formal matrix ring of order n over the ring R, and Σ is a system of factors $\{s_{ijk} \mid i, j, k = 1, \dots, n\}$. The factors s_{ijk}

satisfy relations (5.2). The element that stands on the position (i, j) of the product of matrices (a_{ij}) and (b_{ij}) from the ring $\mathrm{M}(n, R, \Sigma)$ is equal to $\sum_{k=1}^{n} s_{ikj} a_{ik} b_{kj}$.

In Sec. 5, we formulated Problem (I): Which matrices can be matrices of factors for formal matrix rings? Problem (II) is related to the description of the rings $\mathrm{M}(n, R, \Sigma)$ in terms of systems of factors Σ. In the general case, it is difficult to solve these problems; one of the reasons is that the verification of relations (5.2) is difficult.

The situation seems to be more simple if all factors s_{ijk} are integral powers of some central element s of the ring R. We know that for $n = 2$, other possibilities do not exist (see the end of Sec. 5).

For $n \geq 3$, the situation with characterization and classification of rings $\mathrm{M}(n, R, \Sigma)$ becomes more complicated, even if all factors s_{ijk} are powers of the element s. The particular case in which every factor s_{ijk} is equal to s^m for some $m \geq 1$ is interesting. Tang and Zhou [23] explicitly study the rings $\mathrm{M}(n, R, \Sigma)$ such that $s_{iji} = s^2$ for $i \neq j$ and $s_{ijk} = s$ for all pairwise distinct i, j, and k. We call such rings *Tang–Zhou rings*.

Here and in Sec. 8, we consider the case where either $s_{ijk} = 1$ or $s_{ijk} = s$ for all i, j, and k, where s is some central element of the ring R. We impose simple additional restrictions on the element s. Any corresponding matrix ring is denoted by $\mathrm{M}(n, R, s)$; it is clear that for fixed n, there exists only a finite number of distinct such rings. The element s is called a *factor* of each such ring $\mathrm{M}(n, R, s)$.

Further, we assume that we have a ring $\mathrm{M}(n, R, s)$. In addition, we assume that $s^2 \neq 1$ and s is not an idempotent. Without large loss of generality, it can be assumed that s is not an invertible element. Indeed, if the element s is invertible, then $\mathrm{M}(n, R, s) \cong \mathrm{M}(n, R)$ by Proposition 5.2. The following condition is more strong: $s^k \neq s^l$ for all nonnegative distinct k and l.

Thus, we assume that we have a ring $\mathrm{M}(n, R, s)$, where the element s^2 is not equal to 1 or s; in particular, $s \neq 0$ and $s \neq 1$. We present some relations between factors of the form s_{iji}.

Lemma 7.1. *Let i, j, and k be three pairwise distinct subscripts. Then only one of the following three cases is possible for the elements s_{iji}, s_{iki}, and s_{jkj}.*

(1) *All three elements are equal to* 1.
(2) *Some two elements of these three elements are equal to s, and the third element is equal to* 1.
(3) *All three these elements are equal to s.*

Proof. It directly follows from relations (5.4) that the situation is impossible, where some two of three factors s_{iji}, s_{iki}, s_{jkj} are equal to 1, and the third factor is equal to s. Therefore, the cases, mentioned in (1)–(3), remain. Each of these cases actually appears; this follows from what follows. ☐

In Sec. 5, we constructed square matrices $S, S_1, \ldots, S_n$ of order n from factors s_{ijk} of some formal matrix ring. We call these matrices *matrices of factors* of a given ring. In what follows, the words "the matrix of factors" usually mean the matrix S.

It is useful to define the notion of an abstract matrix of factors. As earlier, let s be some central element of the ring R such that $s^2 \neq 1$ and $s^2 \neq s$. Let $T = (t_{ij})$ be a symmetric matrix of order n such that all elements of T are equal to 1 or s, the main diagonal consists of 1, and for all three elements t_{ij}, t_{ik}, and t_{jk}, one of the assertions (1), (2), or (3) in Lemma 7.1 holds. Such a matrix T is called a *matrix of factors*. If τ is some permutation of order n, then $\tau T = (t_{\tau(i)\tau(j)})$ also is a matrix of factors (matrices of the form τT are defined in item (a) of Sec. 5).

Let T be a matrix such that T can be represented in the block form such that the blocks, standing on the main diagonal, consist of 1, and the element s stands on all remaining positions. It is clear that T is a matrix of factors. In such a case, we say that T *has the canonical form*.

Lemma 7.2. *For every matrix of factors T, there exists a permutation σ such that the matrix σT has the canonical form.*

Proof. We use induction on n. For $n = 2$, there exists a unique matrix of factors

$$\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix},$$

which has the canonical form.

We assume that the assertion of the lemma is true for all matrices of order $\le n-1$, where $n \ge 3$. Let T be a matrix of order n. We take the submatrix T' of order $n-1$ that stands in the right lower corner of the matrix T. There exists a permutation τ' of integers $2, \dots, n$ such that the matrix $\tau' T'$ has the canonical form. We take a permutation τ of order n that coincides with τ' on integers $2, \dots, n$. In the right lower corner of the matrix τT, the matrix $\tau' T'$ stands on all remaining positions. If all elements of the first row of the matrix τT, beginning with the second element, are equal to s, then τT already has the canonical form.

Now we assume that the first row of the matrix τT contains 1; of course, we exclude the position $(1,1)$. Let the second block that is on the main diagonal (after the position $(1,1)$) have the order $m-1$, $m \ge 2$. In such a case, the elements $t_{12}, \dots, t_{1m}$ are simultaneously equal to 1 or s. Indeed, if $t_{1i} = 1$ and $t_{1j} = s$, $2 \le i, j, \le m$, then $t_{ij} = s$, which is impossible. We assume that $t_{12} = \cdots = t_{1m} = 1$. If $t_{1l} = 1$ for some l, $m+1 \le l \le n$, then $t_{2l} = 1$, which is also impossible. Consequently, $t_{1\,m+1} = \cdots = t_{1n} = s$. Now if we unite the first block and the second block that stand on the main diagonal, then we obtain the canonical form of the matrix τT.

Thus, we can assume that $t_{12} = \cdots = t_{1m} = s$. What can we say about elements $t_{1\,m+1} = \cdots = t_{1n}$? We assume that $t_{1i} = 1$, $t_{1j} = 1$, and $t_{1k} = 1$, where $m+1 \le i, j, k \le n$ and $i < j < k$. In such a case, we have $t_{ij} = s$, $t_{ik} = 1$; this is a contradiction. Therefore, the sequence $t_{1\,m+1}, \dots, t_{1n}$ can have only one of the following three forms:

(1) $111 \dots sss$;
(2) $sss \dots 111 \dots sss$;
(3) $sss \dots 111$.

The following fact is important: below the sequence $11 \dots 1$ from (1), (2), or (3), the main diagonal contains a block, the order of which is equal to the number of 1's in the given sequence.

In the cases (1) and (2), we act further as follows. We take the submatrix such that the end of its right lower corner is the block that corresponds to the sequence $1\,1 \dots 1$. By the use of some permutation, we reduce this submatrix to the canonical form. Then we apply the same permutation to the whole matrix (we add missing monomial cycles), and we reduce it to the canonical form (under these actions, the submatrix that is placed in the right lower corner does not change).

We consider the remaining case (3). Let the block on the main diagonal, which is placed under the sequence $1\,1 \dots 1$, begin with the row of number $k+1$, where $k \ge 2$. We apply the cycle $(1\,2\, \dots\, k)$ to the matrix. As a result, the block of order $k+1$, which consists of 1's, will appear in the right lower corner. In the first row from the right and in the first column from the left, the element s is placed instead of 1. On the positions $(k,1), \dots, (k, k-1)$ and $(1,k), \dots, (k-1,k)$, the element s is also placed. Thus, to the left and up from the block, consisting of 1's, the element s stands on all positions of the right lower corner. By the use of some permutation, we reduce to the canonical form the submatrix that is placed in rows and columns with numbers $1, 2, \dots, k-1$. After this, the whole matrix will have the canonical form. $\square$

In the situation of the lemma, we say that *the matrix T is reduced to the canonical form σT*. The converse is also true, i.e., if some matrix T is reduced to the canonical form σT, then T is a matrix of factors. Thus, matrices of factors coincide with matrices that can be reduced to the canonical form by permutations.

Let the matrix T have the canonical form. Then blocks that stand on the main diagonal of the matrix can be can arranged in any required order. In other words, there exists a permutation τ such that blocks in the matrix τT are placed in the required order, and τT has the canonical form. For this purpose, it is sufficient to show that if T consists of two blocks, then we can interchange them. Let the first block have the order k, $1 \le k \le n-1$. Then we can take the permutation

$$\begin{pmatrix} 1 & 2 & \dots & n-k & n-k+1 & \dots & n \\ k+1 & k+2 & \dots & n & 1 & \dots & k \end{pmatrix}$$

as τ.

We formulate the main result of the section. First, we remark that one particular case, which appears in the proof, will be considered in the next section.

Theorem 7.3. *Let T be a matrix of factors. There exists a ring $\mathrm{M}(n, R, s)$ such that T is a matrix of factors of $\mathrm{M}(n, R, s)$.*

Proof. We use induction on n. For $n = 2$, only one matrix of factors exists; namely, the matrix

$$\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}.$$

We know that there always exists a ring $\mathrm{M}(2, R, s)$ with matrix of factors

$$\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}.$$

Now we assume that $n > 2$. Let σ be a permutation such that the matrix σT has the canonical form (see Lemma 7.2). The following two cases are possible.

Case 1. All blocks that stand on the main diagonal have the order 1. In Sec. 8, it will be shown that there exists a ring $\mathrm{M}(n, R, s)$ with matrix of factors σT.

Case 2. Not all blocks mentioned in the case 1 have the order 1. Before the theorem, it is remarked that there exists a permutation τ such that the matrix $\tau\sigma T$ has the canonical form, and the order of the lowest block exceeds 1. In $\tau\sigma T$, we take the submatrix T' of order $n-1$ placed in the left upper corner. It has the canonical form. By the induction hypothesis, there exists a ring K of the form $\mathrm{M}(n-1, R, s)$ with matrix of factors T'. We represent this ring as a block matrix ring of order 2:

$$\left(\begin{array}{ccc|c} R & \dots & R & R \\ \cdot & \dots & \cdot & \cdot \\ R & \dots & R & R \\ \hline R & \dots & R & R \end{array}\right).$$

Now we apply the construction of the ring K_4 from Sec. 3 to the ring K. The matrix of factors of the obtained ring K_4 coincides with the matrix $\tau\sigma T$. The ring $\sigma^{-1}\tau^{-1}K_4$ is the required formal matrix ring. Indeed, the matrix T is a matrix of factors of this ring. $\square$

We return to the action of permutations on the set of formal matrix rings (see the item (a) of Sec. 5). It is clear that this action can be restricted to the set of rings $\mathrm{M}(n, R, s)$ of the considered form with fixed factor s. Every permutation τ also acts on every matrix $A = (a_{ij})$; namely, $\tau A = \big(a_{\tau(i)\tau(j)}\big)$. In particular, τ acts on matrices of factors S. In addition, it follows from Lemma 7.2 that the orbits consist of matrices that have the same canonical form. Under the action of permutations, the set of matrices of factors is decomposed into orbits. The number of these orbits is equal to the number of representations of the integer n in the form of sums of positive integers that are less than n.

If the ring $K = \mathrm{M}(n, R, s)$ has the matrix of factors S, then τS is a matrix of factors of the ring τK for every permutation τ. Consequently, if two rings are contained in the same orbit, then the corresponding matrices of factors are also contained in the same orbit. Obviously, the converse is not true. The reason is that a given matrix of factors S can have several series of matrices of factors $S_1, \dots, S_k$. Therefore, the number of orbits for rings is not known. In several particular cases, the number will be presented in the next section.

In Sec. 3, for a given matrix ring

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix},$$

we introduced four methods of constructing matrix rings of larger order. We call these methods *constructions of* K_1*,* K_2*,* K_3*, and* K_4. Now we assume that we have a ring $\mathrm{M}(n, R, s) = L$ with matrices of factors $S, S_1, \ldots, S_n$. We assume that we applied the mentioned constructions to L. The obtained rings are denoted by L_1, L_2, L_3, and L_4, respectively. The matrices of factors $S', S'_1, \ldots, S'_n, S'_{n+1}$ of any these rings are obtained from the matrices $S, S_1, \ldots, S_n$ with the use of a certain regularity. We consider the construction K_4 more explicitly. We must represent the ring L in the block form, as a matrix ring of order 2 (see Sec. 3),

$$L = \begin{pmatrix} P & M \\ N & S \end{pmatrix},$$

where P is a matrix ring of order k, $1 \leq k \leq n-1$; S, M, and N are mentioned in Sec. 3. For simplicity, we take $k = n-1$. We apply the construction K_4 to L and obtain the matrix ring L_4 of order $n+1$:

$$L_4 = \begin{pmatrix} L & \begin{pmatrix} M \\ S \end{pmatrix} \\ \begin{pmatrix} N & S \end{pmatrix} & S \end{pmatrix}.$$

By considering the structure of matrices in the ring L_4, we obtain that the matrices of factors $S', S'_1, \ldots, S'_n$ of the ring L_4 are obtained from the corresponding matrices $S, S_1, \ldots, S_n$ by the same method, and $S'_{n+1} = S'_n$. Namely, to each matrix $S, S_1, \ldots, S_n$, we need to add from the right the last column of this matrix; to each matrix $S, S_1, \ldots, S_n$, we need to add from below the last row of this matrix; on the position $(n+1, n+1)$, we need to put the element that stands on the position (n, n).

We formulate the following question, which will appear again at the end of Sec. 8. We apply the permutation τ of order n to the ring L. Then we apply the construction K_4 to the ring τL; the ring $(\tau L)_4$ will be obtained. Which interrelations exist between the rings L_4 and $(\tau L)_4$? Is it possible to transform one of these rings into the second ring by some permutation?

At the end of the section, we consider excluded values of the factor s: $s^2 = 1$ and $s^2 = s$. If $s^2 = 1$, then the element s is invertible and $\mathrm{M}(n, R, s) \cong \mathrm{M}(n, R)$ by Proposition 5.2. The case $s^2 = s$ (i.e., s is an idempotent) is more interesting. We present one characteristic example. We set $s_{iik} = 1 = s_{ikk}$ and $s_{ijk} = s$ for $i \neq j$ and $j \neq k$. Then $\Sigma = \{s_{ijk}\}$ is a system of factors. Consequently, there exists a ring $\mathrm{M}(n, R, \Sigma)$.

8. Classification of Formal Matrix Rings

We continue to consider the topic considered in the previous section. Our main concern is with Problem (II) formulated in Sec. 5. We will also complete the proof of Theorem 7.3 related to Problem (I). We preserve the notation and the conventions of Sec. 7.

It follows from Theorem 7.3 that for any matrix of factors T, there exists a ring $\mathrm{M}(n, R, s)$ such that its matrix of factors S coincides with T. Several such rings $\mathrm{M}(n, R, s)$ can exist, since for a given matrix S, we can have different families of matrices of factors $S_1, \ldots, S_n$ that correspond to different systems of factors Σ. In the given situation, the classification problem (II) consists in listing all such rings $\mathrm{M}(n, R, s)$ for fixed s.

Depending on the canonical form of the matrix S, we study the following two cases.

(1) The main diagonal of the matrix S consists of two blocks.

(2) All blocks on the main diagonal of the matrix S have the order 1.

In the first case, the matrices $S_1, \ldots, S_n$ are symmetric, and they are uniquely determined by the matrix S.

Similar to Sec. 7, we assume that the matrix of factors of the ring $\mathrm{M}(n, R, s)$ is the matrix S, unless otherwise specified.

Lemma 8.1. *Let us have a ring* $\mathrm{M}(n, R, s)$ *and let* $S, S_1, \ldots, S_n$ *be the matrices of factors of the ring. The following conditions are equivalent.*

(1) *The canonical form of the matrix* S *contains exactly two blocks.*

(2) *For any three elements* s_{iji}, s_{iki}, *and* s_{jkj}, *there is realized either the case* (1), *or the case* (2) *from Lemma* 7.1.
(3) *The matrices* $S_1, \dots, S_n$ *are symmetric.*

Proof. (1) $\Longrightarrow$ (2). Let τ be an arbitrary permutation. If for elements of the matrix S holds (2), then the same is true for elements of the matrix τS, and conversely. Therefore, we can assume that S has the canonical form. Then it is easy to verify that (2) is true.

(2) $\Longrightarrow$ (1). We again assume that S has the canonical form. If we will assume that the number of blocks is not less than 3, then we will quickly obtain a contradiction.

(2) $\Longrightarrow$ (3). We take any three elements s_{iji}, s_{iki}, and s_{jkj}, where the subscripts i, j, and k are pairwise distinct. By (2), either all these elements are equal to 1, or some two of them are equal to s, and the third element is equal to 1. In any case, it follows from relations (5.4) that $s_{ikj} = s_{jki}$.

(3) $\Longrightarrow$ (2). If we assume that

$$s_{iji} = s_{iki} = s_{jkj} = s,$$

then we obtain that all remaining six factors in relation (5.5) are equal to each other. This contradicts relations (5.4). □

Further, we consider rings $\mathrm{M}(n, R, s)$ such that their matrices of factors satisfy equivalent conditions (1)–(3) of Lemma 8.1. It is easy to prove the following assertion: each row of the matrix S uniquely determines the remaining rows. We reformulate this assertion. For this purpose, we pass to the "additive" representation of the matrices $S, S_1, \dots, S_n$; i.e., we replace all elements by corresponding exponents of the element s (as usual, we set $s^0 = 1$). The obtained matrices consist of 0 and 1. (These matrices are examples of Boolean matrices from [11].) We denote by S^+ the matrix constructed from the matrix S. For elements of the matrix S^+, we preserve the notation s_{iji} or, more briefly, s_{ij}. For elements of the matrix S^+ in the field $\mathbb{Z}/2\mathbb{Z}$, the assertion (2) of Lemma 8.1 has the form $s_{ij} + s_{ik} = s_{jk}$. This relation is also true if some two subscripts are equal to each other.

The elements s_{ikj} with distinct i and j (i.e., elements of the matrix S_k) are uniquely determined by elements of the form s_{iji}. More precisely, elements of the main diagonal of the matrix S_k (this diagonal coincides with kth row of the matrix S) determine all remaining elements of this matrix if we take into account relations (5.4). In the field $\mathbb{Z}/2\mathbb{Z}$, the relation $s_{ikj} = s_{ik} \cdot s_{jk}$ is true. It also remains true for equal subscripts. Thus, the matrix S completely determines matrices S_k, $k = 1, \dots, n$.

We can present a complete review of the rings $\mathrm{M}(n, R, s)$ considered in Lemma 8.1.

We specialize that the ring $\mathrm{M}(n, R)$ also falls under the item (3) of the following theorem. We obtain this ring provided all factors s_{ijk} are equal to $1 = s^0$. In the item (2), the matrix, consisting of 1, corresponds to $\mathrm{M}(n, R)$.

Theorem 8.2. *There exists a one-to-one correspondence between the following three sets.*
(1) *The set of sequences of length* $n - 1$ *that consist of* 0 *and* 1.
(2) *The set of matrices of factors of order* n *that satisfy equivalent conditions* (1) *and* (2) *of Lemma* 8.1.
(3) *The set of rings* $\mathrm{M}(n, R, s)$ *whose matrices of factors satisfy conditions* (1), (2), *and* (3) *of Lemma* 8.1.

Proof. In fact, a bijection between sets (1) and (2) is already obtained. Namely, if $T = (t_{ij})$ is some matrix of factors in the set (2), then we associate the sequence $t_{12}, \dots, t_{1n}$ to the matrix T (as indicated above, we replace elements t_{ij} by corresponding exponents of the element s; in what follows, we act similarly). Conversely, let us have some sequence from the set (1). We add 0 to this sequence from the left. We take this extended sequence as the first row of the matrix (t_{ij}) of order n. The remaining elements of the matrix are obtained with the use of the relation $t_{1j} + t_{1k} = t_{jk}$ (in the field $\mathbb{Z}/2\mathbb{Z}$). We also have the relation $t_{ij} + t_{ik} = t_{jk}$ for all $i = 2, \dots, n$ and all j and k. Now we replace 0 (1) by 1 (respectively, s) in this matrix. The constructed matrix is a matrix from the set (2). The above correspondence between sequences in (1) and matrices in (2) is a bijection.

Now we pass to a bijection between sets (2) and (3); this bijection exists even in practice. If T is some matrix in the set (2), then by Theorem 7.3, there exists a ring $\mathrm{M}(n, R, s)$ such that T is a matrix of factors for $\mathrm{M}(n, R, s)$. Now we remark that the coincidence of two rings of the form $\mathrm{M}(n, R, \{s_{ijk}\})$ means that multiplication operations in these rings are equal to each other. It follows from the definition of such rings that systems of factors of these rings $\mathrm{M}(n, R, \{s_{ijk}\})$ are equal to each other. Therefore, it is clear that distinct rings $\mathrm{M}(n, R, s)$ correspond to distinct matrices T.

Conversely, if the ring $\mathrm{M}(n, R, s)$ belongs to the set (3), then we associate the matrix of factors S of this ring with the ring $\mathrm{M}(n, R, s)$. It was remarked earlier that factors s_{ijk} with distinct subscripts i and k are uniquely determined by factors of the form s_{iji}. In other words, matrices of factors $S_1, \dots, S_n$ of the ring $\mathrm{M}(n, R, s)$ are uniquely determined by the matrix S. Therefore, distinct matrices S correspond to distinct rings $\mathrm{M}(n, R, s)$. □

Corollary 8.3. *In Theorem* 8.2, *the number of rings in the set* (3) *is equal to* 2^{n-1}. *Consequently, this number not depends on the ring R and the element s.*

Now we describe one general situation. Let us have an arbitrary set of central elements $\{s_{ijk} \mid i, j, k = 1, \dots, n\}$ of the ring R. We need to verify whether this set satisfies relations (5.2), i.e., whether the set is a system of factors. For this purpose, in particular, we need to verify the relation

$$s_{ijk} \cdot s_{ikl} = s_{ijl} \cdot s_{jkl} \tag{8.1}$$

for all pairwise distinct subscripts i, j, k, and l. A situation is possible in which for any such four subscripts only the mutual disposition of the subscripts i, j, k, l is important, i.e., the order between subscripts is only important. A similar situation appears below. For convenience, we assume that $i = 1$, $j = 2$, $k = 3$, and $l = 4$. By considering all variants of the arrangement of integers i, j, k, l, we obtain that the number of corresponding relations (8.1) is equal to 24. It is convenient to arrange these relations in the form of four series containing six relations in every series. We present the first series such that the subscripts i, j, and k can be equal to any integer from the set $\{1, 2, 3\}$:

$$\begin{aligned} &s_{123} \cdot s_{134} = s_{124} \cdot s_{234}, \quad s_{321} \cdot s_{314} = s_{324} \cdot s_{214}, \\ &s_{312} \cdot s_{324} = s_{314} \cdot s_{124}, \quad s_{213} \cdot s_{234} = s_{214} \cdot s_{134}, \\ &s_{231} \cdot s_{214} = s_{234} \cdot s_{314}, \quad s_{132} \cdot s_{124} = s_{134} \cdot s_{324}. \end{aligned} \tag{8.2}$$

There are three additional series for values of subscripts $\{1, 2, 4\}$, $\{1, 3, 4\}$, and $\{2, 3, 4\}$.

Now we consider the case (2) mentioned at the beginning of the section. Namely, we study another encountered situation for the ring $\mathrm{M}(n, R, s)$, where all factors s_{iji}, $i \neq j$, are equal to s. In other words, the matrix of factors of such a ring has blocks of order 1 on the main diagonal (this matrix already has the canonical form). In particular, we complete the proof of Theorem 7.3. The ring $\mathrm{M}(2, R, s)$ belongs to this type of rings; therefore, we assume that $n \geq 3$.

Theorem 8.4. *Let $T = (t_{ij})$ be a matrix of order $n \geq 3$ such that T has* 1 *on the main diagonal and the element s on remaining positions. There exists a ring $\mathrm{M}(n, R, s)$ such that T is a matrix of factors for $\mathrm{M}(n, R, s)$. Any two such rings can be transformed into each other by permutations, i.e., they are contained in the same orbit.*

Proof. From the text after Corollary 8.3, we know values that we need to give to the factors s_{ijk} to obtain the ring $\mathrm{M}(n, R, s)$. As always, $s_{iik} = 1 = s_{ikk}$. Then we set $s_{iji} = t_{ij}$. Further, for all pairwise distinct subscripts i, j, and k, we assume that $s_{ijk} = 1$ if the permutation (i, j, k) is even, and $s_{ijk} = s$ for the odd permutation (i, j, k). We verify that the set $\Sigma = \{s_{ijk} \mid i, j, k = 1, \dots, n\}$ is a system of factors, i.e., relations (5.2) are true.

If two subscripts in some relation (5.2) (this is relation (8.1)) are equal to each other, then it turns into the relation of the form (5.4) or (5.5). However, these relations are true by the choice of factors s_{ijk}. In particular, the assertion is verified in the case $n = 3$. Further, we assume that $n \geq 4$. It remains to verify the relation (8.1) for all pairwise distinct subscripts i, j, k, and l. Here we are in the situation described after Corollary 8.3, since the mutual disposition of integers i, j, k, and l is only important in the verification. Therefore, it is sufficient to verify that all 24 relations of the form (8.2) are true; this is directly verified.

Thus, Σ is a system of factors; consequently, there exists a ring $\mathrm{M}(n, R, \Sigma)$ with matrix of factors T.

Remark. We interrupt the proof of the theorem; we will return later to the remaining assertion about the orbit.

Conversely, we can assume in the proof that $s_{ijk} = s$ for even permutations (i, j, k), and $s_{ijk} = 1$ for odd permutations (i, j, k). Then we obtain a ring with matrix of factors T. We denote the corresponding rings by L_0 and L_1. The permutation

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix}$$

transforms L_0 into L_1 (we need to take into account that the permutations (i, j, k) and $\big(\sigma(i), \sigma(j), \sigma(k)\big)$ have opposite parities).

Thus, we obtain two rings L_0 and L_1 with matrix of factors T. Do other such rings exist? For $n = 3$, the answer is negative, which follows from the next paragraph.

We take $n = 4$. It directly follows from relations (5.4) and (5.5) that the factors s_{123}, s_{312}, and s_{231} from relations (8.2) are equal to each other, and the same is true for s_{321}, s_{213}, and s_{132}. In addition, elements of these triples need to have opposite values (1 or s). We have a similar situation with the first elements of the remaining 18 relations. The first elements of relations (8.2) coincide with all elements that are considered in these relations. Therefore, we can conclude that the given ring $\mathrm{M}(4, R, s)$ is uniquely determined by the vector (c_1, c_2, c_3, c_4) of length 4, which consists of 0 and 1. Here $c_1 = 0$ if $s_{123} = 1$, and $c_1 = 1$ if $s_{123} = s$, and so on. The vectors $(0, 0, 0, 0)$ and $(1, 1, 1, 1)$ correspond to the rings L_0 and L_1, respectively. Applying cycles $(1\,2\,4\,3)$, $(1\,3\,2\,4)$, and $(1\,4\,2\,3)$ to the rings L_0 and L_1, we obtain four additional rings $\mathrm{M}(4, R, s)$. To these rings, the vectors $(0, 1, 1, 0)$, $(1, 0, 0, 1)$, $(0, 0, 1, 1)$, $(1, 1, 0, 0)$ correspond. The permutation $(1\,4)(2\,3)$ transforms L_0 into L_1 and conversely. There do not exist rings $\mathrm{M}(4, R, s)$ that correspond to the vectors $(0, 1, 0, 1)$, $(1, 0, 1, 0)$, $(0, 1, 1, 1)$, $(1, 0, 1, 1)$, $(1, 1, 0, 1)$, $(1, 1, 1, 0)$, $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$; to verify this, it is sufficient to study relations (8.2). Therefore, we have that there exist six rings $\mathrm{M}(4, R, s)$ with given matrix of factors T. These rings are transformed to each other by permutations, i.e., they are contained in the same orbit. In addition, we remark that

$$L_1 = L_0^t, \quad L_{(0,0,1,1)} = L_{(1,1,0,0)}^t, \quad L_{(0,1,1,0)} = L_{(1,0,0,1)}^t$$

(see the item (d) from Sec. 5).

Now we assume that $n \geq 5$. We already have rings L_0 and L_1 such that T is a matrix of factors for L_0 and L_1. Now let $\mathrm{M}(n, R, s)$ be any other such a ring. We show that it can be transformed into L_0 or L_1 by some permutation.

We fix arbitrary pairwise distinct subscripts i, j, k, and l. The elements standing on the positions

$$(i,i),\ (i,j),\ (i,k),\ (i,l),\ (j,i),\ (j,j),\ (j,k),\ (j,l), (k,i),\ (k,j),\ (k,k),\ (k,l),\ (l,i),\ (l,j),\ (l,k),\ (l,l)$$

in matrices from $\mathrm{M}(n, R, s)$ form some ring $\mathrm{M}(4, R, s)$. We have just proved that there exists a permutation σ of order 4 that transforms this ring $\mathrm{M}(4, R, s)$ into the ring L_0. By thinking that integers not equal to i, j, k, l are fixed, we assume that σ is a permutation of order n. We apply σ to the ring $\mathrm{M}(n, R, s)$. By repeating this action several times, we obtain the permutation of order n that transform the ring $\mathrm{M}(n, R, s)$ into L_0. This completes the proof of Theorem 8.4. □

Leaning on the above, we present the considered matrices of factors S of the rings $\mathrm{M}(n, R, s)$ for $n = 2, 3, 4$. We omit the matrix that consists of 1 and corresponds to the ring $\mathrm{M}(n, R)$.

$n = 2$.

$$S = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}.$$

$n = 3$. Two orbits of matrices of factors and two corresponding orbits of rings.

The first orbit:

$$S = \begin{pmatrix} 1 & s & s \\ s & 1 & s \\ s & s & 1 \end{pmatrix}.$$

The orbit of rings consists of the rings L_0 and L_1.

The second orbit:

$$\begin{pmatrix} 1 & s & s \\ s & 1 & 1 \\ s & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & s \\ 1 & 1 & s \\ s & s & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & s & 1 \\ s & 1 & s \\ 1 & s & 1 \end{pmatrix}.$$

The orbit of rings contains three rings.

$n = 4$. Three orbits of matrices of factors and three corresponding orbits of rings.

The first orbit:

$$S = \begin{pmatrix} 1 & s & s & s \\ s & 1 & s & s \\ s & s & 1 & s \\ s & s & s & 1 \end{pmatrix}.$$

The orbit of rings consists of six rings.

The second orbit:

$$\begin{pmatrix} 1 & s & s & s \\ s & 1 & 1 & 1 \\ s & 1 & 1 & 1 \\ s & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & s \\ 1 & 1 & 1 & s \\ 1 & 1 & 1 & s \\ s & s & s & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & s & 1 & 1 \\ s & 1 & s & s \\ 1 & s & 1 & 1 \\ 1 & s & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & s & 1 \\ 1 & 1 & s & 1 \\ s & s & 1 & s \\ 1 & 1 & s & 1 \end{pmatrix}.$$

The third orbit:

$$\begin{pmatrix} 1 & 1 & s & s \\ 1 & 1 & s & s \\ s & s & 1 & 1 \\ s & s & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & s & s & 1 \\ s & 1 & 1 & s \\ s & 1 & 1 & s \\ 1 & s & s & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & s & 1 & s \\ s & 1 & s & 1 \\ 1 & s & 1 & s \\ s & 1 & s & 1 \end{pmatrix}.$$

The second orbit of rings contains four rings, and the third orbit of rings contains three rings.

The rings L_0 are examples of rings of crossed matrices [2], which are quite useful in the structural theory of some Artinian rings.

We return to Lemma 7.1. In this lemma, we consider possible values of factors in the triple s_{iji}, s_{iki}, s_{jkj}. We considered situations where only the cases (1) and (2) are realized; then we considered the situation where only the case (3) is realized. To these situations correspond some structure of the canonical form of matrices of factors. The question remains where each of the cases (1), (2), and (3) can appear. This is equivalent to the property that the canonical form of the given matrix of factors contains blocks of order > 1 and has more than two blocks on the main diagonal.

How can we obtain all rings $\mathrm{M}(n, R, s)$ for which the canonical form of the matrix of factors has a similar structure? For $n = 2$ and $n = 3$, such rings do not exist. The proof of Theorem 7.3 presents a practical method of constructing the required rings for $n \geq 4$. Namely, we need to take the rings $\mathrm{M}(n, R, s)$, beginning with $n = 3$; then we apply the construction K_4 from Sec. 3 to $\mathrm{M}(n, R, s)$. Before this, we must use some permutation for the transfer to the ring $\mathrm{M}(n, R, s)$, where the matrix of factors has the canonical form. From the rings $\mathrm{M}(n, R, s)$, we need to exclude those rings, whose matrices of factors have the canonical form containing exactly two blocks on the main diagonal. As a result, we obtain the rings $\mathrm{M}(n+1, R, s)$ with the required form of the matrix of factors. Then we apply all possible permutations to these rings. After this, we will have the required rings $\mathrm{M}(n + 1, R, s)$. We remark that if blocks of the matrix of factors of the ring $\mathrm{M}(n, R, s)$ have the orders $k_1, \dots, k_t$ with $t \geq 3$, then blocks of the corresponding ring $\mathrm{M}(n + 1, R, s)$ will have the orders $k_1, \dots, k_{t-1}, k_t + 1$.

We illustrate the given construction method in the case $n = 4$. We take the ring $\mathrm{M}(3, R, s)$, which is denoted by L_0 above. This ring has a matrix of factors

$$\begin{pmatrix} 1 & s & s \\ s & 1 & s \\ s & s & 1 \end{pmatrix}.$$

Applying the construction K_4 to the ring L_0, we obtain the ring $\mathrm{M}(4, R, s)$ with the matrix of factors

$$\begin{pmatrix} 1 & s & s & s \\ s & 1 & s & s \\ s & s & 1 & 1 \\ s & s & 1 & 1 \end{pmatrix}$$

such that $s_{121} = s_{131} = s_{232} = s_{141} = s$ and $s_{343} = 1$. By considering the position of 1 above the main diagonal, we denote the obtained ring by L_{34}. We apply permutations $(3\,1)(4\,2)$, $(1\,3\,4)(2)$, $(1\,3)(2)(4)$, $(2\,3\,4)(1)$, and $(2\,3)(1)(4)$ to the ring L_{34} and obtain the rings L_{12}, L_{13}, L_{14}, L_{23}, and L_{24}, respectively. Above the main diagonal, matrices of factors of these rings have 1 in the positions $(1, 2)$, $(1, 3)$, $(1, 4)$, $(2, 3)$, and $(2, 4)$, respectively.

We can repeat these actions beginning with the ring $L_1 = \mathrm{M}(3, R, s)$. As a result, we obtain the rings L'_{12}, L'_{13}, L'_{14}, L'_{23}, L'_{24}, and L'_{34}. We recall that the ring L_0 can be transformed into the ring L_1 by some permutation (e.g., we can use $(1\,4)(2\,3)$). The ring L_{34} is transformed into the ring L'_{34} by the permutation $(1\,2)(3\,4)$; this gives a partial answer on the question formulated in the end of Sec. 7. Thus, all 12 rings

$$L_{12},\ L_{13},\ L_{14},\ L_{23},\ L_{24},\ L_{34},\ L'_{12},\ L'_{13},\ L'_{14},\ L'_{23},\ L'_{24},\ L'_{34}$$

are contained in the same orbit. In addition, we remark that $L_1 = L_0^t$ and $L'_{ij} = L_{ij}^t$ (see the item (d) of Sec. 5).

The above list of matrices of factors can be supplemented for $n = 4$ by one more orbit:

$$\begin{pmatrix} 1 & 1 & s & s \\ 1 & 1 & s & s \\ s & s & 1 & s \\ s & s & s & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & s & 1 & s \\ s & 1 & s & s \\ 1 & s & 1 & s \\ s & s & s & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & s & s & 1 \\ s & 1 & s & s \\ s & s & 1 & s \\ 1 & s & s & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & s & s & s \\ s & 1 & 1 & s \\ s & 1 & 1 & s \\ s & s & s & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & s & s & s \\ s & 1 & s & 1 \\ s & s & 1 & s \\ s & 1 & s & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & s & s & s \\ s & 1 & s & s \\ s & s & 1 & 1 \\ s & s & 1 & 1 \end{pmatrix}.$$

The corresponding orbit of rings contains 12 rings of the form L_{ij} and L'_{ij}. It is not known whether this orbit contains other rings.

9. Isomorphisms of Rings $\mathrm{M}(n, R, s)$

The following result is directly verified with the use of Lemma 4.5.

Lemma 9.1. *For any formal matrix ring K over the ring R, the relation $C(K) = \{rE \mid r \in C(R)\}$ holds.*

In Sec. 5, the isomorphism problem (III) was formulated. Now we consider the problem for formal matrix rings $\mathrm{M}(n, R, \{s_{ijk}\})$ such that for $i \neq j$ and $j \neq k$, any factor s_{ijk} is equal to s^m for some $m \geq 1$, where s is a fixed central element of the ring R. Similar rings were mentioned at the beginning of Sec. 7. Here we denote such rings by $\mathrm{M}(n, R, s)$; do not confuse this with the symbol $\mathrm{M}(n, R, s)$ from Secs. 7 and 8, where $\mathrm{M}(n, R, s)$ denotes the ring such that $s_{ijk} = 1$ or $s_{ijk} = s$.

In what follows, we denote by $\mathrm{M}(n, R, 0)$ the ring $\mathrm{M}(n, R, \{s_{ijk}\})$ such that all factors s_{ijk} are equal to zero, excepting s_{iik} and s_{ikk}. Further, T is some ring and $\mathrm{M}(n, T, \{t_{ijk}\})$ is an arbitrary formal matrix ring over T.

We recall a familiar definition. A ring R is said to be *normal* if all idempotents of R are central.

Lemma 9.2. *Let R be a normal ring. If $\mathrm{M}(n, R, 0) \cong \mathrm{M}(n, T, \{t_{ijk}\})$, then all factors t_{ijk} are equal to zero, excepting cases where $i = j$ or $j = k$.*

Proof. We set $K_1 = \mathrm{M}(n, R, 0)$ and $K_2 = \mathrm{M}(n, T, \{t_{ijk}\})$. We fix some ring isomorphism $f\colon K_1 \to K_2$. Let I denote the ideal (I_{ij}) of the ring K_1, where $I_{ii} = 0$ and $I_{ij} = R$ for $i \neq j$ (see Proposition 6.3). Then $I^2 = 0$, whence $\big(f(I)\big)^2 = 0$. We assume that there exists a nonzero factor t_{ikj} such that $i \neq k$ and $k \neq j$. Then $E_{ik}E_{kj} = t_{ikj}E_{ij} \neq 0$, where E_{ij} is the matrix unit (see Sec. 5). Consequently, the matrices E_{ik} and E_{kj} cannot be contained in $f(I)$ simultaneously. For definiteness, let $E_{ik} \notin f(I)$. Then

$$E_{ii}(E_{ii} + E_{ik}) - (E_{ii} + E_{ik})E_{ii} = E_{ik} \notin f(I).$$

It is proved that the idempotent $(E_{ii}+E_{ik})+f(I)$ of the factor ring $K_2/f(I)$ is not central in $K_2/f(I)$. Therefore, the ring $K_2/f(I)$ is not normal. On the other hand, the ring $K_2/f(I)$ is normal, since

$$K_2/f(I) \cong K_1/I \cong R \oplus \cdots \oplus R$$

is a finite direct product of normal rings. This is a contradiction. □

Now let s and t be two nonzero central elements of the ring R (by Lemma 9.2, we can assume that $s \neq 0$ and $t \neq 0$). In addition, let $s^k \neq s^l$ for all distinct nonnegative k and l. Further, let $\mathrm{M}(n, R, \{s_{ijk}\})$ and $\mathrm{M}(n, R, \{t_{ijk}\})$ be two rings of the form mentioned at the beginning of the section; namely, every factor s_{ijk} is a positive integral power of the element s, and every factor t_{ijk} is a positive integral power of the element t. We also assume that at least one of the factors s_{ijk} is equal to s and at least one of the factors t_{ijk} is equal to t. In addition, we assume that the systems of factors s_{ijk} and t_{ijk} are "similar" in the following sense:

$$s_{ijk} = s^m \iff t_{ijk} = t^m$$

for all factors s_{ijk} and t_{ijk}. By our convention, we denote the considered rings by $\mathrm{M}(n, R, s)$ and $\mathrm{M}(n, R, t)$, respectively.

Recall that we denote by $J(R)$, $U(R)$, and $Z(R)$ the Jacobson radical, the group of invertible elements, and the set of all (left or right) zero divisors of the ring R.

Theorem 9.3. *Let R be a commutative ring with $Z(R) \subseteq J(R)$. The rings $\mathrm{M}(n, R, s)$ and $\mathrm{M}(n, R, t)$ are isomorphic to each other if and only if $t = v\alpha(s)$, where v is an invertible element in R and α is an automorphism of the ring R.*

Proof. We set $K_1 = \mathrm{M}(n, R, s)$ and $K_2 = \mathrm{M}(n, R, t)$. Given a ring isomorphism $f\colon K_1 \to K_2$, the isomorphism f induces the isomorphism $C(K_1) \to C(K_2)$ of the centers of these rings. We explicitly consider the action of this isomorphism. We take an arbitrary element $a \in R$. By Lemma 9.1, $aE \in C(K_1)$. Consequently, $f(aE) \in C(K_2)$. Further, $f(aE) = bE$ for some $b \in R$. We obtain that f induces an automorphism α of the ring R such that $\alpha(a) = b$. Thus, we have $f(aE) = \alpha(a)E$, $a \in R$.

Now we take the ideal

$$(sE)K_1 = \begin{pmatrix} sR & \dots & sR \\ \dots & \dots & \dots \\ sR & \dots & sR \end{pmatrix}$$

of the ring K_1. Under the action of f, the image of the ideal is the ideal $f\big((sE)K_1\big)$. We have the relations

$$f\big((sE)K_1\big) = f(sE)K_2 = (\alpha(s)E)K_2 = \begin{pmatrix} \alpha(s)R & \dots & \alpha(s)R \\ \dots & \dots & \dots \\ \alpha(s)R & \dots & \alpha(s)R \end{pmatrix}.$$

The isomorphism f induces the isomorphism of the factor rings $K_1/(sE)K_1 \to K_2/(\alpha(s)E)K_2$. The first factor ring is the matrix ring $\mathrm{M}(n, R/sR, 0)$ (see the paragraph before Lemma 9.2). The second factor ring is the matrix ring $\mathrm{M}(n, R/\alpha(s)r, \bar{t})$, where $\bar{t} = t + \alpha(s)R$. Thus, residue classes $t_{ijk} + \alpha(s)R$ are factors of this ring. Applying Lemma 9.2, we obtain that $\bar{t} = 0$ or $t \in \alpha(s)R$ (we must take into account that t is one of the elements t_{ijk}). Consequently, $t = \alpha(s)x$ for some element $x \in R$. Taking the converse isomorphism for f, we obtain that $s = \alpha^{-1}(t)y$, $y \in R$. Then we have

$$t = \alpha(s)x = t\alpha(y)x, \quad t(1 - \alpha(y)x) = 0.$$

By the assumption, we obtain that

$$1 - \alpha(y)x \in J(R), \quad 1 - (1 - \alpha(y)x) = \alpha(y)x \in U(R).$$

Consequently, x is an invertible element. Therefore, $t = v\alpha(s)$, where v is an invertible element, and α is an automorphism of the ring R.

Now we assume that $t = v\alpha(s)$, where v is an invertible element and α is an automorphism of the ring R. We show that the rings K_1 and K_2 are isomorphic to each other. First, the isomorphism $\mathrm{M}(n, R, s) \cong \mathrm{M}\big(n, R, \alpha(s)\big)$ follows from the item (b) of Sec. 5. The set $\{\alpha(s_{ijk}) \mid i, j, k = 1, \dots, n\}$ is a system of factors of the second ring. We can assume that α is the identity automorphism. We take the set $\{v_{ijk} \mid i, j, k = 1, \dots, n\}$, where $v_{ijk} = v^m$ if $s_{ijk} = s^m$. This set is a system of factors. Therefore, we have the ring $\mathrm{M}(n, R, vs)$ with system of factors $\{v_{ijk}s_{ijk} \mid i, j, k = 1, \dots, n\}$. By the item (c) of Sec. 5, there exists an isomorphism $\mathrm{M}(n, R, vs) \cong \mathrm{M}(n, R, s)$. It follows from the relation $t = vs$ and the convention on "the similarity" of systems of factors that $t_{ijk} = v_{ijk}s_{ijk}$ for all i, j, k. Therefore, $\mathrm{M}(n, R, vs) = \mathrm{M}(n, R, t)$ and $\mathrm{M}(n, R, s) \cong \mathrm{M}(n, R, t)$. □

We remark that $Z(R) \subseteq J(R)$ provided R is either a domain or a local ring.

Corollary 9.4. *Let R be a commutative ring that is either a domain or a local ring. The rings* $\mathrm{M}(n, R, s)$ *and* $\mathrm{M}(n, R, t)$ *are isomorphic to each other if and only if $t = v\alpha(s)$, where v is an invertible element, α is an automorphism of the ring R.*

In the papers of Krylov [12] and Tang, Li, and Zhou [21], the authors prove some isomorphism theorems for formal matrix rings of order 2 (e.g., Corollary 9.5). Tang and Zhou [23] prove Theorem 9.3 for rings considered in Sec. 7. In this case, the restrictions on the elements s and t listed before Theorem 9.3, are not required. These restrictions are also not required for $n = 2$ (see the text after Proposition 5.2 about multiplication relations of matrices in such rings).

Corollary 9.5 ([12])**.** *Let R be a commutative ring and let s and t be two elements of this ring such that at least one of these elements is not a zero divisor. The rings* $\mathrm{M}(2, R, s)$ *and* $\mathrm{M}(2, R, t)$ *are isomorphic to each other if and only if $t = v\alpha(s)$, where v is an invertible element, α is an automorphism of the ring R.*

Abyzov and Tapkin [1] study rings $\mathrm{M}(3, R, \Sigma)$ with symmetric matrices S_1, S_2, and S_3, i.e., $s_{ikj} = s_{jki}$ for all $i, j, k = 1, 2, 3$ (see the end of Sec. 5 about matrix rings of order 3). In particular, the authors obtained several results on the isomorphism problem for such rings. In addition, Abyzov and Tapkin introduced systems of factors of a more general (as compared with the case $n = 3$) form as follows. Let $s_1, \dots, s_n$ be arbitrary central elements of the ring R. For all $i, j, k = 1, \dots, n$, we set

$$s_{ijk} = \begin{cases} 1 & \text{if } i = j \text{ or } j = k; \\ s_j & \text{if } i,\ j,\ k \text{ are pairwise distinct;} \\ s_i s_j & \text{if } i = k, \text{ but } i \neq j. \end{cases}$$

It is directly verified that $\Sigma = \{s_{ijk}\}$ is a system of factors, since Σ satisfies relations (5.2). Consequently, there exists a ring $\mathrm{M}(n, R, \Sigma)$. Further, Abyzov and Tapkin [1] transfer results on the isomorphism problem obtained for the rings $\mathrm{M}(3, R, \Sigma)$ to the rings $\mathrm{M}(n, R, \Sigma)$ mentioned above. The class of such rings $\mathrm{M}(n, R, \Sigma)$ contains Tang–Zhou rings from the beginning of Sec. 7.

If s and t are any two invertible elements (without any restrictions), then any two rings $\mathrm{M}(n, R, s)$ and $\mathrm{M}(n, R, t)$ mentioned at the beginning of the section are isomorphic to each other by Proposition 5.2. Therefore, we can assume that one of these elements (for example, s) is not invertible. In this case, if R is a domain, then $s^k \neq s^l$ for $k \neq l$. This is realized in the following two examples. Therefore, we assume in these examples that the corresponding systems of factors $\{s_{ijk}\}$ and $\{t_{ijk}\}$ satisfy only the two conditions formulated before Theorem 9.3.

Example 9.6. Let $s, t \in \mathbb{Z}$. An isomorphism $\mathrm{M}(n, \mathbb{Z}, s) \cong \mathrm{M}(n, \mathbb{Z}, t)$ exists if and only if $t = s$ or $t = -s$.

Proof. First of all, we need to take into account Lemma 9.2 and the remark before Example 9.6. Further, the assertion follows from Theorem 9.3, since the ring $\mathbb{Z}$ has only the identity automorphism and $U(\mathbb{Z}) = \{1, -1\}$. □

Example 9.7. Let R be a commutative domain and let i and j be two positive integers. If $\mathrm{M}(n, R[x], x^i) \cong \mathrm{M}(n, R[x], x^j)$ or $\mathrm{M}(n, R[[x]], x^i) \cong \mathrm{M}(n, R[[x]], x^j)$, then $i = j$.

Proof. Let V be one of the rings $R[x]$, $R[[x]]$. We assume that $i < j$. By Theorem 9.3, we have that $x^i = v(x)\alpha(x^j) = v(x)\alpha(x)^j$, where $v(x)$ is an invertible element of the ring V and α is an automorphism of the ring V. Further, we have $\alpha(x) = x^k(a_0 + a_1 x + \ldots)$, where $k \geq 0$ and $a_0 \neq 0$. Then $x^i = x^{jk} v(x)(a_0 + a_1 x + \ldots)^j$. However, we have $i < j$. Therefore, $k = 0$, and we have the relation $x^i = v(x)(a_0 + a_1 x + \ldots)^j$. Since $a_0 \neq 0$, we have that $v(x)$ is divided by x, which is impossible. □

10. Determinants of Formal Matrices

In the last two sections, R is a commutative ring, and K is some formal matrix ring of order n over the ring R with system of factors $\{s_{ijk} \mid i, j, k = 1, \ldots, n\}$, i.e., $K = \mathrm{M}(n, R, \{s_{ijk}\})$.

We recall the multiplication rules for matrices in the ring K. Let $A = (a_{ij})$, $B = (b_{ij})$, and $AB = (c_{ij})$. Then $c_{ij} = \sum\limits_{k=1}^{n} s_{ikj} a_{ik} b_{kj}$. We will use several times the relations from Sec. 5, which are concerned with the interrelations between factors s_{ijk}. For convenience, we partly repeat these relations. First of all, these are the main relations (5.2):

$$s_{iik} = 1 = s_{ikk}, \quad s_{ijk} \cdot s_{ikl} = s_{ijl} \cdot s_{jkl}.$$

It follows from these relations that we have the following relations:

$$s_{iji} = s_{jij}, \quad s_{iji} = s_{ijl} \cdot s_{jil} = s_{lij} \cdot s_{lji}.$$

In addition, for all $i, j = 1, \ldots, n$, we defined the element t_{ij}, which is equal to s_{ijl} for some $l = 1, \ldots, n$. The relations

$$t_{ij} \cdot t_{ji} = s_{iji}, \quad t_{ij} \cdot t_{jk} = t_{ik} \cdot s_{ijk}$$

hold. Now we introduce the notion of the determinant of an arbitrary matrix in K and show that such a determinant satisfies properties that are similar to the main properties of the ordinary determinant of matrices from the ring $\mathrm{M}(n, R)$. We use properties of the ordinary determinant without additional explanations.

In several cases, determinants of matrices from the ring K are considered in [1, 6, 7, 22, 23]. The paper [15] contains a general approach to the notion of the determinant of an arbitrary formal matrix of order 2.

We are interesting in the following question: Which transformations of rows and columns of matrices in K are possible? We take an arbitrary matrix $A = (a_{ij})$ in K.

(a) Rows of the matrix A can be multiplied by elements of R. Therefore, we can consider common factors of elements of rows.

We have the homomorphism

$$\varphi_{ijk} \colon R_{ij} \otimes_{R_j} R_{jk} \to R_{ik}, \quad \varphi_{ijk}(x \otimes y) = s_{ijk} xy = x \circ y$$

(see Secs. 3 and 5). The symbol $\circ$ can be considered as an operation in R; however, the result of the operation depends on subscripts i and j. To emphasize that the element r of the ring R is used as an element of R_{ij}, we will add the subscripts i and j to this element. Namely, we set $r_{ij} = r$. Let $A_1, \ldots, A_n$

be rows of the matrix A. Then $r_{ij} \circ A_j$ denotes the vector-row $(r_{ij} \circ a_{j1}, \dots, r_{ij} \circ a_{jn})$. In fact, we deal with some R-module $(R, \dots, R)$.

(b) We can multiply (in the sense of the operation $\circ$) the jth row of the matrix A by the element r_{ij} and add the product to the ith row. Such transformation is briefly presented in the form $r_{ij} \circ A_j + A_i$.

(c) We can replace the ith row of the matrix A by the row $1_{ij} \circ A_j = (s_{ij1}a_{j1}, \dots, s_{ijn}a_{jn})$, and we can also replace the jth row of the matrix A by the row $1_{ji} \circ A_i = (s_{ji1}a_{i1}, \dots, s_{jin}a_{in})$. The transfer from the matrix A to the obtained matrix is called the *interchange* of the ith row and the jth row of the matrix A. Similar transformations can be performed over columns of the matrix A.

We will say that the ith row and the jth row of the matrix A are *proportional* if $A_j = r_{ji} \circ A_i$ or $A_i = r_{ij} \circ A_j$ for some elements $r_{ji}, r_{ij} \in R$.

Let η be one of the homomorphisms

$$\mathrm{M}(n, R, \{s_{ijk}\}) \to \mathrm{M}(n, R)$$

defined before Proposition 5.2. It acts with the use of the relation $(a_{ij}) \to (t_{ij}a_{ij})$, where $t_{ij} = s_{ijl}$ for some fixed integer $l = 1, \dots, n$.

The ordinary determinant of the matrix $C \in \mathrm{M}(n, R)$ is denoted by $|C|$. For every matrix A from the ring K, we set $d(A) = |\eta A|$. We call the element $d(A)$ the *determinant* of the matrix A in the ring K, and the mapping $d\colon K \to R$, $A \to d(A)$ is called the *determinant of the ring* K.

We present the equivalent method of defining determinants of matrices in K. First, we return to the operation $\circ$. If $x_{ij}, x_{jk} \in R$, then $x_{ij} \circ x_{jk} = s_{ijk}x_{ij}x_{jk}$ by our convention. We will assign subscripts i, k to the element $x_{ij}x_{jk}$.

Now, given the elements $a_{i_1i_2}, \dots, a_{i_{k-1}i_k}$ of the ring R, the expression

$$a_{i_1i_2} \circ a_{i_2i_3} \circ \dots \circ a_{i_{k-1}i_k} \tag{10.1}$$

has an exact sense. Indeed, any arrangement of parentheses in (10.1) is rightful. In addition, the result does not depend on the arrangement of parentheses, i.e., $\circ$ is an associative operation. The proof uses the induction on the number of elements. For $k = 3$, the assertion follows from the main relations mentioned at the beginning of the section.

With the use of the tensor product $R_{i_1i_2} \otimes_R \dots \otimes_R R_{i_{k-1}i_k}$, we can give a value to the expression (10.1). The homomorphisms φ_{ijk} induce the homomorphism φ from this product into R. Then

$$a_{i_1i_2} \circ \dots \circ a_{i_{k-1}i_k} = \varphi(a_{i_1i_2} \otimes \dots \otimes a_{i_{k-1}i_k}).$$

We also can act as follows. As earlier, let E_{ij} be a matrix unit. Then $E_{ij}E_{jk} = s_{ijk}E_{ik}$. If

$$(a_{i_1i_2}E_{i_1i_2}) \cdot \ldots \cdot (a_{i_{k-1}i_k}E_{i_{k-1}i_k}) = cE_{i_1i_k}, \quad c \in R,$$

then $a_{i_1i_2} \circ \dots \circ a_{i_{k-1}i_k} = c$.

Let us have one more element $a_{i_ki_1}$. Subscripts of elements $a_{i_1i_2}, \dots, a_{i_{k-1}i_k}, a_{i_ki_1}$ form a cycle; it is denoted by σ. We represent it, beginning with another element. In such a case, the expression $a_{i_1i_2} \circ \dots \circ a_{i_{k-1}i_k} \circ a_{i_ki_1}$ is equal to the corresponding expression for another representation of the cycle σ (we must take into account the relation $s_{iji} = s_{jij}$).

Now we can give an exact meaning to the expression $a_{1i_1} \circ \dots \circ a_{ni_n}$ provided the second subscripts of the factors form a permutation of integers $1, \dots, n$. For this purpose, we represent the permutation

$$\tau = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}$$

as the product of independent cycles $\sigma_1, \dots, \sigma_m$. If $c_1, \dots, c_m$ are products (in the sense of the operation $\circ$) of elements whose subscripts are included in cycles $\sigma_1, \dots, \sigma_m$, respectively, then we assume that

$$a_{1i_1} \circ \dots \circ a_{ni_n} = c_1 \cdot \ldots \cdot c_m.$$

We state that

$$d(A) = \sum_{n!} (-1)^q a_{1i_1} \circ \dots \circ a_{ni_n},$$

where q is the number of inversions in the permutation $i_1, \dots, i_n$. Since

$$d(A) = |\eta A| = \sum_{n!} (-1)^q t_{1i_1} a_{1i_1} \cdot \dots \cdot t_{ni_n} a_{ni_n},$$

it is sufficient to verify that corresponding summands of two sums are equal to each other. We take two such summands. We decompose the permutation of subscripts of factors of these summands into the product of independent cycles. We show that products of elements whose subscripts form some cycle are equal to each other. We take one from cycles $(i_1 i_2 \dots i_k)$. The product $a_{i_1 i_2} \circ \dots \circ a_{i_k i_1}$ is equal to

$$s_{i_1 i_2 i_1} \cdot s_{i_2 i_3 i_1} \cdot \dots \cdot s_{i_{k-1} i_k i_1} \cdot a_{i_1 i_2} \cdot \dots \cdot a_{i_k i_1}.$$

The corresponding product for the determinant ηA is

$$t_{i_1 i_2} \cdot \dots \cdot t_{i_k i_1} \cdot a_{i_1 i_2} \cdot \dots \cdot a_{i_k i_1}.$$

With the use of induction on the length of the cycle, we prove that

$$t_{i_1 i_2} \cdot \dots \cdot t_{i_k i_1} = s_{i_1 i_2 i_1} \cdot \dots \cdot s_{i_{k-1} i_k i_1}.$$

If $k = 2$, then $t_{i_1 i_2} t_{i_2 i_1} = s_{i_1 i_2 i_1}$. Let $k \geq 3$. We have $t_{i_{k-1} i_k} t_{i_k i_1} = t_{i_{k-1} i_1} s_{i_{k-1} i_k i_1}$; then we use the induction hypothesis. As a result, we obtain the relation

$$d(A) = \sum_{n!} (-1)^q a_{1i_1} \circ \dots \circ a_{ni_n}.$$

We can say that the *complete development formula* holds for the determinant $d(A)$. As a corollary, we obtain that the first definition of the determinant $d(A)$ does not depend on the choice of the homomorphism η.

We present several main properties of the determinant $d(A)$. Each of them can be verified with the use of either the first definition or the second definition of the determinant. We more frequently use the second definition to avoid the repetition of well-known arguments.

(1) $d(E) = 1$.

(2) The determinant d is a polylinear function of matrix rows.

Proof. The property (2) follows from the relation $d(A) = \eta A$ and a similar property of the ordinary determinant. □

(3) If the matrix A' is obtained from the matrix A by interchanging the ith row and jth row, then $d(A') = -s_{iji} d(A)$.

Proof. For the matrix A', the ith row is equal to $1_{ij} \circ A_j$ and the jth row is equal to $1_{ji} \circ A_i$. For the matrix $\eta(A')$, the ith row is equal to $(t_{i1} s_{ij1} a_{j1}, \dots, t_{in} s_{ijn} a_{jn})$. For every k, we obtain

$$t_{ik} s_{ijk} = s_{ikl} s_{ijk} = s_{ijl} s_{jkl} = s_{ijl} t_{jk}.$$

We can repeat the same action for the jth row of the matrix $\eta(A')$. We obtain that

$$|\eta(A')| = s_{ijl} s_{jil} |A''| = s_{iji} |A''|,$$

where the matrix A'' is obtained from ηA by the interchange of the ith row and the jth row. Therefore, $|A''| = -|\eta A|$. Now we obtain

$$d(A') = |\eta(A')| = s_{iji} |A''| = -s_{iji} |\eta A| = -s_{iji} d(A). \quad \square$$

(4) If some two rows of the matrix A are proportional, then $d(A) = 0$.

Proof. Let the jth row of the matrix A be equal to $r_{ji} \circ A_i$ for some $r_{ji} \in R$. By property (2), we can assume that this row has the form $(s_{ji1}a_{i1}, \dots, s_{ijn}a_{in})$. For the matrix ηA, the row with number i is equal to $(s_{i1l}a_{i1}, \dots, s_{inl}a_{in})$ for some l, and the row with number j is equal to $(s_{j1l}s_{ji1}a_{i1}, \dots, s_{jnl}s_{jin}a_{in})$. For every $k = 1, \dots, n$, we have $s_{jik}s_{jkl} = s_{jil}s_{ikl}$. Now it is clear that the ith row of the matrix ηA is proportional to the jth row of ηA. Consequently, $d(A) = |\eta A| = 0$. □

(5) If we multiply (in the sense of the operation $\circ$) the jth row of the matrix A by some element $r_{ij} \in R$ and add the product to the ith row of the matrix A, then the determinant of the obtained matrix will be equal to $d(A)$.

Proof. In the proof of (5), we can use standard methods based on properties (2) and (4). □

(6) For any two matrices A and B, the relation $d(AB) = d(A)d(B)$ holds.

Proof. It is clear that the relations

$$d(AB) = |\eta(AB)| = |\eta(A)\eta(B)| = |\eta(A)|\,|\eta(B)| = d(A)d(B)$$

are true. □

(7) If $s_{ikj} = s_{jki}$ for all i, j, and k, then $d(A) = d(A^t)$ for every matrix A. If all elements s_{iji} are nonzero divisors in R, then the converse is also true.

Proof. We can use the homomorphism η or the complete development formula; this is practically the same method. We use the first method. We have $d(A) = |\eta A|$ and $d(A^t) = |\eta(A^t)|$. There exists a familiar correspondence between the summand determinants $|\eta A|$ and $|\eta(A^t)|$. We take some summand c of the determinant $|\eta A|$. Let τ be the permutation of subscripts of this summand. We have the relation $\tau = \sigma_1 \cdot \ldots \cdot \sigma_m$, where σ_i are pairwise independent cycles (in particular, of length 1). Further, let c_i be the product of factors whose subscripts occur in σ_i, $i = 1, \dots, n$. Then $c = c_1 \cdot \ldots \cdot c_m$.

Let the summand d of the determinant $|\eta(A^t)|$ correspond to c. The permutation of its subscripts is $\tau^{-1} = \sigma_m^{-1} \cdot \ldots \cdot \sigma_1^{-1}$. For the element d, we have the corresponding representation $d = d_1 \cdot \ldots \cdot d_m$. We verify that $c_1 = d_1, \dots, c_m = d_m$. For this purpose, we take some cycle $\sigma = (i_1 i_2 \dots i_k)$. Without loss of generality, we can assume that $k \geq 2$. Then $\sigma^{-1} = (i_k, i_{k-1}, \dots, i_1)$. Now it is sufficient to verify that the product $t_{i_1i_2}t_{i_2i_3} \cdot \ldots \cdot t_{i_{k-1}i_k}t_{i_ki_1}$ is equal to the product $t_{i_ki_{k-1}}t_{i_{k-1}i_{k-2}} \cdot \ldots \cdot t_{i_2i_1}t_{i_1i_k}$. These products are equal to

$$(t_{i_1i_2}t_{i_2i_3} \cdot \ldots \cdot t_{i_{k-2}i_{k-1}}t_{i_{k-1}i_1})s_{i_{k-1}i_ki_1}$$

and

$$(t_{i_2i_1}t_{i_3i_2} \cdot \ldots \cdot t_{i_{k-1}i_{k-2}}t_{i_1i_{k-1}})s_{i_1i_ki_{k-1}},$$

respectively. The form of the expressions in parentheses prompts us to use induction on the length of the cycle σ. The case $k = 2$ is obvious.

Now we assume that all factors s_{iji} are nonzero divisors and $d(A) = d(A^t)$ for every matrix A. If some two of three subscripts i, j, k are equal to each other, then $s_{ikj} = s_{jki}$. Therefore, we assume that subscripts i, j, k are pairwise distinct We take the matrix

$$A = E + E_{ik} + E_{kj} + E_{ji} - E_{ii} - E_{kk} - E_{jj}.$$

The determinant $d(A)$ is equal to $1_{ik} \circ 1_{kj} \circ 1_{ji} = s_{ikj}s_{iji}$, and the determinant $d(A^t)$ is equal to $s_{jki}s_{jij}$. Consequently, $s_{ikj} = s_{jki}$. □

There exists a familiar relation for a determinant with corner consisting of zeros.

(8) Let the matrix A have the form

$$\begin{pmatrix} B & D \\ 0 & C \end{pmatrix},$$

where B and C are matrices of order m and $n - m$, respectively. Then $d(A) = d(B)d(C)$.

Proof. We give some explanations. The matrix B has a system of factors $\{s_{ijk} \mid 1 \le i,j,k \le m\}$, and the matrix C has a system of factors $\{s_{ijk} \mid m+1 \le i,j,k \le n\}$ (all subscripts can be reduced by m). For the rings $\mathrm{M}(m, R, \{s_{ijk}\})$ and $\mathrm{M}(n-m, R, \{s_{ijk}\})$, we assume that corresponding homomorphisms η are restrictions of the homomorphism η to the ring $\mathrm{M}(n, R, \{s_{ijk}\})$. We mean that the matrix B is identified with the matrix

$$\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix},$$

and the matrix C is identified with the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}.$$

Now we obtain the relations

$$d(A) = |\eta A| = |\eta B| \cdot |\eta C| = d(B)d(C). \qquad \square$$

If the set of factors s_{ijk} contains zero divisors, this often hinders the work with matrices. We define one formal matrix ring that sometimes helps to avoid these difficulties. In one case, a similar ring is introduced in [23] (see below).

We fix a commutative ring R and an integer $n \ge 2$. Let $X = \{x_{ijk}\}$ be a set of $n(n^2-1)$ variables, where $1 \le i,j,k \le n$, $i \ne j$, $j \ne k$. Further, let $R[X]$ be the polynomial ring in variables x_{ijk} with coefficients in the ring R.

Let I be the ideal of the ring $R[X]$ generated by all differences of the form $x_{ijk}x_{ikl} - x_{ijl}x_{jkl}$. We denote by $\overline{R[X]}$ the factor ring $R[X]/I$. We identify elements of the ring R with their images in $\overline{R[X]}$. For simplicity of representations, we denote the residue class $x_{ijk} + I$ by x_{ijk}. In what follows, it will be important that the elements x_{ijk} are nonzero divisors in $\overline{R[X]}$.

We set $x_{iik} = 1 = x_{ikk}$ for all $i,k = 1,\dots,n$. In the ring $\overline{R[X]}$, we have the relations $x_{ijk}x_{ikl} = x_{ijl}x_{jkl}$ for all values of subscripts. Therefore, there exists a formal matrix ring $\mathrm{M}(n, \overline{R(X)}, \{x_{ijk}\})$. This ring is denoted by $\mathrm{M}(n, \overline{R[X]}, X)$. There exists a familiar homomorphism

$$\eta\colon \mathrm{M}(n, \overline{R[X]}, X) \to \mathrm{M}(n, \overline{R[X]})$$

(see the paragraph before Proposition 5.2). In addition, the determinant $d\colon \mathrm{M}(n, \overline{R[X]}, X) \to \overline{R[X]}$ was defined earlier. In a certain sense, the ring $\mathrm{M}(n, \overline{R[X]}, X)$ and the determinant d form a couniversal object for formal matrix rings of order n over R and their determinants. We give an exact meaning to these words.

Let us have a concrete formal matrix ring $\mathrm{M}(n, R, \{s_{ijk}\})$. In such a case, several homomorphisms appear, we call them the *permutation homomorphisms*; they are denoted by the same symbol θ. Every such a homomorphism replaces the symbol x_{ijk} by the element s_{ijk}. First of all, this the homomorphism $\theta\colon R[X] \to R$. Since R can be embedded in $R[X]$, we have that θ splits, i.e., $R[X] = R \oplus \mathrm{Ker}(\theta)$.

It follows from main relations (5.2) that $I \subseteq \mathrm{Ker}(\theta)$. After identification, we can assume that $\overline{R[X]} = R \oplus \big(\mathrm{Ker}(\theta)\big)/I$. In addition, θ induces the homomorphism $\overline{R[X]} \to R$; we denote it by the same symbol θ.

More generally, there exists a split permutation homomorphism $\theta\colon \mathrm{M}(n, \overline{R[X]}) \to \mathrm{M}(n, R)$ such that θ is applied to each element of the matrix, and there also exists a decomposition $\mathrm{M}(n, \overline{R[X]}) = \mathrm{M}(n,R) \oplus \mathrm{Ker}(\theta)$. Further, θ induces the permutation homomorphism

$$\theta\colon \mathrm{M}(n, \overline{R[X]}) \to \mathrm{M}(n, R)$$

and the decomposition

$$\mathrm{M}(n, \overline{R[X]}) = \mathrm{M}(n, R) \oplus \big(\mathrm{Ker}(\theta)/\mathrm{M}(n, I)\big). \tag{10.2}$$

The last homomorphism θ is also a homomorphism of formal matrix rings

$$\mathrm{M}(n, \overline{R[X]}, X) \to \mathrm{M}(n, R, \{s_{ijk}\}).$$

Remarks.

(1) The decomposition (10.2) is only additive.

(2) The homomorphism θ is surjective. More precisely, every matrix $(a_{ij}) \in \mathrm{M}(n, R)$ is the image of the matrix (a_{ij}) in $\mathrm{M}(n, \overline{R[X]})$ (we identify matrices (a_{ij}) and $(a_{ij} + I)$).

Now we can write two commutative diagrams:

$$\begin{array}{ccc} \mathrm{M}(n, \overline{R[X]}, X) & \xrightarrow{\eta} & \mathrm{M}(n, \overline{R[X]}) \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta} \\ \mathrm{M}(n, R, \{s_{ijk}\}) & \xrightarrow{\eta} & \mathrm{M}(n, R) \end{array}, \qquad \begin{array}{ccc} \mathrm{M}(n, \overline{R[X]}) & \xrightarrow{\det} & \overline{R[X]} \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta} \\ \mathrm{M}(n, R) & \xrightarrow{\det} & R \end{array},$$

where det is the ordinary determinant. Commutativity of the second diagram is directly verified with the use of the complete development formula for the determinant. We also have the commutative diagram that unites two previous diagrams:

$$\begin{array}{ccc} \mathrm{M}(n, \overline{R[X]}, X) & \xrightarrow{d} & \overline{R[X]} \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta} \\ \mathrm{M}(n, R, \{s_{ijk}\}) & \xrightarrow{d} & R \end{array}.$$

We meant the existence of the last diagram when we said that the pair $(\mathrm{M}(n, \overline{R[X]}, X), d)$ is couniversal.

In one important case, we can simplify the construction of the ring $\mathrm{M}(n, \overline{R[X]}, X)$. We consider formal matrix rings $\mathrm{M}(n, R, \{s_{ijk}\})$ such that every factor s_{ijk} is a nonnegative integral power of some nonzero element s. We denote by $\mathrm{M}(n, R, s)$ some such ring. In Secs. 7 and 8, we circumstantially considered rings $\mathrm{M}(n, R, s)$ such that every factor s_{ijk} is equal to 1 or s. In Sec. 9, we assumed that $s_{ijk} = s^m$, $m \geq 1$, for $i \neq j$ and $j \neq k$.

For the element s, we also assume that $s^k \neq s^l$ for all distinct nonnegative k and l. Let x be a variable. For all i, j, and k, we set $x_{ijk} = x^m$ provided $s_{ijk} = s^m$. The set $\{x_{ijk} \mid i, j, k = 1, \dots, n\}$ is a system of factors in the polynomial ring $R[x]$. Consequently, there exists a formal matrix ring $\mathrm{M}(n, R[x], \{x_{ijk}\})$; it is denoted by $\mathrm{M}(n, R[x], x)$. The above three diagrams turn into diagrams

$$\begin{array}{ccc} \mathrm{M}(n, R[x], x) & \xrightarrow{\eta} & \mathrm{M}(n, R[x]) \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta} \\ \mathrm{M}(n, R, s) & \xrightarrow{\eta} & \mathrm{M}(n, R) \end{array}, \qquad \begin{array}{ccc} \mathrm{M}(n, R[x], x) & \xrightarrow{d} & R[x] \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta} \\ \mathrm{M}(n, R, s) & \xrightarrow{d} & R \end{array}.$$

In these diagrams, d is the determinant, and θ is the permutation homomorphism that replaces the symbol x by the symbol s. Tang and Zhou defined and used the ring $\mathrm{M}(n, R[x], x)$ in [23].

11. Some Theorems on Formal Matrices

We preserve the notation of the previous section. As before, we assume that we have some formal matrix ring $\mathrm{M}(n, R, \{s_{ijk}\})$, where R is a commutative ring. We show that there are analogues of the Hamilton–Cayley theorem and one familiar invertibility criterion of the matrix.

We recall that $t_{ij} = s_{ijl}$ for some fixed l. In what follows, we will use the homomorphism

$$\eta\colon \mathrm{M}(n, R, \{s_{ijk}\}) \to \mathrm{M}(n, R), \quad (a_{ij}) \to (t_{ij} a_{ij})$$

(see Sec. 5).

Let $A = (a_{ij})$ be some matrix. We denote by $\eta(A)^*$ the adjoint matrix for ηA. Then

$$\eta(A)\eta(A)^* = \eta(A)^*\eta(A) = |\eta(A)|E.$$

Further, we have $\eta(A)^* = (A'_{ji})$, where A'_{ji} is the algebraic adjunct of the element $t_{ji}a_{ji}$. We recall that A'_{ji} is the determinant of the matrix that is obtained from the matrix ηA after the replacement of the element $t_{ji}a_{ji}$ by 1 and the replacement of all remaining elements of the jth row and the ith column by 0.

We temporarily assume that all elements s_{ijk} are nonzero divisors. First, we also assume that $n \geq 3$. We take an arbitrary summand of the determinant A'_{ji}, where $j \neq i$. This summand necessarily contains the factor $t_{ik}a_{ik}t_{kj}a_{kj}$ for some $k \neq i, j$. Since $t_{ik}t_{kj} = s_{ikj}t_{ij}$, we can write $A'_{ji} = t_{ji}A_{ji}$, where A_{ji} is a certain element of the ring R canonically obtained from A'_{ji}. If $i = j$, then $t_{ij} = 1$ and $A_{ji} = A'_{ji}$. For $n = 2$, the existence of such an element t_{ij} is directly verified.

We form the matrix $A^* = (A_{ji})$. We have $\eta(A^*) = \eta(A)^*$. Further,

$$\eta(AA^*) = \eta(A)\eta(A^*) = \eta(A)\eta(A)^* = |\eta(A)| \cdot E = d(A)E = \eta(d(A)E);$$

similarly, we have

$$\eta(AA^*) = \eta(d(A)E).$$

Since all s_{ijk} are nonzero divisors, $\mathrm{Ker}(\eta) = 0$ by Proposition 5.2. Consequently, $AA^* = A^*A = d(A)E$.

How can we construct the matrix A^* if not all elements s_{ijk} are nonzero divisors? For this purpose, we use the commutative diagram from Sec. 10:

$$\begin{array}{ccc} \mathrm{M}(n, \overline{R[X]}, X) & \xrightarrow{d_X} & \overline{R[X]} \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} \\ \mathrm{M}(n, R, \{s_{ijk}\}) & \xrightarrow{d} & R \end{array} \qquad (11.1)$$

(we have replaced d by d_X in the first row of the diagram). Since each x_{ijk} is a nonzero divisor, it follows from the above that there exists a unique matrix $A^*_X \in \mathrm{M}(n, \overline{R[X]}, X)$ such that $AA^*_X = A^*_X A = d_X(A)E$. Therefore, we have that

$$\theta(A)\theta(A^*_X) = \theta(A^*_X)\theta(A) = \theta(d_X(A)E)$$

or

$$A\theta(A^*_X) = \theta(A^*_X)A = \theta(d_X(A)E).$$

It follows from the diagram that $\theta d_X(A) = d\theta(A) = d(A)$, whence $\theta(d_X(A)E) = d(A)E$. As a result, we obtain the relations

$$A\theta(A^*_X) = \theta(A^*_X)A = d(A)E.$$

It remains to set $A^* = \theta(A^*_X)$. If all elements s_{ijk} are nonzero divisors, then this matrix A^* coincides with the matrix A^* defined in the text before the diagram. If we compare similar construction methods of the matrix A^* before the diagram (11.1) and of the matrix A^*_X, then it is clear that $A^* = \theta(A^*_X)$.

We finish the first part of Sec. 11 with the following results.

Theorem 11.1. *Let us have a formal matrix ring* $\mathrm{M}(n, R, \{s_{ijk}\})$ *and let A be a matrix in this ring.*

(1) $AA^* = AA^* = d(A)E$.

(2) *The matrix A is invertible if and only if $d(A)$ is an invertible element of the ring A.*

(3) *If A is an invertible matrix, then* $A^{-1} = d(A)^{-1}A^*$.

For the determinant $d(A)$, there exist analogues of the decomposition of a determinant relative to elements of a row and the orthogonality property of rows and algebraic adjuncts. We mean the relations

$$a_{i1} \circ A_{i1} + a_{i2} \circ A_{i2} + \cdots + a_{in} \circ A_{in} = d(A),$$
$$a_{i1} \circ A_{j1} + a_{i2} \circ A_{j2} + \cdots + a_{in} \circ A_{jn} = 0, \quad i \neq j,$$

following from the relation $AA^* = d(A)E$.

The familiar Hamilton-Cayley theorem states that every matrix is a root of its characteristic polynomial. We extend this theorem on matrices in the matrix ring $\mathrm{M}(n, R, \{s_{ijk}\})$.

Let x be a variable. We have the formal matrix ring $\mathrm{M}(n, R[x], \{s_{ijk}\})$ and the homomorphism $\eta\colon \mathrm{M}(n, R[x], \{s_{ijk}\}) \to \mathrm{M}(n, R[x])$ from Proposition 5.2. This homomorphism extends the homomorphism $\eta\colon \mathrm{M}(n, R, \{s_{ijk}\}) \to \mathrm{M}(n, R)$.

Now we take some symbol λ. Let A be a matrix in the ring $\mathrm{M}(n, R, \{s_{ijk}\})$. With the use of the relation $f(\lambda) = d(\lambda E - A)$, we define the characteristic polynomial $f(\lambda)$ of the matrix A (with respect to the ring $\mathrm{M}(n, R, \{s_{ijk}\})$), where $d(\lambda E - A)$ is the determinant of the matrix $\lambda E - A$ in the ring $\mathrm{M}(n, R[\lambda], \{s_{ijk}\})$. Since

$$d(\lambda E - A) = |\eta(\lambda E - A)| = |\lambda E - \eta A|,$$

we have that $f(\lambda)$ is the characteristic polynomial of the matrix ηA in the ring $\mathrm{M}(n, R)$. By the Hamilton–Cayley theorem, $f(\eta A) = 0$. By the use of the relation

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n, \quad a_i \in R,$$

we obtain that $\eta(fA) = f(\eta A) = 0$. If η is an injective mapping (i.e., all s_{ijk} are nonzero divisors), then $f(A) = 0$, which is required.

If the mapping η is not injective, then we again consider the ring $\mathrm{M}(n, \overline{R[X]}, X)$ from Sec. 10; for this ring, the corresponding homomorphism η is injective. The ring $\mathrm{M}(n, (\overline{R[X]})[\lambda], X)$ exists. The homomorphism $\theta\colon \overline{R[X]} \to R$ from Sec. 10 induces the homomorphism $\theta\colon (\overline{R[X]})[\lambda] \to R[\lambda]$ that applies θ to coefficients of polynomials. Finally, the last homomorphism θ induces the homomorphism

$$\theta\colon \mathrm{M}(n, (\overline{R[X]})[\lambda], X) \to \mathrm{M}(n, R[\lambda], \{s_{ijk}\}).$$

We also have the following commutative diagram, which is similar to the diagram from Sec. 10:

$$\begin{array}{ccc} \mathrm{M}(n, (\overline{R[X]})[\lambda], X) & \xrightarrow{d} & (\overline{R[X]})[\lambda] \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} \\ \mathrm{M}(n, R[\lambda], \{s_{ijk}\}) & \xrightarrow{d} & R[\lambda] \end{array} \quad . \qquad (11.2)$$

We return to the characteristic polynomial $f(\lambda)$ of the matrix $A = (a_{ij})$. We take the matrix $\bar{A} = (\bar{a}_{ij})$ from the ring $\mathrm{M}(n, \overline{R[X]}, X)$ with $\theta(\bar{A}) = A$, where θ is taken from the diagram (11.1). Let $F(\lambda)$ be the characteristic polynomial of the matrix $\bar{A}$, i.e., $F(\lambda) = d(\lambda E - \bar{A})$, where $\lambda E - \bar{A} \in \mathrm{M}(n, (\overline{R[X]})[\lambda], X)$. Then $\theta(\lambda E - \bar{A}) = \lambda E - A$, where θ is taken from the diagram (11.2). It follows from the diagram (11.2) that

$$\theta d(\lambda E - \bar{A}) = d\theta(\lambda E - \bar{A}) = d(\lambda E - A).$$

In other words, $\theta\big(F(\lambda)\big) = f(\lambda)$, where

$$F(\lambda) \in (\overline{R[X]})[\lambda], \quad f(\lambda) \in R[\lambda], \quad \theta\colon (\overline{R[X]})[\lambda] \to R[\lambda].$$

We represent the polynomial $F[\lambda]$ more explicitly:

$$F(\lambda) = \lambda^n + \bar{a}_1\lambda^{n-1} + \cdots + \bar{a}_{n-1}\lambda + \bar{a}_n, \quad \bar{a}_i \in \overline{R[X]}.$$

Then we have

$$f(\lambda) = \theta\big(F(\lambda)\big) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n.$$

Now it follows from the above that

$$F(\bar{A}) = \bar{A}^n + \bar{a}_1\bar{A}^{n-1} + \cdots + \bar{a}_{n-1}\bar{A} + \bar{a}_n E = 0.$$

Therefore,

$$0 = \theta\big(F(\bar{A})\big) = A^n + a_1A^{n-1} + \cdots + a_{n-1}A + a_nE = f(A).$$

We have proved the following result.

Theorem 11.2. *If A is a matrix in the ring $\mathrm{M}(n, R, \{s_{ijk}\})$ and $f(\lambda)$ is the characteristic polynomial of the matrix A, then $f(A) = 0$.*

Tang and Zhou [23] proved Theorem 11.2 for their formal matrix rings (see the beginning of Sec. 7).

Finally, we consider the uniqueness property of a determinant. For the ordinary determinant of matrices, some combinations of properties (1)–(8) of Sec. 10 are characteristic combinations. We mean that if some mapping $f\colon \mathrm{M}(n, R) \to R$ satisfies some of the properties (1)–(8), then f coincides with the determinant. We show that under some restrictions introduced in Sec. 10 the determinant properties (1)–(3) are characteristic properties in the mentioned sense.

Theorem 11.3. *Let $\mathrm{M}(n, R, \{s_{ijk}\})$ be a formal matrix ring, where the ring R does not have elements of additive order 2 and all factors s_{ijk} are nonzero divisors. In addition, let $f\colon \mathrm{M}(n, R, \{s_{ijk}\}) \to R$ be a mapping such that the following properties are true:*

(1) $f(E) = 1$;

(2) *f is a polylinear function of rows of the matrix*;

(3) *if we interchange rows with numbers i and j in an arbitrary matrix $A \in \mathrm{M}(n, R, \{s_{ijk}\})$ and denote the obtained matrix by A', then $f(A') = -s_{iji} f(A)$.*

Then f coincides with the determinant d of the ring $\mathrm{M}(n, R, \{s_{ijk}\})$.

Proof. Let $A = (a_{ij})$ be an arbitrary matrix. We use several times the polylinearity of the function f for the representation of the element $f(A)$ as a sum of elements of the form $f(C)$, where C is a matrix such that the ith row of the matrix C contains the element a_{ij} on the position (i, j), and the remaining elements are equal to zero. Then from every row of the matrix C, we interchange this element a_{ij} with the sign of the mapping f. We denote by D the obtained matrix (every row of the matrix D has 1 on at most one position, and D has 0 on the remaining positions). Considering (2), we obtain that $f(C) = a_{1j1} \cdot \ldots \cdot a_{njn} f(D)$.

We show that if some of the subscripts $j_1, \ldots, j_n$ are equal to each other, then $f(D) = 0$. For example, let us assume that 1 stands on the positions (i, k) and (j, k), $i \neq j$. In the matrix D, we interchange rows with subscripts i and j; we denote the obtained matrix by D'. By (3), $f(D') = -s_{iji} f(D)$. It follows from (2) that

$$f(D') = s_{ijk} s_{jik} f(D) = s_{iji} f(D).$$

Therefore, we have the relations

$$s_{iji} f(D) = -s_{iji} f(D), \quad 2s_{iji} f(D) = 0, \quad 2f(D) = 0, \quad f(D) = 0.$$

Thus, there only remain elements of $f(C)$ such that subscripts of elements $a_{1j_1}, \ldots, a_{nj_n}$ form a permutation. In addition, if some row of the matrix D consists of zeros, then $f(D) = 0$ by property (2). Therefore, we assume that every row of any matrix D contains exactly one element that is equal to 1, and the remaining elements are equal to zero; this means that D is a permutation matrix. Let τ be a permutation of subscripts of nonzero elements of the matrix D. We have $\tau = \sigma_1 \cdot \ldots \cdot \sigma_m$, where σ_i are pairwise independent cycles. We denote by σ one of these cycles; let $\sigma = (i_1 i_2 \ldots i_k)$. By interchanging rows with numbers $i_1, i_2, \ldots, i_k$, we obtain the situation such that each 1 from these rows is placed on the main diagonal. How will the element $f(D)$ be changed in this situation? We will know this with the use of induction on the length of the cycle σ.

For $m = 2$ and $m = 3$, the induction hypothesis is directly verified. We assume that for any cycle σ of length m with $3 \leq m < k$, the relation

$$f(D) = (-1)^{m-1} s_{i_1 i_2 i_3} s_{i_1 i_3 i_4} \cdot \ldots \cdot s_{i_1 i_{m-1} i_m} s_{i_1 i_m i_1} f(V)$$

holds. The difference between the matrices V and D consists in the property that the rows of the matrix V with numbers $i_1, \ldots, i_m$ have 1 on the main diagonal.

Now let σ have the length k. We interchange rows of the matrix D with numbers i_1 and i_2. The obtained matrix is denoted by D'. The matrices D and D' are interrelated by the relation $f(D') =$

$-s_{i_1i_2i_1}f(D)$. In the matrix D', we take out of the sign f the element $s_{i_1i_2i_3}$ of the row i_1 and the element $s_{i_2i_1i_3}$ of the row i_2. We denote by D'' the obtained matrix. We have the relation $f(D') = s_{i_2i_1i_2}s_{i_1i_2i_3}f(D'')$. It follows from the last two relations that $f(D) = -s_{i_1i_2i_3}f(D'')$. On the position (i_2, i_2) of the matrix D'', we have 1, and subscripts of elements (i.e., of 1) in the remaining rows form a cycle $(i_1i_3 \dots i_k)$ of length $k-1$. By the induction hypothesis, we obtain the relation

$$f(D'') = (-1)^{k-2}s_{i_1i_3i_4}s_{i_1i_4i_5} \cdot \ldots \cdot s_{i_1i_{k-1}i_k}s_{i_1i_ki_1}f(W),$$

where W is the matrix that has 1 in rows with numbers $i_1, \dots, i_k$ on the main diagonal, and the remaining rows coincide with the corresponding rows of the matrix D. Finally, we obtain the relation

$$f(D) = (-1)^{k-1}s_{i_1i_2i_3}s_{i_1i_3i_4} \cdot \ldots \cdot s_{i_1i_{k-1}i_k}s_{i_1i_ki_1}f(W).$$

Since the determinant function d satisfies properties (1)–(3), a similar relation holds for d.

Further, we deal similarly with the remaining cycles σ_i in the decomposition of the permutation τ. Finally, we obtain that the corresponding matrix W coincides with E. Therefore, $f(W) = d(W)$. Thus, we obtain that

$$f(D) = d(D), \quad f(C) = d(C), \quad f(A) = d(A). \qquad \square$$

Remarks.

(1) A matrix $A = (a_{ij})$ that is not invertible in the ring $\mathrm{M}(n, R)$ can be invertible in the ring $\mathrm{M}(n, R, \Sigma)$, where Σ is some system of factors. We formulate the following problem: Characterize matrices A that are invertible in some ring $\mathrm{M}(n, R, \Sigma)$. By considering the existence of the homomorphism

$$\eta\colon \mathrm{M}(n, R, \Sigma) \to \mathrm{M}(n, R)$$

from Sec. 5, we can give another form to this problem. For which matrices A and systems of factors $\Sigma = \{s_{ijk}\}$ is the matrix $A' = (s_{ijl}a_{ij})$ invertible in the ring $\mathrm{M}(n, R)$ for some integer $l = 1, 2, \dots, n$?

(2) We can define and study the permanent of the matrix A in the ring $\mathrm{M}(n, R, \Sigma)$. It is useful to remark that for several problems related to determinants and permanents, there is the possibility of variations related to the existence of many systems of factors Σ. For example, in the case of the familiar Polya problem about conversion of permanents and determinants, we can require that the permanent of the matrix A in the ring $\mathrm{M}(n, R, \Sigma)$ coincides with the determinant of some other matrix in the ring $\mathrm{M}(n, R, \Sigma')$ for some system of factors Σ' (the system is new in the general case).

(3) Let F be a field and let $\Sigma = \{s_{ijk}\}$ be some system of factors $s_{ijk} \in F$. If all s_{ijk} are not equal to zero, then it follows from Proposition 5.2 that there is an isomorphism $\mathrm{M}(n, F, \Sigma) \cong \mathrm{M}(n, F)$. This case is not very interesting. If some factors s_{ijk} are equal to zero, then properties of the ring $\mathrm{M}(n, F, \Sigma)$ may strongly differ from properties of the ring $\mathrm{M}(n, F)$. We distinguish a particular case where every factor s_{ijk} is equal to 1 or 0. For the corresponding rings $\mathrm{M}(n, F, \Sigma)$, we can study Problems (I)–(III) formulated in Sec. 5. In addition, we can consider the above questions (1) and (2).

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