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EXPLICIT FORM OF THE FUNDAMENTAL SOLUTION TO A SECOND ORDER PARABOLIC OPERATOR

A. G. Chechkin
Lomonosov Moscow State University
Moscow 119991, Russia
a.g.chechkin@gmail.com

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We propose and justify an explicit representation of the fundamental solution to a system of parabolic equations with special initial conditions. Bibliography: 5 *titles.*

1 Statement of the Problem and the Main Results

Suppose that $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $t \in \mathbb{R}_+ = [0, +\infty)$, and $u(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is a real-valued function in $\mathbb{R}_+ \times \mathbb{R}^n$ with continuous partial derivatives ∂_t, ∂_{x_k}, $\partial^2_{x_k x_l}$, $k, l = 1, \dots, n$. The class of such functions is denoted by $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R})$. We say that an operator $\mathfrak{L}$ is a *second order operator* if it has the form

$$\mathfrak{L}[v] = \frac{1}{2} \sum_{ij} A_{ij}(t) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_i \Big(\sum_j B_{ij}(t) x_j + c_i(t) \Big) \frac{\partial v}{\partial x_i}$$
$$+ \Big(\sum_{ij} F_{ij}(t) x_i x_j + \sum_i g_i(t) x_i + h(t) \Big) v,$$

where $A_{ij}(t)$, $B_{ij}(t)$, $c_i(t)$, $F_{ij}(t)$, $g_i(t)$, $i, j = 1, \dots, n$, and $h(t)$ are some functions. Respectively, we consider the *second order equation*

$$\dot{u} = \mathfrak{L}[u] \tag{1.1}$$

Such operators are used in Kolmogorov, Bellman type equations and so on (cf., for example, [1] and the bibliography therein).

1.1. Assumptions.

Assumption A. The coefficients $A(t)$, $B(t)$, $F(t) : \mathbb{R}_+ \to \mathbf{M}_{n\times n}(\mathbb{R})$ of the second order operator $\mathfrak{L}$ are continuous functions in $\mathbb{R}_+$ and have finite limits A_0, B_0, $F_0 \in \mathbf{M}_{n\times n}(\mathbb{R})$ as $t \to 0$. The matrix $A(t)$ is symmetric, and the matrix A_0 is positive definite.

Let a solution to the Cauchy problem

$$P' = -P(2AS + B^T) - (2AS + B)P - 2A, \quad P|_{t=0} = \mathbf{0} \in \mathbf{M}_{n\times n}(\mathbb{R}), \tag{1.2}$$

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where $S(t)$ is a symmetric solution to the problem

$$S' = 2SAS + SB + B^T S + \frac{1}{2}(F + F^T), \quad S|_{t=0} = \mathbf{0} \in \mathbf{M}_{n\times n}(\mathbb{R}), \tag{1.3}$$

is represented as

$$P(t) = -2tA_0 + R(t) \tag{1.4}$$

in a neighborhood of zero, where $R(t)$ is a matrix defined on $[0, \varepsilon]$, $0 < \varepsilon \ll 1$.

Remark 1.1. The existence of solutions to the problems (1.2) and (1.3) and their properties are discussed in Section 2 (cf. Propositions 2.2 and 2.4).

We introduce the notation

$$Q(t) = -\frac{1}{2t} R(t) A_0^{-1}, \tag{1.5}$$

$$\overline{Q}(t) = [E + Q(t)]^{-1} - E, \tag{1.6}$$

$$\widetilde{Q}(t) = \overline{Q}(t) + (A(t) - A_0)A_0^{-1}[E + \overline{Q}], \tag{1.7}$$

$$\widetilde{q}(t) = \frac{1}{n} \operatorname{tr} \widetilde{Q}. \tag{1.8}$$

Assumption B. The following improper integral exists:

$$\int_0^t \frac{\widetilde{q}(s)}{s} ds < +\infty, \quad 0 \leqslant t < \varepsilon.$$

Remark 1.2. Assumption B is satisfied if $\widetilde{q}(t)$ is $O(t^\gamma)$ as $t \to 0$, where $\gamma > 0$,

1.2. The main theorem. We consider two systems of differential equations

$$\begin{cases} S' = 2SAS + SB + B^T S + \dfrac{1}{2}(F + F^T), \\ q' = (2SA + B^T)q + 2Sc + g, \\ r' = \operatorname{tr}(AS) + \dfrac{1}{2} q^T A q + q^T c + h, \end{cases} \tag{1.9}$$

$$\begin{cases} P' = -P(2SA + B^T) - (2AS + B)P - 2A, \\ m' = -(2AS + B)m - Aq - c, \\ C' = C \cdot (\operatorname{tr}(AP^{-1})) \end{cases} \tag{1.10}$$

and the Cauchy problem

$$\frac{\partial u}{\partial t} = \mathfrak{L}[u], \quad u\big|_{t=0} = \delta_y(x), \tag{1.11}$$

where $\delta_y(x)$ is a delta-function with singularity at $y \in \mathbb{R}^n$.

Theorem 1.1. *Suppose that Assumptions* A *and* B *are satisfied. Then the solution* $u(t, x)$ *to the Cauchy problem* (1.11) *has the form*

$$\exp\left\{x^T S(t)x + q^T(t)x + r(t)\right\} C(t) \exp\left\{\left\langle P^{-1}(t)(x - m(t; y)), (x - m(t; y))\right\rangle\right\}, \tag{1.12}$$

where $S(t)$, $q(t)$, $r(t)$ are solutions to the system (1.9) *with the initial conditions $S_{ij}(0) = q_k(0) = r(0) = 0$, $i, j, k \in \overline{1, n}$, $P(t)$, $m(t; y)$ are solutions of the first two equations of the system* (1.10) *with the initial conditions $P_{ij}(0) = 0$, $i, j \in \overline{1, n}$, $m(0) = y$, and $C(t)$ is a partial solution of the third equation of the system* (1.10) *of the form*

$$C(t) = \frac{1}{\sqrt{(2\pi t)^n \det A_0}} \exp\left\{ -\frac{n}{2} \int_0^t \frac{\widetilde{q}(s)}{s} ds \right\}.$$

2 Auxiliaries

We use the method for studying parabolic equations proposed in [2]).

2.1. Integral of exponential of quadratic functions.

Proposition 2.1. *Let $v(x) = \langle Sx, x\rangle + \langle q, x\rangle + r$ be a quadratic form, where $S \in \mathbf{M}_{n\times n}(\mathbb{R})$ is a symmetric matrix, $q \in \mathbb{R}^n$, and $r \in \mathbb{R}$. The integral*

$$\int_{\mathbb{R}^n} \exp v(x)\ dx_1 \dots dx_n \tag{2.1}$$

exists if and only if the matrix $S(t)$ is negative definite; *moreover,*

$$\int_{\mathbb{R}^n} \exp\{\langle Sx, x\rangle + \langle q, x\rangle + r\} dx_1 \dots dx_n = \pi^{n/2} |\det S|^{-1/2} \exp\left\{ r - \frac{1}{4}\langle S^{-1}q, q\rangle \right\}. \tag{2.2}$$

Proof. As is known, the quadratic form with a symmetric matrix with real entries can be reduced to the diagonal form by an orthogonal transformation, i.e., for a symmetric matrix $S = S^T$ there exists an orthogonal matrix $O : O^{-1} = O^T$ such that $O^T SO = \Lambda$, $\Lambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$. Making the change of variables $x = Oy$, we can write the quadratic form $\langle Sx, x\rangle$ in the form

$$\langle Sx, x\rangle = \langle SOy, Oy\rangle = \langle O^T SOy, y\rangle = \langle \Lambda y, y\rangle = \sum_{k=1}^n \lambda_k y_k^2,$$

where λ_k, $k = 1, \dots, n$, are eigenvalues of the matrix S. Note that λ_k, $k = 1, \dots, n$, are real because the matrix S is symmetric. It is obvious that $\det S = \lambda_1 \cdot \ldots \cdot \lambda_n$.

In the variables $y = (y_1, \dots, y_n)$, the function $v(x)$ takes the form

$$v(x) = \langle \Lambda y, y\rangle + \langle O^T q, y\rangle + r = \sum_{k=1}^n [\lambda_k y_k^2 + \widetilde{q}_k y_k] + r,$$

where $\widetilde{q}_k$ is the kth component of the vector $\widetilde{q} = O^T q = O^{-1} q$. Hence the integral (2.1) (of multiplicity n) exists if and only if the following n single integrals

$$I_k = \int_{-\infty}^{+\infty} \exp\{\lambda_k y_k^2 + \widetilde{q}_k y_k\}\ dy_k.$$

simultaneously exist. The last condition is satisfied if and only if $\lambda_k < 0$ for all $k = 1, \dots, n$. Moreover,

$$\lambda_k y_k^2 + \widetilde{q}_k y_k = -\Big[(-\lambda_k)y_k^2 - \widetilde{q}_k y_k + \Big(-\frac{1}{4\lambda_k}\Big)\widetilde{q}_k^2\Big] - \frac{1}{4\lambda_k}\widetilde{q}_k^2$$
$$= -\Big[\sqrt{-\lambda_k}y_k - \frac{1}{2\sqrt{-\lambda_k}}\widetilde{q}_k\Big]^2 - \frac{1}{4\lambda_k}\widetilde{q}_k^2.$$

Consequently,

$$I_k = \exp\Big\{-\frac{1}{4\lambda_k}\widetilde{q}_k^2\Big\} \cdot \frac{\sqrt{\pi}}{\sqrt{-\lambda_k}} \tag{2.3}$$

since (independently of b in the case $a > 0$)

$$\int_{-\infty}^{+\infty} e^{-(ay+b)^2}dy = \frac{\sqrt{\pi}}{a},$$

From (2.3) and the equality $dx_1 \dots dx_n = dy_1 \dots dy_n$, valid in view of the orthogonality of the matrix O, it follows that

$$\int_{\mathbb{R}^n} \exp v(x)dx_1 \dots dx_n = \exp\Big\{r - \sum_{k=1}^n \frac{1}{4\lambda_k}\widetilde{q}_k^2\Big\} \prod_{k=1}^n \frac{\sqrt{\pi}}{\sqrt{-\lambda_k}}. \tag{2.4}$$

We note that the matrix $\operatorname{diag}\{1/\lambda_1, \dots, 1/\lambda_n\}$ is the inverse of Λ. Therefore,

$$\sum_{k=1}^n \frac{1}{4\lambda_k}\widetilde{q}_k^2 = \frac{1}{4}\langle \Lambda^{-1}\widetilde{q}, \widetilde{q}\rangle = \frac{1}{4}\langle (O^T S O)^{-1}\widetilde{q}, \widetilde{q}\rangle$$
$$= \frac{1}{4}\langle O^T S^{-1} O\widetilde{q}, \widetilde{q}\rangle = \frac{1}{4}\langle S^{-1}O\widetilde{q}, O\widetilde{q}\rangle = \frac{1}{4}\langle S^{-1}q, q\rangle;$$

moreover,

$$\prod_{k=1}^n \frac{\sqrt{\pi}}{\sqrt{-\lambda_k}} = \frac{(\sqrt{\pi})^n}{\sqrt{(-1)^n \det S}} = \pi^{n/2}|\det S|^{-1/2}.$$

Therefore, the equality (2.4) can be written in the form

$$\int_{\mathbb{R}^n} \exp\{\langle Sx, x\rangle + \langle q, x\rangle + r\}dx_1 \dots dx_n = \pi^{n/2}|\det S|^{-1/2} \exp\Big\{r - \frac{1}{4}\langle S^{-1}q, q\rangle\Big\}, \tag{2.5}$$

which is a formula for computing the integral of exponential of quadratic functions, valid provided that all the eigenvalues of the matrix S are negative. □

2.2. Parabolic equation. We consider the equation

$$\frac{\partial u(t,x)}{\partial t} = \mathfrak{I}[u(t,x)], \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{2.6}$$

where

$$\mathfrak{I}[u] = \sum_{k,l=1}^n \frac{1}{2}a_{kl}(t,x)\frac{\partial^2 u}{\partial x_k \partial x_l} + \sum_{k=1}^n b_k(t,x)\frac{\partial u}{\partial x_k} + C(t,x)\cdot u, \tag{2.7}$$

or, in the vector form,

$$\frac{\partial u}{\partial t} = \Big\langle \frac{1}{2}A, \frac{\partial^2 u}{\partial x^2}\Big\rangle + \Big\langle b, \frac{\partial u}{\partial x}\Big\rangle + Cu. \tag{2.8}$$

Lemma 2.1. *Let $\rho \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R})$, $\rho(t,x) > 0$, be a positive solution to Equation* (2.6)*. Then a function $v \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R})$ of the form*

$$v(t,x) = \frac{u(t,x)}{\rho(t,x)}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \tag{2.9}$$

satisfies the equation

$$\frac{\partial v}{\partial t} = \left\langle \frac{1}{2} A, \frac{\partial^2 v}{\partial x^2} \right\rangle + \left\langle A \frac{\partial \ln \rho}{\partial x} + b, \frac{\partial v}{\partial x} \right\rangle. \tag{2.10}$$

Proof. Differentiating the identity (2.9), we get

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial \rho}{\partial t} \cdot v + \rho \cdot \frac{\partial v}{\partial t}, \\
\frac{\partial u}{\partial x_k} &= \frac{\partial \rho}{\partial x_k} \cdot v + \rho \cdot \frac{\partial v}{\partial x_k}, \\
\frac{\partial^2 u}{\partial x_k \partial x_l} &= \frac{\partial^2 \rho}{\partial x_k \partial x_l} \cdot v + \frac{\partial \rho}{\partial x_k} \frac{\partial v}{\partial x_l} + \frac{\partial \rho}{\partial x_l} \frac{\partial v}{\partial x_k} + \rho \cdot \frac{\partial^2 v}{\partial x_k \partial x_l}.
\end{aligned}$$

Substituting the obtained expressions into Equation (2.8), we find

$$\begin{aligned}
\frac{\partial \rho}{\partial t} \cdot v + \rho \cdot \frac{\partial v}{\partial t} = &\Big[\Big\langle \frac{1}{2} A, \frac{\partial^2 \rho}{\partial x^2} \Big\rangle \cdot v + \Big\langle A \frac{\partial \rho}{\partial} x, \frac{\partial v}{\partial x} \Big\rangle + \rho \Big\langle \frac{1}{2} A, \frac{\partial^2 v}{\partial x^2} \Big\rangle \Big] \\
&+ \Big[\Big\langle v, \frac{\partial \rho}{\partial x} \Big\rangle \cdot v + \rho \cdot \Big\langle b, \frac{\partial v}{\partial x} \Big\rangle \Big] + C \cdot [\rho \cdot v].
\end{aligned}$$

Rearranging the terms, we find

$$\rho \cdot \frac{\partial v}{\partial t} = \rho \cdot \Big\langle \frac{1}{2} A, \frac{\partial^2 v}{\partial x^2} \Big\rangle + \Big\langle A \frac{\partial \rho}{\partial x}, \frac{\partial v}{\partial x} \Big\rangle + \rho \cdot \Big\langle b, \frac{\partial v}{\partial x} \Big\rangle + \Big(- \frac{\partial \rho}{\partial t} + \mathfrak{L}[\rho] \Big) \cdot v. \tag{2.11}$$

The last term on the right-hand side of (2.11) vanishes since ρ satisfies Equation (2.6). Dividing both sides of (2.11) by $\rho > 0$, we obtain (2.10). □

2.3. Second order equation. The positive functions

$$\rho(t,x) = \exp \left\{ x^T S(t) x + q^T(t) x + r(t) \right\}, \tag{2.12}$$

where $S \in \mathbf{M}_{n \times n}(\mathbb{R})$, satisfy an equation of the form (1.1) (cf. [2]), and the following assertion holds (cf. [2, Theorem 3.1]).

Lemma 2.2. *A function $\rho \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R})$ of the form* (2.12) *is a solution to Equation* (1.1) *if and only if the coefficients $S(t)$, $q(t)$, $r(t)$ satisfy the system of equations*

$$\begin{aligned}
S' &= \frac{1}{2} S^T A S + \frac{1}{2} S A S^T + S A S + S B + B^T S^T + \frac{1}{2}(F + F^T), \\
q' &= (S A + S^T A + B^T) q + S c + S^T c + g, \\
r' &= \operatorname{tr}(A S) + \frac{1}{2} q^T A q + q^T c + h.
\end{aligned} \tag{2.13}$$

We consider the Cauchy problem

$$S' = \frac{1}{2}S^T AS + \frac{1}{2}SAS^T + SAS + SB + B^T S^T + \frac{1}{2}(F + F^T),$$
$$S|_{t=0} = \mathbf{0} \in \mathbf{M}_{n\times n}(\mathbb{R}).$$

We look for a solution to this problem in the space of symmetric matrices $\mathbf{M}_{n\times n}(\mathbb{R})$. The Cauchy problem can be written in the form (1.3), and the system (2.13) takes the form (1.9).

Proposition 2.2. *Let Assumption* A *hold. Then the solution* $S(t)$ *to the problem* (1.3) *exists in an* ϵ_1*-neighborhood of zero.*

Proof. Indeed, by Assumption A, $\lim\limits_{t\to 0} S'(t) = \frac{1}{2}(F_0 + F_0^T)$ exists, i.e., $S'(t) = \frac{1}{2}(F_0 + F_0^T) + o(1)$. This means that

$$S(t) = \frac{1}{2}t(F_0 + F_0^T) + K(t), \tag{2.14}$$

where $K(t) = (k_{ij}(t))$ is a symmetric matrix; moreover, $k_{ij}(t) = o(t)$ as $t \to 0$, i.e., there exists $\epsilon_1 > 0$ such that the representation (2.14) holds. □

Definition 2.1. A system of the form (2.13) is called a *Riccati type system* (cf. [3]–[5]).

Corollary 2.1. *Let* $S(t)$*,* $q(t)$*,* $r(t)$ *satisfy the system* (1.9)*. Then the function* v *of the form* (2.9) *in Lemma* 2.1 *with* ρ *of the form* (2.12) *in Equation* (1.1) *satisfies the equation*

$$\frac{\partial v}{\partial t} = \left\langle \frac{1}{2}A, \frac{\partial^2 v}{\partial x^2} \right\rangle + \left\langle \widehat{B}x + \widehat{c}, \frac{\partial v}{\partial x} \right\rangle, \tag{2.15}$$

where $\widehat{B} = 2AS + B$ *and* $\widehat{c} = Aq + c$.

To justify formulas for $\widehat{B}$ and $\widehat{c}$, we recall that a function ρ of the form (2.12) satisfies

$$\frac{\partial \ln \rho}{\partial x} = 2Sx + q.$$

2.4. A Riccati type system with singularity. Consider functions $u(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ of class $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R})$ satisfying (1.1). We look for a solution to Equation (1.1) in the form

$$u(t,x) = \exp\{x^T P^{-1}(t)x + q^T(t)x + r(t)\}, \tag{2.16}$$

where P^{-1} is a symmetric matrix. According to Lemma 2.2, we have the system

$$\begin{aligned}
(P^{-1})' &= 2P^{-1}AP^{-1} + P^{-1}B + B^T P^{-1} + \frac{1}{2}(F + F^T),\\
q' &= (2P^{-1}A + B^T)q + 2P^{-1}c + g,\\
r' &= \operatorname{tr}(AP^{-1}) + \frac{1}{2}q^T Aq + q^T c + h.
\end{aligned} \tag{2.17}$$

Proposition 2.3. *If entries of an invertible matrix* $P(t)$ *are differentiable, then the entries of the inverse matrix* $P^{-1}(t)$ *are also differentiable*; *moreover,*

$$(P^{-1}(t))' = -P^{-1}(t)P'(t)P^{-1}(t). \tag{2.18}$$

Proof. Each entry of the matrix $P^{-1}(t)$ can be expressed as the ratio of two polynomials in entries of $P(t)$ taken with a suitable sign: the cofactor corresponding to a chosen element and the determinant of $P(t)$ different from zero since the matrix $P(t)$ is nonsingular. Therefore, entries of $P^{-1}(t)$ are differentiable. Differentiating the identity $P(t)P^{-1}(t) \equiv E$ with respect to t, we obtain the relation $P'(t)P^{-1}(t) + P(t)(P^{-1}(t))' = 0$ which immediately implies (2.18). □

It is easy to show that a function $u(t, x)$ of the form (2.16) is also represented as

$$u(t,x) = C(t) \cdot \exp\{\langle P^{-1}(t)(x - m(t)), (x - m(t))\rangle\}, \tag{2.19}$$

where

$$m = -\frac{1}{2} Pq, \tag{2.20}$$

$$C = \exp\Big\{r - \frac{1}{4}\langle Pq, q\rangle\Big\}. \tag{2.21}$$

Note that the nonsingularity of P^{-1} in a neighborhood of zero follows from Proposition 2.6. We set $\widetilde{B} = -B^T$.

Lemma 2.3. *A function* $u \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R})$ *of the form* (2.19) *is a solution to Equation* (1.1) *if and only if* $P(t)$, $m(t)$, $C(t)$ *satisfy the equations*

$$\begin{aligned} P' &= -\frac{1}{2} P(F + F^T)P + P\widetilde{B} + \widetilde{B}^T P - 2A, \\ m' &= -\frac{1}{2} P(F + F^T)m + \widetilde{B}^T m - c - \frac{1}{2} Pg, \\ C' &= C \cdot [\mathrm{tr}\,(AP^{-1}) + \frac{1}{2}\langle (F + F^T)m, m\rangle + \langle g, m\rangle + h]. \end{aligned} \tag{2.22}$$

Proof. Using (2.18), we transform the first equation of the system (2.17) as follows:

$$-P^{-1}P'P^{-1} = 2P^{-1}AP^{-1} + P^{-1}B + B^T P^{-1} + \frac{1}{2}(F + F^T).$$

Multiplying both sides of this equality from the left and from the right by the matrix P, we obtain the first equation of (2.22). Differentiating the identity (2.20), we find

$$m' = -\frac{1}{2}(P'q + Pq').$$

Substituting the expressions for P' and q' from (2.22) and (2.17) respectively, we get

$$m' = -\frac{1}{2}\Big(-\frac{1}{2}P(F + F^T)Pq + \widetilde{B}^T Pq + 2c + Pg\Big) = -\frac{1}{2}P(F + F^T)m + \widetilde{B}^T m - c - \frac{1}{2}Pg.$$

Hence we obtain the second equation of (2.22). Finally, differentiating (2.21), we have

$$C' = \exp\Big\{r - \frac{1}{4}\langle Pq, q\rangle\Big\} \cdot \Big(r - \frac{1}{4}\langle Pq, q\rangle\Big)' = C \cdot \Big(r - \frac{1}{4}\langle Pq, q\rangle\Big)'.$$

However, $(\langle Pq, q\rangle)' = \langle P'q, q\rangle + 2\langle Pq, q'\rangle$ since the matrix P is symmetric. Hence

$$C' = C \cdot \Big(r' - \frac{1}{4}\langle P'q, q\rangle - \frac{1}{2}\langle Pq, q'\rangle\Big).$$

Substituting the expressions for P', q', and r' from (2.22) and (2.17), we obtain the equality

$$C' = C \cdot \Big(\operatorname{tr}(AP^{-1}) + \frac{1}{8}\langle P(F+F^T)Pq, q\rangle - \frac{1}{2}\langle Pg, q\rangle + h \Big)$$
$$= C \cdot \Big(\operatorname{tr}(AP^{-1}) + \frac{1}{2}\langle (F+F^T)m, m\rangle + \langle g, m\rangle + h \Big)$$

which coincides with the third equation of the system (2.22). □

Definition 2.2. The system (2.22) is referred to as a *Riccati type system with singularity.*

Proposition 2.4. *Let Assumption* A *be satisfied. Then a solution* $P(t)$ *to the Cauchy problem* (1.2) *exists in an* ϵ_2*-neighborhood of zero.*

Proof. By Assumption A, $\lim\limits_{t\to 0} P'(t) = -2A_0$ exists, i.e., $P'(t) = -2A_0 + o(1)$. Hence

$$P(t) = -2tA_0 + R(t), \tag{2.23}$$

where $R(t) = (r_{kl}(t))$ is a symmetric matrix; moreover, $r_{kl}(t) = o(t)$ as $t \to 0$, i.e., there exists $\epsilon_2 > 0$ such that (2.23) holds. □

We note that $\epsilon_2 \leqslant \epsilon_1$ and the constant ε in Assumption A is equal to ϵ_2.

2.5. Additional Facts. The proof of the following assertion is obvious by Lemma 2.3.

Proposition 2.5. *A function* $v \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R})$ *of the form* (2.19) *is a solution to Equation* (2.15) *if and only if its coefficients satisfy the system* (1.10).

Proposition 2.6. *The matrix* $P(t)$ *is negative definite in a neighborhood of zero.*

Proof. Let $\lambda_0 > 0$ be the least eigenvalue of the matrix A_0. By Assumption A,

$$\langle A_0 x, x\rangle \geqslant \lambda_0 \langle x, x\rangle, \quad x \in \mathbb{R}^n.$$

We choose a small $t_0 \in \mathbb{R}_+$ such that $|r_{kl}(t)| < \frac{1}{2}\lambda_0 t$ for all $t \in (0, t_0)$, where $R(t) = (r_{kl}(t))$ is defined in (2.23). Then for $t \in (0, t_0)$

$$|\langle R(t)x, x\rangle| \leqslant \frac{1}{2}\lambda_0 t \langle x, x\rangle, \quad x \in \mathbb{R}^n.$$

However, $\langle P(t)x, x\rangle = -2t\langle A_0 x, x\rangle + \langle R(t)x, x\rangle$ in view of (2.23). Combining the last two inequalities, we conclude that for $t \in (0, t_0)$

$$\langle P(t)x, x\rangle \leqslant -\frac{3}{2}\lambda_0 t \langle x, x\rangle, \quad x \in \mathbb{R}^n.$$

Consequently, $\langle P(t)x, x\rangle < 0$ for $t \in (0, t_0)$ and all $x \neq 0$, i.e., the matrix $P(t)$ is negative definite on $(0, t_0)$ and all its eigenvalues are negative. □

Proposition 2.7. $\lim\limits_{t\to 0} I(t) = 1$, *where*

$$I(t) = \int\limits_{\mathbb{R}^n} C(t) \cdot \exp\{\langle P^{-1}(t)(x - m(t;y)), (x - m(t;y))\rangle\} dx_1 \dots dx_n,$$

$P(t)$ and $m(t;y)$ are solutions to the first two equations of the system (1.10) *with the initial conditions $P_{ij}(0) = 0$, $i,j \in \overline{1,n}$, and $m(0) = y$, whereas $C(t)$ is a partial solution to the third equation of the system* (1.10) *of the form*

$$C(t) = \frac{1}{\sqrt{(2\pi t)^n \det A_0}} \ \exp\left\{ -\frac{n}{2} \int_0^t \frac{\widetilde{q}(s)}{s} ds \right\}.$$

Proof. Replacing $\nu(t) = \ln C(t)$, we reduce the third equation of (1.10) to the form

$$\nu'(t) = \operatorname{tr}(AP^{-1}). \tag{2.24}$$

For (1.5) we have $Q(t) = (q_{kl}(t))$; moreover, $q_{kl}(t) = o(1)$ as $t \to 0$. From (2.23) it follows that

$$P(t) = -2t[E + Q(t)]A_0. \tag{2.25}$$

Consequently,

$$P^{-1}(t) = -\frac{1}{2t} A_0^{-1}[E + Q(t)]^{-1} = -\frac{1}{2t} A_0^{-1}[E + \overline{Q}(t)],$$

where $\overline{Q}(t) = (\overline{q}_{kl}(t))$ is given in (1.6). The matrix $\overline{Q}(t)$ exists by Assumption A. Moreover, $\overline{q}_{kl}(t) = o(1)$ as $t \to 0$. We set $W(t) = A(t)P^{-1}(t)$. Then

$$\begin{aligned} -2tW(t) &= A(t)A_0^{-1}[E + \overline{Q}(t)] = [A_0 + (A(t) - A_0)]A_0^{-1}[E + \overline{Q}(t)] \\ &= E + \overline{Q}(t) + (A(t) - A_0)A_0^{-1}[E + \overline{Q}(t)] = E + \widetilde{Q}(t), \end{aligned}$$

where $\widetilde{Q}(t) = (\widetilde{q}_{kl}(t))$ is given by (1.7); moreover, $\widetilde{q}_{kl}(t) = o(1)$ as $t \to 0$ by Assumption A.

Taking into account the above notation, we transform the right-hand side of (2.24) as follows:

$$\operatorname{tr}(AP^{-1}) = \operatorname{tr} W(t) = -\frac{1}{2t} \operatorname{tr}(E + \widetilde{Q}(t)) = -\frac{n}{2t}(1 + \widetilde{q}(t)),$$

where $\widetilde{q}(t)$ is defined in (1.8); moreover, $\widetilde{q}(t) = o(1)$ as $t \to 0$ by the definition and properties of the matrix $\widetilde{Q}(t)$. Equation (2.24) takes the form

$$\nu'(t) = -\frac{n}{2t}(1 + \widetilde{q}(t)). \tag{2.26}$$

The function $\nu_0(t)$ defined by the equality

$$\nu_0(t) = -\frac{n}{2} \ln t - \frac{n}{2} \int_0^t \frac{\widetilde{q}(s)}{s} ds,$$

is a partial solution to this equation. Thus, the general solution to Equation (2.26) has the form $\nu(t) = \nu_0(t) + N_0$, $N_0 \in \mathbb{R}$. As a consequence, we find

$$\nu(t) = N_0 - \frac{n}{2} \ln t + o(1), \quad t \to 0. \tag{2.27}$$

Hence $C(t) = C_0 \exp \nu_0(t)$ and $C_0 = \exp N_0 > 0$ are solutions to the third equation of (1.10).

By (2.5) and (2.21), we have $I(t) = C(t) \cdot \pi^{n/2} |\det P(t)|^{1/2}$ (independently of $y \in \mathbb{R}^n$). Hence

$$\ln I(t) = \nu(t) + \frac{n}{2} \ln \pi + \frac{1}{2} \ln |\det P(t)|. \tag{2.28}$$

From (2.25) it follows that

$$|\det P(t)| = (2t)^n \det A_0 \cdot (1 + o(1)). \tag{2.29}$$

Based on (2.27) and (2.29), from (2.28) we find

$$\ln I(t) = N_0 - \frac{n}{2} \ln t + \frac{n}{2} \ln \pi + \frac{1}{2} \ln((2t)^n \det A_0(1 + o(1))) = N_0 + \ln \sqrt{(2\pi)^n \det A_0} + o(1).$$

Setting $N_0 = -\ln \sqrt{(2\pi)^n \det A_0}$, we obtain the required equality $\lim_{t \to 0} I(t) = 1$. □

We define $G(t, x; y) : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by the formula

$$G(t, x; y) = C(t) \cdot \exp\{\langle P^{-1}(t)(x - m(t; y)), (x - m(t; y))\rangle\},$$

where $P(t)$, $m(t)$, $C(t)$ satisfy the assumptions of Proposition 2.7.

Lemma 2.4. *Let $\varphi(x) : \mathbb{R}^n \to \mathbb{R}$ be a function of class $C_0^\infty(\mathbb{R}^n)$, and let*

$$f(t, x) = \int_{\mathbb{R}^n} \varphi(y) G(t, y; x) dy.$$

Then $f(t, x) \to \varphi(x)$ as $t \to 0$.

Proof. For any fixed $t > 0$ the matrix $P = P(t)$ is symmetric and negative definite by construction and Propositions 2.6 and 2.4. Hence there exists an orthogonal matrix $O : O^{-1} = O^T$ such that $O^T P(t) O = \Lambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$; moreover, all λ_k are negative.

Denote by $\sqrt{-\Lambda}$ the matrix $\operatorname{diag}\{\sqrt{-\lambda_1}, \dots, \sqrt{-\lambda_n}\}$ and by $\sqrt{-P(t)}$ the matrix $O\sqrt{-\Lambda}O^T$. Making the change of variables $y = \sqrt{-P(t)}z + m(t; x)$, we find

$$\begin{aligned}
f(t, x) &= \int_{\mathbb{R}^n} \varphi(y) C(t) \exp\{\langle P^{-1}(t)(y - m(t; x)), (y - m(t; x))\rangle\} dy \\
&= \int_{\mathbb{R}^n} \varphi(y) C(t) \exp\{\langle O\sqrt{-\Lambda}O^T O \Lambda^{-1} O^T O \sqrt{-\Lambda} O^T z, z\rangle\} d(O\sqrt{-\Lambda}O^T z) \\
&= \int_{\mathbb{R}^n} C(t) \varphi(\sqrt{-P(t)}z + m(t; x)) \exp\{-\langle z, z\rangle\} \cdot |\det P(t)|^{1/2} dz \\
&= \pi^{-n/2} I(t) \int_{\mathbb{R}^n} \varphi(\sqrt{-P(t)}z + m(t; x)) \exp\{-\langle z, z\rangle\} dz.
\end{aligned}$$

The last integral converges to $\pi^{n/2}\varphi(x)$ since $P(t) \to 0$ and $m(t; x) \to x$ as $t \to 0$. Hence $f(t, x) \to \varphi(x)$ as $t \to 0$ by Proposition 2.7. □

3 Proof of Theorem 1.1

We represent a solution to the Cauchy problem (1.11) as the product $u(t,x) = \rho(t,x)v(t,x)$. By Corollary 2.1, the problem splits into two ones:

$$\frac{\partial \rho}{\partial t} = \mathfrak{L}[\rho], \quad \rho|_{t=0} = 1, \tag{3.1}$$

$$\frac{\partial v}{\partial t} = \mathfrak{F}[v], \quad v|_{t=0} = \delta_y(x), \tag{3.2}$$

where $\mathfrak{L}$ is a second order operator and $\mathfrak{F}$ is the operator corresponding to the right-hand side of Equation (2.15). By Lemma 2.2, the problem (3.1) has a solution $\rho(t,x)$ of the form (2.12), where $S(t)$, $q(t)$, $r(t)$ are solutions to the system (1.9) with the initial conditions $S_{ij}(0) = q_k(0) = r(0) = 0$, $i,j,k \in \overline{1,n}$. By Proposition 2.5, there exists a function $v(t,x)$ of the form (2.19), where $P(t)$, $m(t)$, $C(t)$ are solutions to the system (1.10). From Lemma 2.4 it follows that for the initial conditions $P_{ij}(0) = 0$, $i,j \in \overline{1,n}$, $m(0) = y$, and

$$C(1) = \exp\left\{ -\frac{n}{2}\int_0^{+\infty} \frac{\widetilde{q}(s)}{s} ds - \ln\sqrt{(2\pi)^n \det A_0} \right\}$$

we have $v(t,x) \to \delta_y(x)$ as $t \to 0$. It is obvious that for such initial conditions on $S(t)$, $q(t)$, $r(t)$, $P(t)$, $m(t)$, $C(t)$ the product $\rho(t,x)v(t,x)$ also converges to $\delta_y(x)$ as $t \to 0$.

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