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WEIGHTED CALDERÓN–ZYGMUND DECOMPOSITION WITH SOME APPLICATIONS TO INTERPOLATION

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Let X be an A_1-regular lattice of measurable functions and let Q be a projection that is also a Calderón–Zygmund operator. In this case, it is possible to define a space X^Q consisting of functions $f \in X$ for which $Qf = f$ in a certain sense. By using the Bourgain approach to interpolation, we show that the couple $\left(\mathrm{L}_1^Q, X^Q\right)$ is K-closed in (L_1, X). This result is sharp in the sense that, in general, A_1-regularity cannot be replaced by weaker conditions such as A_p-regularity for $p > 1$. Bibliography: 13 titles.

0. Introduction

For simplicity, we only consider the case of spaces S of homogeneous type, $S = \mathbb{R}^n$ or $S = \mathbb{T}^n$. A natural setting for these results is consideration of lattices of measurable functions on a measurable space $S \times \mathcal{X}$, where $\mathcal{X}$ is a σ-finite measurable space. Mostly, we treat operators like the Hardy-Littlewood maximal operator M that act in the first variable. More details on lattices of measurable functions can be found, e.g., in [10].

Let Q be a projection that is also a Calderón–Zygmund operator. For a lattice X, we define a space X^Q of functions f from X for which $Qf = f$ in a certain sense. We show that natural attempts to make the definition of X^Q precise in a fairly general setting lead to various technical difficulties.

Many interesting spaces arising in harmonic analysis, such as the real Hardy and Sobolev spaces, can be defined in terms of L_p^Q with a suitable projection Q (acting in a space of vector-valued functions), e.g., see [11]. We consider the following problem: When is the couple $\left(\mathrm{L}_1^Q, X^Q\right)$ K-closed in (L_1, X)? The K-closedness means that there exists a constant C such that for any function $h \in \mathrm{L}_1^Q + X^Q$ and any decomposition $h = f + g$, $f \in \mathrm{L}_1$, $g \in X$, there exists another decomposition $h = F + G$, $F \in \mathrm{L}_1^Q$, $G \in X^Q$, such that $\|F\|_{\mathrm{L}_1} \le C\|f\|_{\mathrm{L}_1}$ and $\|G\|_X \le C\|g\|_X$. This property was established earlier in [11] in the case where $X = \mathrm{L}_p$ for all $1 < p \le \infty$ using the so-called Bourgain method, and this result later proved to be useful; see also [3].

A weighted Calderón–Zygmund decomposition was developed in [9], which allowed to extend in [6, Chap. 4] the K-closedness to the case of A_1-regular lattices X (see the definition below).

Theorem 1. *Assume that X is an A_1-regular lattice of measurable functions satisfying the Fatou property, and let Q be a projection that is also a Calderón–Zygmund operator. Then the couple $\left(\mathrm{L}_1^Q, X^Q\right)$ is* K*-closed in (L_1, X) with a constant depending only on the A_1-regularity constant of X and properties of Q.*

A typical example, as well as a particular case on which the proof of this result is based, is the case of the lattices

$$X = \mathrm{L}_\infty(w) = \{g \mid |g| \le Cw \text{ a. e.}\};$$

such a lattice is A_1-regular if and only if $w \in \mathrm{A}_1$. In the present paper, we give a somewhat simplified proof of Theorem 1 and discuss some related problems.

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For generalities on the Muckenhoupt A_p weights see, e.g., [8, Chap. 5]. A lattice X is called A_p-regular (respectively, BMO-regular) with constants (C, m) if for any $f \in X$ there exists a majorant $w \geq |f|$ such that $\|w\|_X \leq m\|f\|_X$ and $w \in \mathrm{A}_p$ (respectively, $\log w \in \mathrm{BMO}$) with a constant C. It was noted that such conditions, and especially A_1-regularity, often characterize lattices of measurable functions that are interesting in the study of behavior of various harmonic analysis operators; see, e.g., [7, 12]. In particular, it is known that the A_1-regularity of a lattice X is equivalent to the boundedness of the maximal operator M in X, and for a wide class of lattices X and Calderón–Zygmund operators Q, the boundedness of Q in X is equivalent to the A_1-regularity of both X and X'.

It may seem that more subtle and mysterious interpolation properties of the spaces X^Q can hardly be characterized in such a simple way. But gradually it was discovered that in the important case of Hardy-type spaces X_A (which corresponds to the Riesz projection $Q = \mathbb{P}$), there is a rather intimate connection of such conditions with interpolation properties; for example, at least for $p \in \{1, 2, \infty\}$, the K-closedness of (H_p, X_A) in (L_p, X) is equivalent to the BMO-regularity of X, see [3] and [13].

However, in the general case of Calderón–Zygmund projections Q, such a characterization may be false, as the following simple example demonstrates. We can define the space of constant functions on the unit circle $\mathbb{T}$ with the normalized Lebesgue measure via a singular integral operator Q with kernel $K(x, y) = 1$, which surely is a Calderón–Zygmund operator. Let w be a summable weight such that $w > 0$ almost everywhere. In this case, the space $\mathrm{L}_\infty(w)^Q$ is nontrivial if and only if

$$\operatorname*{ess\,inf}_{t \in \mathbb{T}} w(t) > 0. \tag{1}$$

However, since the set $\mathrm{L}_\infty(w)$ is dense in L_1, it is easy to see that the K-closedness of $\left(\mathrm{L}_1^Q, \mathrm{L}_\infty(w)^Q\right)$ in $(\mathrm{L}_1, \mathrm{L}_\infty(w))$ implies that $\mathrm{L}_\infty(w)^Q$ is nontrivial. This means that for $p > 1$ and such a projection Q, the conclusion of Theorem 1 does not hold for A_p-regular lattices $\mathrm{L}_\infty(w)$ and all weights $w \in \mathrm{A}_p$ since many of such weights do not satisfy (1).

Nevertheless, this example may be only an exception to the rule; essentially, it has more to do (by duality) with the rather comprehensively studied interpolation of spaces of codimension 1 (see, e.g., [1]) and not with interpolation of Hardy-type spaces. We also note that this question is closely related to general problems of stability of decompositions under action of various singular operators; see, e.g., [4].

1. Weighted Calderón–Zygmund decomposition and the Bourgain method

Any almost everywhere nonnegative measurable function w is called here a weight. For a lattice X, the weighted space $X(w)$ is naturally defined by $X(w) = \{wf \mid f \in X\}$ with the norm $\|g\|_{X(w)} = \left\|gw^{-1}\right\|_X$. This definition, however, leads to a somewhat confusing notation $\mathrm{L}_p\left(w^{-1/p}\right)$ for the "standard" weighted Lebesgue spaces with the norm

$$\|h\|_{\mathrm{L}_p\left(w^{-1/p}\right)} = \left(\int |h|^p w\right)^{\frac{1}{p}}.$$

For measurable sets E and weights w, we denote by $w(E) = \int_E w$ the corresponding measure. We need a weighted version of the following well-known notion; we state it for general linear operators since it will be applied to certain maps which are derived from the Calderón–Zygmund operators. We also use an additional variable.

Definition 2. *Let P be a linear operator on $\mathrm{L}_1\left(a^{-1}\right)$, where a is a weight. We say that P admits a Calderón–Zygmund type decomposition with weight a if for any $\lambda > 0$ and any*

$f \in \mathrm{L}_1\left(a^{-1}\right)$ *there exists a decomposition* $f = g + b$ *and a measurable set* $\Omega \subset S \times \mathcal{X}$ *such that the following statements are valid:*

(1) $\|g\|_{\mathrm{L}_\infty} \le C\lambda$;
(2) $\|g\|_{\mathrm{L}_1(a^{-1})} \le C\|f\|_{\mathrm{L}_1(a^{-1})}$;
(3) $\|b\|_{\mathrm{L}_1(a^{-1})} \le C\|f\|_{\mathrm{L}_1(a^{-1})}$;
(4) $a(\Omega) \le \frac{C}{\lambda}\|f\|_{\mathrm{L}_1(a^{-1})}$;
(5) $\int\limits_{(S\times\mathcal{X})\setminus\Omega} |Pb|a \le C\|f\|_{\mathrm{L}_1(a^{-1})}$

with a constant C *independent of* f.

The property introduced by Definition 2 together with the (assumed) boundedness of P in $\mathrm{L}_t\left(a^{-\frac{1}{t}}\right)$ for some $t > 1$ implies (by the well-known routine argument) the weak type of P, which already has many interesting applications, see [9, Sec. 2]. It is important to note that under these assumptions, P is correctly defined on the entire $\mathrm{L}_1\left(a^{-1}\right)$.

We now provide a couple of simple definitions of the space X^Q which are expanded later. The simplest case arises when the projection Q acts boundedly in X; then

$$X^Q = \{f \in X \mid Qf = f\} \tag{2}$$

correctly defines a closed subspace of X. However, in interesting cases, Q is not bounded in X. If Q is defined on $\mathrm{L}_1\left(a^{-1}\right)$ (but not necessarily takes values in this space) and Q is a projection (i.e., $Q^2 = Q$ on a dense subset of $\mathrm{L}_1\left(a^{-1}\right)$), then for a lattice X of measurable functions on S such that $X \cap \mathrm{L}_1\left(a^{-1}\right)$ is dense in X we may define X^Q as the closure in X of the set

$$\left\{f \in X \cap \mathrm{L}_1\left(a^{-1}\right) \mid Qf = f\right\}. \tag{3}$$

The Bourgain method, which is the main application of Calderón–Zygmund type decompositions to interpolation, can be stated in the following form in our setting.

Proposition 3. *Assume that* Q *is a linear operator on* $\mathrm{L}_1\left(a^{-1}\right)$, Q *is a projection that admits a Calderón–Zygmund type decomposition with weight* a, *and* Q *is bounded in* $\mathrm{L}_t\left(a^{-\frac{1}{t}}\right)$ *with some* $t > 1$. *Then the couple*

$$\left(\mathrm{L}_1^Q\left(a^{-1}\right), \mathrm{L}_t^Q\left(a^{-\frac{1}{t}}\right)\right)$$

is K*-closed in* $\left(\mathrm{L}_1\left(a^{-1}\right), \mathrm{L}_t\left(a^{-\frac{1}{t}}\right)\right)$.

Indeed, assume that some function $f \in \mathrm{L}_1^Q\left(a^{-1}\right) + \mathrm{L}_t^Q\left(a^{-\frac{1}{t}}\right)$ admits a decomposition $f = f_0 + f_1$, $f_0 \in \mathrm{L}_1\left(a^{-1}\right)$, $f_1 \in \mathrm{L}_t\left(a^{-\frac{1}{t}}\right)$. By the assumptions, for f_0 and

$$\lambda = \left(\|f_1\|^t_{\mathrm{L}_t\left(a^{-\frac{1}{t}}\right)} \|f_0\|^{-1}_{\mathrm{L}_1(a^{-1})}\right)^{\frac{1}{t-1}}$$

there exists a Calderón–Zygmund type decomposition $f_0 = g + b$ with weight a. Simple estimates show that the decomposition $f = g_0 + g_1$, $g_0 = Qb$, $g_1 = Q(g + f_1)$ satisfies the stated K-closedness property. Indeed, conditions (1) and (2) of Definition 2 imply that

$$\|g_1\|_{\mathrm{L}_t\left(a^{-\frac{1}{t}}\right)} \le c\|g + f_1\|_{\mathrm{L}_t\left(a^{-\frac{1}{t}}\right)} \le c\|f_1\|_{\mathrm{L}_t\left(a^{-\frac{1}{t}}\right)} + c\left(\|g\|^{t-1}_{\mathrm{L}_\infty}\|g\|_{\mathrm{L}_1(a^{-1})}\right)^{\frac{1}{t}} \le c'\|f_1\|_{\mathrm{L}_t\left(a^{-\frac{1}{t}}\right)}$$

with some c and c' independent of f_0 and f_1, and since $f = b + g + f_1$ and $(I - Q)f = 0$, conditions (3)–(5) of Definition 2 imply that

$$\|Qb\|_{\mathrm{L}_1(a^{-1})} = \int\limits_{(S\times\mathcal{X})\setminus\Omega} |Qb|a + \int\limits_{\Omega} |b + (I-Q)(g+f_1)|a$$

$$\leq c\|f_0\|_{\mathrm{L}_1(a^{-1})} + ca(\Omega)^{\frac{t-1}{t}} \|g + f_1\|_{\mathrm{L}_t\left(a^{-\frac{1}{t}}\right)} \leq c'\|f_0\|_{\mathrm{L}_1(a^{-1})}$$

with some c and c' independent of f_0 and f_1.

Let Q be a Calderón–Zygmund operator on S, i.e., Q is a singular integral operator such that Q is bounded in L_q with some $1 < q < \infty$ and the kernel $K(x, y)$ of Q together with the kernel $\tilde{K}(x, y) = K(y, x)$ of the conjugate operator Q^* satisfy the estimate

$$|K(x,s) - K(x,t)| \leq C_K \frac{|s-t|^\gamma}{|x-s|^{n+\gamma}}, \quad |x-s| > 2|s-t|, \tag{4}$$

where x, s, t belong to $S = \mathbb{R}^n$ or $S = \mathbb{T}^n$. For generalities on such operators, see, e.g., [8].

For an operator T and a weight u we define an operator T_u corresponding to a density change by $T_u f = \frac{1}{u}T(uf)$. The main problem can be stated as follows: For what u and a, does the operator Q_u admit a Calderón–Zygmund decomposition with weight a? In [9], the following result was obtained (without the additional variable and in a somewhat implicit manner).

Theorem 4 ([9, Sec. 2])**.** *Assume that Q is a Calderón–Zygmund operator, $a \in \mathrm{A}_\infty$, $w \in \mathrm{A}_1$, and $u = \frac{a}{w}$. Then Q_u admits a Calderón–Zygmund type decomposition with weight a, and Q_u is bounded in $\mathrm{L}_t\left(a^{-\frac{1}{t}}\right)$ for all sufficiently small $t > 1$. The operator Q_u has weak type $(1,1)$ with weight a.*

The construction of the corresponding decomposition is based on the standard Calderón–Zygmund decomposition carried out with the weighted maximal operator

$$M_{[a]}f(t) = \sup_{Q\ni t} \frac{1}{a(Q)} \int\limits_Q |f|a,$$

where the supremum is taken over all dyadic cubes. Naturally, this result also holds with an additional variable; moreover, the case with an additional variable is easily recovered from the one-variable case by integrating the respective estimates in Definition 2. The boundedness of Q_u in $\mathrm{L}_t\left(a^{-\frac{1}{t}}\right)$ follows from properties of the Muckenhoupt weights (see [9, Lemma 2]).

We note that precise conditions on the weights a and u that correspond to the part of the conclusion of Theorem 4 concerning the weak type of Q_u are not clear. However, it can be shown that in many cases, the condition $w = \frac{a}{u} \in \mathrm{A}_1$ is necessary. It is well known that this condition is necessary in the classical case $u = 1$, $a = w$ (see, e.g., [8, Chap. 5, Sec. 4.6]); thus, any interesting generalizations seem unlikely.

2. The case of the couple $\left(\mathrm{L}_1^Q\left(w_0^{-1}\right), \mathrm{L}_\infty^Q\left(w_1\right)\right)$

Let Q be a Calderón–Zygmund operator that is a projection. In this section, we consider the following question: For what weights w_0 and w_1 is the couple

$$\left(\mathrm{L}_1^Q\left(w_0^{-1}\right), \mathrm{L}_\infty^Q\left(w_1\right)\right)$$

K-closed in $\left(\mathrm{L}_1\left(w_0^{-1}\right), \mathrm{L}_\infty\left(w_1\right)\right)$? In the case without weights, an answer was given in [11], see also [3, Sec. 4]. The main idea is to prove the K-closedness on the entire interval $(1, \infty)$

by gluing the K-closedness on three overlapping intervals. We apply the same scheme to the weighted Calderón–Zygmund decomposition.

Assume that $a_0 \in \mathrm{A}_\infty$, $w_0 \in \mathrm{A}_1$, and $u_0 = \frac{a_0}{w_0}$. Then, by Theorem 4, the operator Q_{u_0} and weight a_0 satisfy the assumptions of Proposition 3; thus, the couple $\left(\mathrm{L}_1^{Q_{u_0}}\left(a_0^{-1}\right), \mathrm{L}_t^{Q_{u_0}}\left(a_0^{-\frac{1}{t}}\right)\right)$ is K-closed in $\left(\mathrm{L}_1\left(a_0^{-1}\right), \mathrm{L}_t\left(a_0^{-\frac{1}{t}}\right)\right)$ for all sufficiently small $t > 1$ with suitable estimates on t and the constant of K-closedness via the constants of the weights a_0 and w_0 and properties of Q.

It is easy to see that $[X(u)]^Q = uX^{Q_u}$ for any normed lattice X and any weight u (coincidence of the sets and norms). This implies at once that (since $a_0 = u_0 w_0$) the couple

$$\left(\mathrm{L}_1^Q\left(w_0^{-1}\right), \mathrm{L}_t^Q\left(a_0^{1-\frac{1}{t}} w_0^{-1}\right)\right) \tag{5}$$

is K-closed in $\left(\mathrm{L}_1\left(w_0^{-1}\right), \mathrm{L}_t\left(a_0^{1-\frac{1}{t}} w_0^{-1}\right)\right)$ and Q is bounded in the second space.

We need to use duality in order to obtain the K-closedness on the interval covering the end of the scale $(1, \infty)$ and also to properly define the corresponding space X^Q. For a Banach space X and its subspace $Y \subset X$, the annihilator $Y^\perp$ is the set

$$Y^\perp = \{f \in X^* \mid f(y) = 0 \text{ for all } y \in Y\}.$$

Lemma 5 ([3, Lemma 1.2])**.** *Let (Y_0, Y_1) be a subcouple of a compatible couple of Banach spaces (X_0, X_1). If $X_0 \cap X_1$ is dense in both X_0 and X_1, then the following conditions are equivalent*:

(1) (Y_0, Y_1) *is* K-*closed in* (X_0, X_1);
(2) $\left(Y_0^\perp, Y_1^\perp\right)$ *is* K-*closed in* (X_0^*, X_1^*).

It is easy to see that if Q is bounded in a weighted space $\mathrm{L}_p(\omega)$, $1 < p < \infty$, then

$$\left[\mathrm{L}_{p'}^{I-Q^*}\left(\omega^{-1}\right)\right]^\perp = \mathrm{L}_p^Q(\omega).$$

This observation allows us to extend the definition of $X = \mathrm{L}_p^Q(\omega)$ in a natural manner to the case $p = \infty$ in which one cannot state that Q is defined on a dense set. Let

$$\mathrm{L}_\infty^Q(\omega) = \left[\mathrm{L}_1^{I-Q^*}\left(\omega^{-1}\right)\right]^\perp$$

for all suitable weights ω. For spaces S having finite measure, this definition coincides with the previous one and even with definition (2) (see, e.g., [3, Corollary 4.4]). Sets (3) may be not dense in the strong topology of $X = L_\infty(\omega)$ (see [3, Proposition 4.1]).

By duality, Lemma 5 shows that for all weights $a_1 \in \mathrm{A}_\infty$ and $w_1 \in \mathrm{A}_1$, the couple

$$\left(\mathrm{L}_{t'}^Q\left(a_1^{-\frac{1}{t'}} w_1\right), \mathrm{L}_\infty^Q(w_1)\right) \tag{6}$$

is K-closed in $\left(\mathrm{L}_{t'}\left(a_1^{-\frac{1}{t'}} w_1\right), \mathrm{L}_\infty(w_1)\right)$ for all sufficiently small $t > 1$ with suitable estimates on t and on the K-closedness constant in terms of the constants of the weights a_1 and w_1 and properties of Q. Moreover, Q is bounded in the first space of the couple.

The boundedness of Q at the respective ends of the intervals (5) and (6) implies the K-closedness for a middle interval that overlaps them with a suitable choice of weights.

We formulate a most general Wolff-type result concerning the "gluing of scales" as it is applied to K-closedness. Let (X_0, X_1) be an interpolation couple of quasi-Banach spaces and let $0<\theta<1$. An intermediate space E for (X_0,X_1) is said to belong to $\mathcal{C}(\theta;X_0,X_1)$ if there is a continuous inclusion $E\subset (X_0,X_1)_{\theta,\infty}$ and $\|x\|_E\le C_E\|x\|_{X_0}^{1-\theta}\|x\|_{X_1}^{\theta}$ for all $x\in E$ with a constant C_E independent of x.

Proposition 6 ([5, Proposition 5])**.** *Let (Y_0,Y_1) be a closed subcouple of an interpolation couple (X_0,X_1) of quasi-Banach spaces and let $0<\theta<\delta<1$. Assume also that we are given some spaces $E_0\in\mathcal{C}(\theta;X_0,X_1)$ and $E_1\in\mathcal{C}(\delta;X_0,X_1)$ and F_0 and F_1 are their respective subspaces such that both F_0 and F_1 contain $Y_0\cap Y_1$, $F_0\subset Y_0+F_1$, and $F_1\subset Y_1+F_0$. If (Y_0,F_1) is* K*-closed in (X_0,E_1) and (F_0,Y_1) is* K*-closed in (E_0,X_1), then (Y_0,Y_1) is* K*-closed in (X_0,X_1).*

For weighted Lebesgue spaces, it is well known (see, e.g., [2]) that

$$\mathrm{L}_p(\omega)\in\mathcal{C}\left(\theta;\mathrm{L}_{p_0}(\omega_0),\mathrm{L}_{p_1}(\omega_1)\right)$$

for $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$ and $\omega=\omega_0^{1-\theta}\omega_1^{\theta}$, at least if $1\le p_0<p_1\le\infty$ or $1\le p_0=p_1<\infty$. Since the endpoint spaces of the scale $\left(\mathrm{L}_1\left(w_0^{-1}\right),\mathrm{L}_\infty(w_1)\right)$ are already defined, for gluing of scales we need to find some points

$$0<\theta_1<\theta_2<\theta_3<\theta_4<1$$

of the interval $(0,1)$ such that the corresponding K-closedness is valid for the respective couples

$$\left(\mathrm{L}_1\left(w_0^{-1}\right),\mathrm{L}_{\frac{1}{1-\theta_2}}\left(w_0^{-(1-\theta_2)}w_1^{\theta_2}\right)\right),$$

$$\left(\mathrm{L}_{\frac{1}{1-\theta_1}}\left(w_0^{-(1-\theta_1)}w_1^{\theta_1}\right),\mathrm{L}_{\frac{1}{1-\theta_4}}\left(w_0^{-(1-\theta_4)}w_1^{\theta_4}\right)\right),$$

and

$$\left(\mathrm{L}_{\frac{1}{1-\theta_3}}\left(w_0^{-(1-\theta_3)}w_1^{\theta_3}\right),\mathrm{L}_\infty(w_1)\right).$$

This means that in order to apply the above results to intervals (5) and (6), we need to find some weights $a_0,a_1\in\mathrm{A}_\infty$ such that θ_2 is sufficiently small, θ_3 is sufficiently close to 1,

$$w_0^{-(1-\theta_j)}w_1^{\theta_j}=a_0^{\theta_j}w_0^{-1}$$

for $j\in\{1,2\}$, and

$$w_0^{-(1-\theta_j)}w_1^{\theta_j}=a_1^{\theta_j-1}w_1^{-1}$$

for $j\in\{3,4\}$. It is easy to see that these conditions are satisfied if and only if $a_0=a_1=w_0w_1$. The assumptions of Theorem 4 require that $a_0,a_1\in\mathrm{A}_\infty$. Thus, we arrive at the following result.

Proposition 7. *Assume that weights $w_0,w_1\in\mathrm{A}_1$ are such that $w_0w_1\in\mathrm{A}_\infty$. Then the couple*

$$\left(\mathrm{L}_1^Q\left(w_0^{-1}\right),\mathrm{L}_\infty^Q(w_1)\right)$$

is K*-closed in $\left(\mathrm{L}_1\left(w_0^{-1}\right),\mathrm{L}_\infty(w_1)\right)$ with an estimate for the constant depending only on the A_1-constants of the weights w_0, w_1, the A_∞-constant of the weight w_0w_1, and properties of Q.*

We note that it is not clear whether the condition $w_0w_1\in\mathrm{A}_\infty$ is necessary for the conclusion of Proposition 7.

3. The case of the couple $\left(\mathrm{L}_1^Q, X^Q\right)$

Proposition 7 easily implies Theorem 1 if X is A_1-regular and the space X^Q satisfies the condition

$$X^Q \cap \mathrm{L}_\infty(w) = \mathrm{L}_\infty^Q(w) \tag{7}$$

for all $w \in X \cap \mathrm{A}_1$. Indeed, assume that $f \in \mathrm{L}_1^Q + X^Q$ and $f = g_0 + h_0$, where $g_0 \in \mathrm{L}_1$ and $h_0 \in X$. There exists an A_1-majorant $w_0 \in \mathrm{A}_1$ for h_0, $\|w_0\|_X \leq m\|h_0\|_X$, with some constants (C, m) independent of h_0. Then $h_0 \in \mathrm{L}_\infty(w_0)$. In the case $S = \mathbb{T}^n$, the condition $f \in \mathrm{L}_1^Q \subset \mathrm{L}_1^Q + \mathrm{L}_\infty^Q(w_0)$ is always satisfied, and by Proposition 7, there exists a decomposition $f = g + h$ such that $g \in \mathrm{L}_1^Q$ and $h \in \mathrm{L}_\infty^Q(w_0)$ with

$$\|g\|_{\mathrm{L}_1} \leq C\|g_0\|_{\mathrm{L}_1}$$

and

$$\|h\|_X \leq \|h\|_{\mathrm{L}_\infty(w_0)}\|w_0\|_X \leq Cm\|h_0\|_X$$

for some C independent of g_0 and h_0, which proves the K-closedness in this case.

In the case $S = \mathbb{R}^n$, the inclusion

$$f \in \mathrm{L}_1^Q + \mathrm{L}_\infty^Q(w_0)$$

is not guaranteed. However, by the assumptions, $f = g_1 + h_1$ with some $g_1 \in \mathrm{L}_1^Q$ and $h_1 \in X^Q$. Let w_1 be an A_1-majorant for h_1 and let $\varepsilon > 0$. Then

$$f \in \mathrm{L}_1^Q + \mathrm{L}_\infty^Q(w)$$

with $w = w_0 + \varepsilon w_1$, and, by Proposition 7, there exists a decomposition

$$f = g_\varepsilon + h_\varepsilon$$

such that $g_\varepsilon \in \mathrm{L}_1^Q$ and $h_\varepsilon \in \mathrm{L}_\infty^Q(w)$ with the corresponding estimates

$$\|g_\varepsilon\|_{\mathrm{L}_1} \leq C\|g_0\|_{\mathrm{L}_1}$$

and $\|h_\varepsilon\|_{\mathrm{L}_\infty(w)} \leq C$ and with some C independent of ε, g_0, and h_0. We choose ε small enough so that $\|w\|_X \leq 2\|w_0\|_X$. Then the functions g_ε and h_ε satisfy the estimates $\|g_\varepsilon\|_{\mathrm{L}_1} \leq C\|g_0\|_{\mathrm{L}_1}$ and $\|h_\varepsilon\|_X \leq \|h_\varepsilon\|_{\mathrm{L}_\infty(w)}\|w\|_X \leq 2Cm\|h_0\|_X$, which proves the K-closedness in the general case. Thus, we have established Theorem 1 under the additional assumption (7).

Let us finally give a suitable definition for the space X^Q. Specifically, we construct the *smallest* subspace X^Q of X satisfying (7). Assume that Q is a projection that is a Calderón–Zygmund operator and that X is an A_1-regular lattice with constants (C, m) satisfying the Fatou property. Similarly to the definition of the space L_∞^Q, for any nonzero weight $w \in X \cap \mathrm{A}_1$ we set

$$X_w^Q = \left(\mathrm{L}_1^{I-Q^*}\left(w^{-1}\right)\right)^\perp \subset \mathrm{L}_\infty(w)$$

with the topology of X. This definition is correct since by Theorem 4, the projection $I - Q^*$ is properly defined on $\mathrm{L}_1\left(w^{-1}\right)$. The balls of the space X_w^Q are closed in X since it is easily seen that these balls are closed with respect to convergence in measure.

The spaces X_w^Q are monotone in w: If $w_1 \leq w_2$ almost everywhere, then

$$\mathrm{L}_1^{I-Q^*}\left(w_2^{-1}\right) \subset \mathrm{L}_1^{I-Q^*}\left(w_1^{-1}\right)$$

and $X_{w_1}^Q \subset X_{w_2}^Q$. Now let

$$X^Q = \bigcup_{\substack{w \in X, w \neq 0, \\ w \in \mathrm{A}_1 \text{ with constant } C}} X_w^Q \tag{8}$$

with the topology of X. It is easy to see that X^Q is a linear space since for any $w_0, w_1 \in X \cap \mathrm{A}_1$ and $w = w_0 + w_1$, the relations

$$\mathrm{L}_1\left(w_0^{-1}\right) \cap \mathrm{L}_1\left(w_1^{-1}\right) \supset \mathrm{L}_1\left(w^{-1}\right)$$

and $\mathrm{L}_1^{I-Q^*}\left(w_0^{-1}\right) \cap \mathrm{L}_1^{I-Q^*}\left(w_1^{-1}\right) \supset \mathrm{L}_1^{I-Q^*}\left(w^{-1}\right)$ hold, i.e.,

$$\left[X_{w_0}^Q\right]^{\perp} \cap \left[X_{w_1}^Q\right]^{\perp} \supset \left[X_w^Q\right]^{\perp};$$

passing to the annihilators, we see that $X_{w_0}^Q + X_{w_1}^Q \subset X_w^Q$.

Let us now verify that X^Q is a closed subspace of X. Indeed, assume that a sequence $f_n \in X^Q$, $n \in \mathbb{N}$, is such that the series $\sum_n f_n$ converges in X; it suffices to verify that $g = \sum_n f_n \in X^Q$. Grouping functions in this series together, we may further assume that $\|f_n\|_X \le 2^{-n}$ for $n \ge 2$. The A_1-regularity of X implies that there exist some A_1-majorants w_n for the functions $2^n f_n$. By the monotonicity (or using a more general Proposition 3.4 of [12]), it is easy to see that $w = \sum\limits_n 2^{-n} w_n$ belongs to A_1 with constant C, and w is an A_1-majorant for g. We also have the estimates

$$|f_n| \le 2^{-n} w_n \le w$$

for all $n \in \mathbb{N}$, which implies that $f_n \in X_w^Q$ for all $n \in \mathbb{N}$; thus, $g \in X_w^Q \subset X^Q$ because of the closedness of balls of X_w^Q.

It is easy to see that this definition coincides with the earlier one in the case where Q is a Calderón–Zygmund operator defined on $X \cap \mathrm{L}_1$ if we additionally assume that this set is dense in X. Indeed, if $f \in X^Q \cap \mathrm{L}_1$ and Q is defined on f, then $f \in X_w^Q = \left[\mathrm{L}_1^{I-Q^*}\left(w^{-1}\right)\right]^{\perp}$ with a weight $w \in \mathrm{A}_1$. Thus, for any bounded function g supported on a set of finite measure, we have the equalities

$$0 = \int f[(I - Q^*)g] = \int [(I - Q)f]g \tag{9}$$

(recall that we assume that Q is defined in L_q for some $1 < q < \infty$), which implies that $Qf = f$ and

$$X^Q \subset \operatorname*{clos}_X \{f \in X \cap \mathrm{L}_1 \mid Qf = f\}.$$

On the other hand, if $f \in X \cap \mathrm{L}_1$ and $Qf = f$, then $f \in X_w^Q$ for any A_1-majorant w of f due to relation (9), which proves the converse inclusion

$$X^Q \supset \operatorname*{clos}_X \{f \in X \cap \mathrm{L}_1 \mid Qf = f\}.$$

We see that in this case, both definitions yield the same space X^Q.

In conclusion, we note that definition (8) can be written in the form $X^Q = X \cap N_Q$, where $N_Q = \bigcup_{w \in \mathrm{A}_1} \left(\mathrm{L}_1^{I-Q^*}\left(w^{-1}\right)\right)^{\perp}$ is an analog of the Smirnov class N_+ used in the definition of Hardy-type spaces. However, this particular definition seems to work only for A_1-regular lattices since the set N_Q, in contrast to N_+, is not necessarily closed with respect to convergence in measure; the set A_1 is itself dense with respect to convergence in measure in the set of almost everywhere nonnegative measurable functions. This motivates the following question: Is it possible to extend the set N_Q in a natural way to cover most interesting cases? It is desirable for this space to cover at least all the spaces X^Q appearing in the results of the present work together with their annihilators. The simple approach based on taking the closure of N_Q with respect to convergence in measure does not always work; for example, in the case of codimension 1 spaces on the unit circle $S = \mathbb{T}$ defined by the projection $Qf = f - \int f$ (this is the dual example to that described in the Introduction), taking the closure with respect to convergence in measure completely destroys the condition $Qf = f$.

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