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CHEBYSHEV POLYNOMIALS, ZOLOTAREV POLYNOMIALS, AND PLANE TREES

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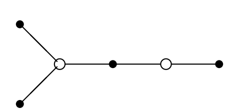
Abstract. A polynomial with exactly two critical values is called a generalized Chebyshev polynomial (or Shabat polynomial). A polynomial with exactly three critical values is called a Zolotarev polynomial. Two Chebyshev polynomials f and g are called Z-homotopic if there exists a family p_α, $\alpha \in [0,1]$, where $p_0 = f$, $p_1 = g$, and p_α is a Zolotarev polynomial if $\alpha \in (0,1)$. As each Chebyshev polynomial defines a plane tree (and vice versa), Z-homotopy can be defined for plane trees. In this work, we prove some necessary geometric conditions for the existence of Z-homotopy of plane trees, describe Z-homotopy for trees with five and six edges, and study one interesting example in the class of trees with seven edges.

1. Introduction

1.1. Generalized Chebyshev Polynomials. A polynomial $p(z) \in \mathbb{C}[z]$ is called a generalized Chebyshev polynomial if it has exactly two finite critical values: α and β (in what follows, we will call such a polynomial simply a Chebyshev polynomial). If $p(z)$ is a Chebyshev polynomial, then the set $p^{-1}[\alpha,\beta]$ is a plane connected tree T_p (see, e.g., [1]). Inverse images of the points α and β are vertices of the tree T_p, and the degree of a vertex equals the multiplicity of the corresponding critical point (a vertex of degree 1 is a simple root of the polynomial $p(z) - \alpha$ or $p(z) - \beta$). Also for each plane tree T there exists a Chebyshev polynomial $p(z)$ defined up to linear change of variable z and variable $u = p(z)$ such that the trees $p^{-1}[\alpha,\beta]$ and T are isotopic. Such a polynomial $p(z)$ will be called a *polynomial that defines the tree T*.

Vertices of a plane tree T can be colored in two colors—black and white—so that colors of any two adjacent vertices are different. Such a coloring will be called a *binary structure* of T. Obviously, vertices of one color are inverse images of α and vertices of the other color are inverse images of β.

The type (or passport) of a plane tree with binary structure consists of two sequences of multiplicities of white vertices and black vertices in nonincreasing order. Thus, the type of the tree



is $\langle 3,2 \mid 2,1,1,1\rangle$.

Remark 1. Often it is assumed that the numbers α and β are 0 and 1.

1.2. Zolotarev Polynomials. A polynomial $p \in \mathbb{C}[z]$ is called a *Zolotarev polynomial* if it has exactly three finite critical values. If p is a Zolotarev polynomial, $\deg(p) = n$, α, β, and γ are its critical values, and C is a simple arc $C \subset \mathbb{C}$ connecting the points α, β, and γ, then $p^{-1}(C)$ is a connected plane tree with $2n$ edges. Here points from the set $p^{-1}\{\alpha,\beta,\gamma\}$ are vertices of this tree and the degree of a vertex v, $p(v) = \alpha$, is equal to the multiplicity of the critical point v if α is an endpoint of C, or to the double multiplicity if α is an interior point. Vertices of the tree $p^{-1}(C)$ can be colored in three colors: white, black, and grey, where white vertices are inverse images of the interior (with respect to arc C) critical value. One vertex of each edge is white and the other one is black or grey.

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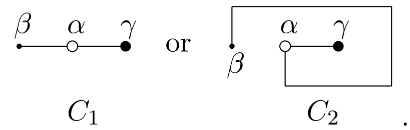
Remark 2. Arcs C_1 and C_2 connecting points α, β, and γ can be isotopically nonequivalent: for example,



In this case, the trees $p^{-1}(C_1)$ and $p^{-1}(C_2)$ can also be isotopically nonequivalent.

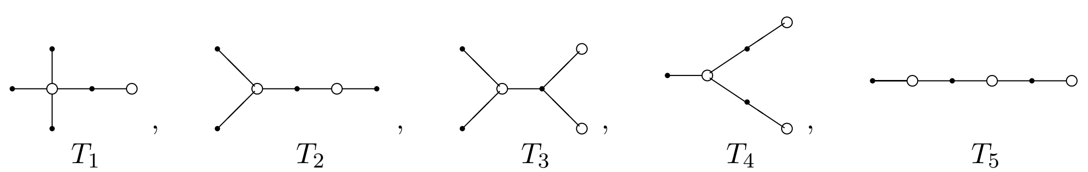
The *passport* of a Zolotarev polynomial consists of three sequences of multiplicities of its critical points that correspond to the first, the second, and the third critical value. Multiplicity sequences will be written in the nonincreasing order: $\langle k_1, k_2, \ldots \mid l_1, l_2, \ldots \mid m_1, m_2, \ldots \rangle$. For example, the critical points of the polynomial $p = x^2(x-1)^2(3x-1)$ are 0, 1, 2/3, and 1/5 with values 0, 0, 4/81, and $-32/3125$, respectively. So $\langle 2, 2 \mid 2 \mid 2 \rangle$ is the passport of p.

2. Z-Homotopy

Definition 1. Two trees T_1 and T_2 will be called Z-homotopic if there exists a continuous family $p_\lambda \in \mathbb{C}[z]$, $\lambda \in [0,1]$, such that

- all polynomials p_λ have the same degree;
- the polynomial p_0 is a Chebyshev polynomial and defines the tree T_1;
- the polynomial p_1 is a Chebyshev polynomial and defines the tree T_2;
- the polynomials p_λ, $\lambda \neq 0, 1$, are Zolotarev polynomials, but *not* Chebyshev polynomials.

Example 1. Let us study the Z-homotopy problem on the set of 5-edge trees. There are five of them:



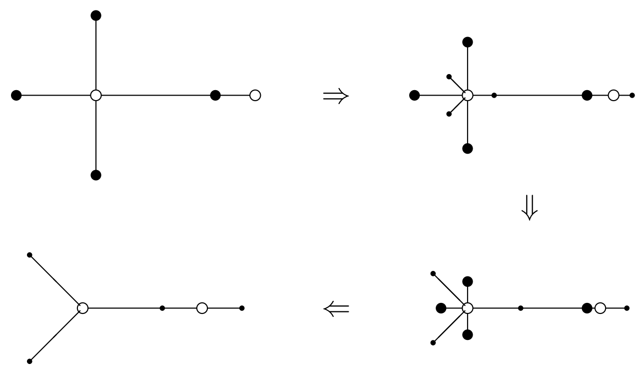
Let

$$p = \int x^2(x-1)(x-a)\,dx.$$

Critical points of p are 0, 1, and a, and the corresponding critical values are 0, $5a-3$, and $a^4(5-3a)$.

- If $a = 0$, then p is a Chebyshev polynomial that defines the tree T_1.
- If $a = 1$, then p is a Chebyshev polynomial that defines the tree T_3.
- If $a = 3/5$, then $p(1) = 0$ and p is a Chebyshev polynomial that defines the tree T_2.
- If $a = 5/3$, then $p(a) = 0$ and p is a Chebyshev polynomial that defines the tree T_2.
- If $a = (-2 \pm \sqrt{5}\,i)/3$, then $p(a) = p(1)$, and p is a Chebyshev polynomial that defines the tree T_4.

For all other values of the parameter a, the polynomial p is a Zolotarev polynomial. Thus, deformations of the parameter a allow one to realize pairwise Z-homotopies between the trees T_1, T_2, T_3, and T_4. For example, the following deformation of the tree corresponds to the increase of the parameter a from 0 to 3/5 (the arc C in this case is the segment that connects the critical values $5a-3$ and $a^3(5-3a)$).



The trees T_1, T_2, and T_4 are Z-homotopic to the tree T_5. Indeed, let us consider the polynomial

$$p(x) = \int x(x-1)(x-a)(x-b)\,dx.$$

If $p(a) = p(0)$, where $a \neq 2$, then this polynomial is a Zolotarev polynomial (here $b = (3a^2-5a)/(5a-10)$). However, for some values of the parameter a the polynomial p degenerates into a Chebyshev polynomial. Indeed,

(1) if $a = 0$, then $b = 0$, and we have a Chebyshev polynomial defining the tree T_1;
(2) if $a = 1$, then $b = 2/5$ and $p(1) = 0$, and we have a Chebyshev polynomial defining the tree T_2;
(3) if $a = 5/3$, then $b = 0$, and we have a Chebyshev polynomial defining the tree T_2;
(4) if $a = \pm\sqrt{5}$, then $b = 1 \pm \sqrt{5}$ and $p(1) = p(b)$, and we have a Chebyshev polynomial defining the tree T_5;
(5) if $a = (5 \pm \sqrt{5})/4$, then $b = -(1 \pm \sqrt{5})/4$ and $p(1) = p(b)$, and we have a Chebyshev polynomial defining the tree T_5;
(6) if $a = (5 \pm \sqrt{5}\,i)/3$, then $b = 1$, and we have a Chebyshev polynomial defining the tree T_4.

Thus, a deformation of the parameter a allows us to construct a Z-homotopy between the trees T_1 and T_5, T_2 and T_5, and T_4 and T_5.

The trees T_3 and T_5 are not Z-homotopic. This statement will be proved in the next section. Also it is a consequence of results in Sec. 4.

3. Geometry of Space of Zolotarev Polynomials of Degree 5

Let

$$q = x^4 + ax^2 + bx + c, \quad p = \int q\,dx.$$

The polynomial p is a Zolotarev polynomial if among the numbers $p(x_1)$, $p(x_2)$, $p(x_3)$, and $p(x_4)$, where x_1, x_2, x_3, and x_4 are roots of q, there are only three different values. In this case, the polynomial $s(y) = \big(y-p(x_1)\big)\big(y-p(x_2)\big)\big(y-p(x_3)\big)\big(y-p(x_4)\big)$ has a multiple root, i.e., its discriminant is zero. This discriminant is reducible:

$$(1280a^6 - 32256a^4c + 9504a^3b^2 + 269568a^2c^2 - 69984ab^2c - 19683b^4 - 746496c^3)$$
$$\times\,(16a^4c - 4a^3b^2 - 128a^2c^2 + 144ab^2c - 27b^4 + 256c^3) = 0.$$

We see that the variety of Zolotarev polynomials of degree 5 is reducible and has two components C_1 and C_2. The second factor, which defines the component C_2, is simply the discriminant of the polynomial q.

The intersection $C_1 \cap C_2$ is the union of three components:

$$C_1 \cap C_2 = C_3 \cup C_4 \cup C_5.$$

- Polynomials that belong to C_3 are Chebyshev polynomials that define the tree T_4.
- Polynomials that belong to C_4 are Chebyshev polynomials that define the tree T_2.
- Polynomials that belong to C_5 are Chebyshev polynomials that define the tree T_1.

A Chebyshev polynomial p_0 that defines T_5 belongs only to the first component C_1, and a Chebyshev polynomial p_1 that defines T_3 belongs only to the second component C_2. Thus, a family of Zolotarev polynomials that connect p_0 and p_1 must also contain one of Chebyshev polynomials in $C_1 \cap C_2$. But then this family is not a Z-homotopy.

4. Theorems

In this section, we will prove a sufficient condition, when a tree cannot be Z-homotopic to a "chain."

Lemma 1. *Let p_λ, $0 < \lambda < 1$, be a continuous family of Zolotarev polynomials of degree n. Then passports of all these polynomials are the same.*

Proof. Let a_λ, b_λ, and c_λ be critical values of the polynomial p_λ. They are continuous functions of the parameter λ. A change of passport during increase or decrease of the parameter λ can occur only in the case of collision of roots of the polynomial $p_\lambda - a_\lambda$ (or $p_\lambda - b_\lambda$, or $p_\lambda - c_\lambda$): two roots x'_λ and x''_λ of the polynomial $p_\lambda - a_\lambda$ of multiplicities k' and k'', respectively, approach each other, when $\lambda \to \mu$, and generate a root x_μ of the polynomial $p_\mu - a_\mu$ of multiplicity $k' + k'' - 1$.

Let the passport of p_λ be $\langle k_1, \ldots, k_r \mid l_1, \ldots, l_s \mid m_1, \ldots, m_t \rangle$. Then

$$\sum_{i=1}^{r} k_i = n, \quad \sum_{i=1}^{s} l_i = n, \quad \sum_{i=1}^{t} m_i = n,$$

$$\sum_{i=1}^{r}(k_i - 1) + \sum_{i=1}^{s}(l_i - 1) + \sum_{i=1}^{t}(m_i - 1) = n - 1.$$

Hence, $r + s + t = 2n + 1$. But the collision of roots diminishes the number r and violates the above equality. □

Remark 3. We see that it is more correct to speak not about Z-homotopy, but about Z-homotopy in the class of Zolotarev polynomials with a given passport. Thus, the trees T_1, T_2, T_3, and T_4 with five edges are pairwise Z-homotopic in the class of Zolotarev polynomials with the passport $\langle 3 \mid 2 \mid 2 \rangle$, and the trees T_1 and T_5, T_2 and T_5, and T_4 and T_5 are Z-homotopic in the class of Zolotarev polynomials with the passport $\langle 2, 2 \mid 2 \mid 2 \rangle$.

Lemma 2. *Let p_λ, $0 \le \lambda < 1$, be a continuous family of polynomials of degree n, where p_0 is a Chebyshev polynomial and p_λ, $\lambda > 0$, are Zolotarev polynomials (but not Chebyshev polynomials). Let us assume that a critical point a of the polynomial p_0 of multiplicity k generates m critical points $a_1(\lambda), \ldots, a_m(\lambda)$ in the family p_λ with multiplicities $k_1, \ldots, k_m$, $m > 1$. Then the numbers $p_\lambda\big(a_1(\lambda)\big), \ldots, p_\lambda\big(a_m(\lambda)\big)$ cannot all be equal.*

Proof. Let us assume that the opposite is true:

$$p_\lambda\big(a_1(\lambda)\big) = \ldots = p_\lambda\big(a_m(\lambda)\big) = \alpha(\lambda).$$

Let $\lambda \to 0$. Then

$$a_i(\lambda) \to a, \quad i = 1, \ldots, m, \quad \alpha(\lambda) \to \alpha = p_0(a).$$

But $k - 1 = (k_1 - 1) + \ldots + (k_m - 1)$, so a is a root of the polynomial $p_0 - \alpha$ of multiplicity $k + m - 1$. We have a contradiction. □

Definition 2. A tree is called a *chain* if valencies of all its vertices are ≤ 2.

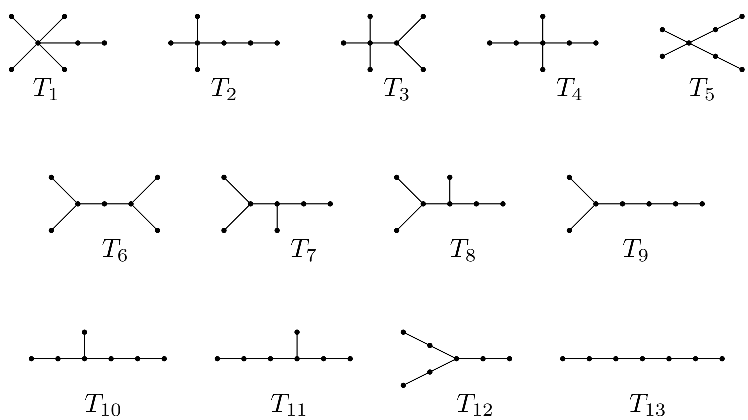
Theorem 1. *If a tree T has a white vertex a of degree ≥ 3 and a black vertex b of degree ≥ 3, then it cannot be Z-homotopic to a chain.*

Proof. Let us assume that the opposite is true. Then there exists a Z-homotopy connecting a Chebyshev polynomial p_0 that defines T with a Chebyshev polynomial p_1 that defines a chain. This means that critical points a and b in the family p_λ generated critical points $a_1, \dots, a_m$ and $b_1, \dots, b_n$, respectively, all of them of multiplicity 2. Let $p_0(a) = \alpha$ and $p_0(b) = \beta$. If the parameter λ is small, then the values $p_\lambda(a_1), \dots, p_\lambda(a_m)$ are close to α and among them there are at least two different values. Analogously, the values $p_\lambda(b_1), \dots, p_\lambda(b_n)$ are close to β and among them are at least two different values. But then a polynomial p_λ, $\lambda \ll 1$, has at least four critical values. We have a contradiction. ☐

Corollary 1. *The trees T_3 and T_5 cannot be Z-homotopic.*

5. Trees with Six Edges

Below are all plane 6-edge trees.



By Theorem 1, the trees T_3 and T_{13}, T_7 and T_{13}, and T_8 and T_{13} are not Z-homotopic. However, there is one more nonhomotopic pair.

Proposition 1. *The trees T_6 and T_{12} are not Z-homotopic.*

Proof. Let the opposite be true, and let a and b be white (for example) vertices of degree 3 of the tree T_6.

THE FIRST CASE. Let polynomials p_λ have a critical point a_λ of multiplicity 3 and all other critical points be of multiplicity 2. Thus, the vertex b generates two critical points b_1 and b_2 of multiplicity 2, $p_\lambda(b_1) \neq p_\lambda(b_2)$, and the value $p_\lambda(a)$ coincides with the value $p_\lambda(b_1)$ or with the value $p_\lambda(b_2)$. But then the tree T_{12} has a white vertex of degree 2 in addition to the white vertex of degree 3.

THE SECOND CASE. Polynomials p_λ have critical points only of multiplicity 2. Thus, vertices a and b generate critical points a_1, a_2 and b_1, b_2, respectively. Moreover, $p_\lambda(a_1) = p_\lambda(b_1)$, $p_\lambda(a_2) = p_\lambda(b_2)$, and $p_\lambda(b_1) \neq p_\lambda(b_2)$. Let the fifth critical point be $c = c_\lambda$. The vertex of T_{12} of degree 3 cannot be generated by junction of the points a_1 and b_1 (or a_2 and b_2), because otherwise during the change of the parameter λ from 1 to 0 the vertex of degree 3 of T_{12} generates two critical points with the same values. Also, this vertex cannot be generated by junction of the points c and a_1 (for example), because then T_{12} has a vertex of degree 3 and a vertex of degree 2 of the same color. ☐

All other pairs of trees are Z-homotopic. The construction of the corresponding Z-homotopy usually is quite straightforward. Let us describe some interesting cases.

- The tree T_4 and the tree T_{12}. Let vertices of degree 2 of T_4 be in points ± 1, its vertex of degree 4 be in origin, the vertex of degree 3 of T_{12} be in origin, and its vertices of degree 2 be in cubic roots of 1.

  Let us consider the polynomial

$$p = \int x^2(x-1)(x-a)(x-b)\,dx$$

with the condition $p(a) = p(b)$. Then p is a Zolotarev polynomial with passport $\langle 3 \mid 2, 2 \mid 2\rangle$. If $a = 0$ and $b = -1$, then p degenerates into a Chebyshev polynomial that corresponds to the tree T_4. The change of the parameter a from 0 to $-i$, to $2 - i$, to 2, to $2 + \sqrt{3}\,i/2$, and to $(-1 + \sqrt{3}\,i)/2$ induces the change of the parameter b from -1 to $(-1 - \sqrt{3}\,i)/2$.

- The tree T_{10} and the tree T_{13}. Let the vertex of degree 3 of T_{10} be in origin, its vertices of degree 2 be in points 1, $a_1 \approx 1.57 - 0.03\,i$ and $b_1 \approx -0.57 + 0.58\,i$, vertices of degree 2 of T_{13} be in points 0, ± 1, and $\pm\sqrt{3}$.

  Let us consider the polynomial

$$p = \int x(x-1)(x-a)(x-b)(x-c)\,dx$$

  with conditions $p(a) = 0$ and $p(b) = p(c)$. Then p is a Zolotarev polynomial with passport $\langle 2, 2 \mid 2, 2 \mid 2\rangle$. If $a = a_1$, $b = b_1$, and $c = 0$, then p degenerates into a Chebyshev polynomial that corresponds to the tree T_{10}. The change of the parameter b from b_1 to -1 induces the change of the parameter a from a_1 to $\sqrt{3}$ and the change of the parameter c from 0 to $-\sqrt{3}$ (here c moves along the arc in the lower half plane).

- The tree T_{12} and the tree T_{13}. Let the vertex of degree 3 of T_{12} be in the point $i/\sqrt{3}$, its vertices of degree 2 be in the points ± 1 and $\sqrt{3}\,i$, vertices of degree 2 of T_{13} be in points 0, ± 1, and $\pm 1/\sqrt{3}$.

  Let us consider the polynomial

$$p = \int (x^2-1)(x-a)(x-b)(x-c)\,dx$$

  with conditions $p(-1) = p(1) = p(c)$. Then p is a Zolotarev polynomial with passport $\langle 2, 2, 2 \mid 2 \mid 2\rangle$. If $a = b = i/\sqrt{3}$ and $c = \sqrt{3}\,i$, then p degenerates into a Chebyshev polynomial that corresponds to the tree T_{12}. The change of the parameter a from $i/\sqrt{3}$ to $1/\sqrt{3}$ induces the change of the parameter b from $i/\sqrt{3}$ to $-1/\sqrt{3}$ and the change of the parameter c from $\sqrt{3}\,i$ to 0.

6. Trees with Seven Edges

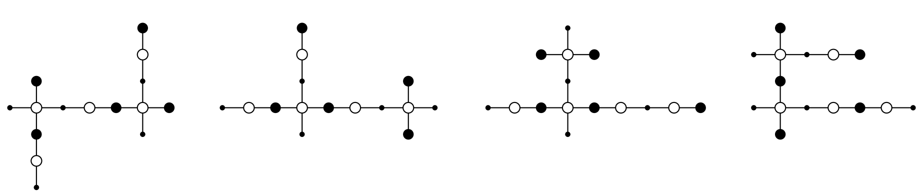
Zolotarev polynomials of degree 7 with passport $\langle 2, 2 \mid 2, 2 \mid 2, 2\rangle$ give a nontrivial example of absence of Z-homotopy (nontrivial in the sense that this absence cannot be explained by Lemma 2 or Theorem 1). Without loss of generality, we can assume that the first critical value is 0 and that the corresponding critical points are 0 and 1. Then such a polynomial is of the form

$$p(x) = \int x(x-1)(x-a)(x-b)(x-c)(x-d)\,dx,$$

where

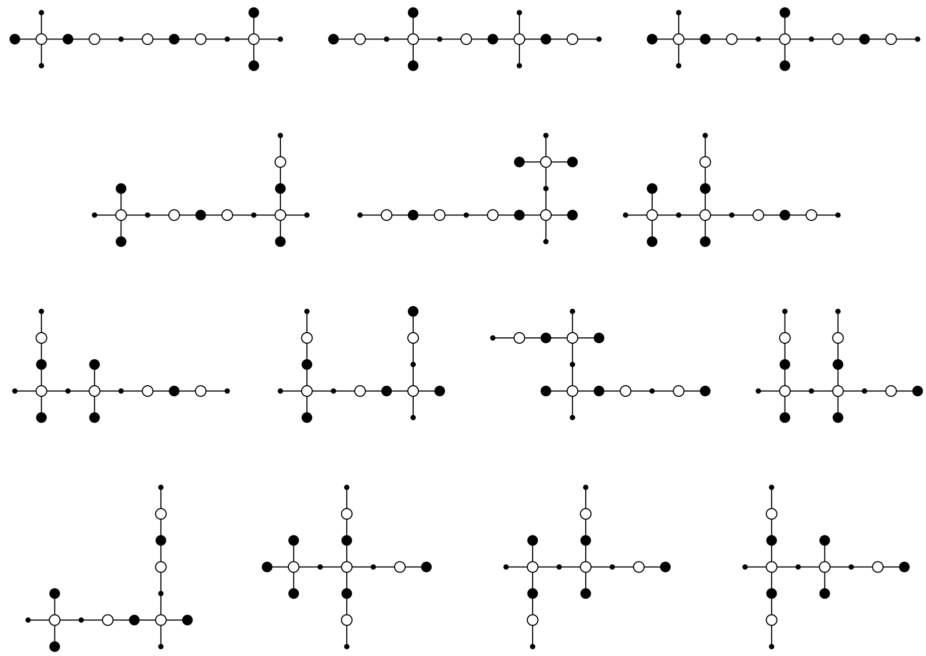
$$p(1) = 0, \quad p(a) = p(b), \quad p(c) = p(d).$$

The algebraic variety C in 4-dimensional space with coordinates a, b, c, and d defined by these conditions is reducible: it is the union of two components $C = C_1 \cup C_2$ of degrees 8 and 16. Trees (up to mirror symmetry) that correspond to Zolotarev polynomials from the first component can be seen in the picture below.



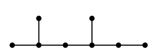
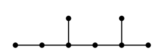
The order of monodromy group of Zolotarev polynomials from C_1 is 168.

Trees (up to mirror symmetry) that correspond to Zolotarev polynomials from the second component can be seen in the picture below.

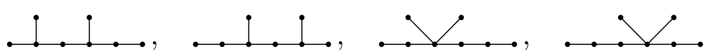


The order of the monodromy group of Zolotarev polynomials from C_2 is 2520.

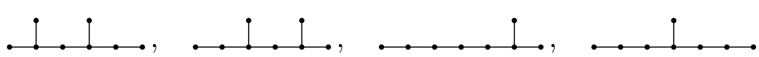
The intersection $C_1 \cap C_2$ consists of Chebyshev polynomials that correspond to trees

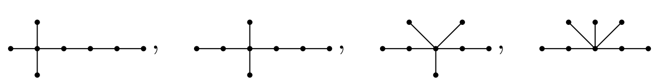
 or .

However, the component C_1 contains Chebyshev polynomials that correspond to trees

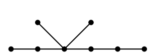
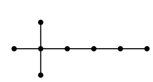


and the component C_2 contains Chebyshev polynomials that correspond to trees





Thus, we see that trees

 and ,

for example, are not Z-homotopic in the class of Zolotarev polynomials with the passport $\langle 2,2 \mid 2,2 \mid 2,2 \rangle$ (although they are Z-homotopic in the class with the passport $\langle 4 \mid 2 \mid 2 \rangle$).

REFERENCES

1. S. Lando and A. Zvonkin, *Graphs on Surfaces and Their Applications*, Springer, Berlin (2004).

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