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APPLICATION OF FUZZY SETS IN SOLVING SOME MANAGEMENT PROBLEM. PART 2

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Abstract. This paper continues another our work, which is the first of two parts where the approach to the processing of quantitative expert evaluations in the process of group decision-making under uncertainty is considered. In the second part, represented by this paper, an approach is proposed for the processing of qualitative expert evaluations in the process of group decision-making. The approach is based on the use of triangular fuzzy numbers. In group decision-making the opinions of experts are expressed by linguistic variables like *very bad*, *not very bad*, *problematic*, *good*, and so on. The technique of conversion of expert quantitative opinions to triangular fuzzy numbers is considered. A simple method of expressing expert opinions by triangular fuzzy numbers is introduced. A new approach to determining the expert degrees of importance is proposed. The proposed methodology is discussed in full detail and its algorithm is described. An illustrative example is given.

CONTENTS

1. Introduction

As in Part 1 (see [10]), our aim in this second part of the study is to try to solve frequently encountered control problems in situations in which there is no previous experience and the initial information is incomplete. It is generally accepted that fuzzy numbers are a good tool to express uncertain, limited knowledge (see, e.g., [4, 6, 8]). For the solution of the considered problems we propose the use of triangular fuzzy numbers due to their computational efficiency (see [14]). We also rely on expert evaluations and the process of group decision-making, but, in contrast to [10], here we consider such situations where the evaluation of decision-making alternatives by quantitative criteria is practically impossible. We propose the following methodology.

Experts are requested to evaluate alternatives by the scale of linguistic variables: bad, average, good and so on (for details, see Sec. 3). Then to each linguistic variable we put into correspondence a triangular fuzzy number according to the specially prepared table (see, e.g., [1, 7]). As a result, we obtain a finite collection of triangular fuzzy numbers and attempts to solve the problem of alternatives aggregation. We propose a new method for the solution of this problem.

The essence of our proposed method consists in the following. As in [11], on the set of triangular fuzzy numbers in the universe X, the degree of agreement between expert estimates is measured by

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the general metric defined by means of the isotone valuation introduced in the paper and embracing the whole class of distances between triangular fuzzy numbers.

Further, the concepts of increasing shuffling and the representative of a finite collection of triangular fuzzy numbers are introduced and the motivation of their introduction is explained. A representative is defined as a triangular fuzzy number such that the sum of distances between this number and all other members of the considered finite collection of triangular fuzzy numbers is minimal. Speaking in general, a representative may take an infinite number of values, but it can be defined uniquely by using a special fuzzy aggregation operator. The basic idea of the proposed method is: the smaller the distance between the subjective estimate of an expert and the representative, the greater the weight of his importance. Upon assessing the weight of the expert's importance, the desired result of group decision-making is obtained by the standard procedure.

The paper consists of five sections. In Sec. 2, the necessary information for understanding the paper is presented and some original results are obtained. Several theorems and propositions are proved, and a special fuzzy aggregation operator is introduced in the metric space of triangular fuzzy numbers. In Sec. 3, the methodology for group decision-making, where expert opinions are expressed by linguistic variables, is proposed. Firstly, a simple way of converting expert quantitative opinions to triangular fuzzy numbers is considered. Further, a method for group decision-making, where expert opinions are expressed by triangular fuzzy numbers, is introduced. A new approach for determining the degrees of importance of experts is proposed. The proposed methodology is discussed in full detail and its algorithm is presented. Section 4 is devoted to the practical application of the proposed methodology. Section 5, the final one, summarizes the results of the paper.

2. Essential Notions and the Theoretical Background

A *triangular fuzzy number* $\widetilde{R} = (a, b, c)$, $a, b, c \in \mathbb{R}$, $a \le b \le c$, is a fuzzy set on the universe X with the following membership function:

$$\mu_{\widetilde{R}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & \text{if } a < x < b, \\ \dfrac{c-x}{c-b}, & \text{if } b \le x < c, \\ 0, & \text{otherwise,} \end{cases} \qquad x \in X,$$

$$\Psi(X) = \left\{ \widetilde{R}_i = (a_i, b_i, c_i),\ a_i \le b_i \le c_i,\ i \in \mathbb{N} \right\}$$

is the set of all triangular fuzzy numbers on the universe X.

$\widetilde{R}_1 = \widetilde{R}_2 \Longleftrightarrow a_1 = a_2,\ b_1 = b_2,\ c_1 = c_2,\ \widetilde{R}_1, \widetilde{R}_2 \in \Psi(X)$.

$\widetilde{R}_1 \oplus \widetilde{R}_2 = (a_1 + b_1, a_2 + b_2, c_1 + c_2),\ \widetilde{R}_1, \widetilde{R}_2 \in \Psi(X)$.

$\alpha \odot \widetilde{R} = (\alpha a, \alpha b, \alpha c),\ \alpha > 0,\ \widetilde{R} \in \Psi(X)$.

Definition 2.1. The triangular fuzzy number $\widetilde{R}_1 = (a_i)$ is less than or equal to the triangular fuzzy number $\widetilde{R}_2 = (b_i)$, $i = \overline{1,3}$, i.e., $\widetilde{R}_1 \preceq \widetilde{R}_2$ if and only if

$$a_1 \le b_1, \quad a_2 \le b_2, \quad a_3 \le b_3. \tag{2.1}$$

It is known that (see [4])

$$\begin{cases} \widetilde{\min}\{\widetilde{R}_1, \widetilde{R}_2\} = \Big(\min\{a_1, b_1\}, \min\{a_2, b_2\}, \min\{a_3, b_3\}\Big), \\ \widetilde{\max}\{\widetilde{R}_1, \widetilde{R}_2\} = \Big(\max\{a_1, b_1\}, \max\{a_2, b_2\}, \max\{a_3, b_3\}\Big). \end{cases} \tag{2.2}$$

Hence it follows that the above definition is equivalent to those given in the literature (see, e.g., [4]):

$$\widetilde{R}_1 \preceq \widetilde{R}_2 \Longleftrightarrow \begin{cases} \widetilde{\min}\{\widetilde{R}_1, \widetilde{R}_2\} = \widetilde{R}_1, \\ \widetilde{\max}\{\widetilde{R}_1, \widetilde{R}_2\} = \widetilde{R}_2 \end{cases} \qquad \widetilde{R}_1, \widetilde{R}_2 \in \Psi(X).$$

It is not difficult to verify that the distributivity of $\widetilde{\min}$ and $\widetilde{\max}$ holds in $\Psi(X)$:

$$\begin{aligned} \widetilde{\min}\big\{\widetilde{R}_1, \widetilde{\max}\{\widetilde{R}_2, \widetilde{R}_3\}\big\} &= \widetilde{\max}\Big\{\widetilde{\min}\{\widetilde{R}_1, \widetilde{R}_2\}, \widetilde{\min}\{\widetilde{R}_1, \widetilde{R}_3\}\Big\}, \\ \widetilde{\max}\big\{\widetilde{R}_1, \widetilde{\min}\{\widetilde{R}_2, \widetilde{R}_3\}\big\} &= \widetilde{\min}\Big\{\widetilde{\max}\{\widetilde{R}_1, \widetilde{R}_2\}, \widetilde{\max}\{\widetilde{R}_1, \widetilde{R}_3\}\Big\}. \end{aligned} \tag{2.3}$$

To determine the distances between triangular fuzzy numbers, we need to introduce the metric on $\Psi(X)$. Depending on the metric, the distances between two objects can be measured in many ways. There exists a vast amount of literature on this issue (see, e.g., [6]). Here we have chosen the following approach.

We say that the function $v(\widetilde{R}) : \Psi(X) \to \mathcal{R}^+$ is an *isotone valuation on* $\Psi(X)$ if

$$\begin{gathered} v\big(\widetilde{\max}\{\widetilde{R}_1, \widetilde{R}_2\}\big) + v\big(\widetilde{\min}\{\widetilde{R}_1, \widetilde{R}_2\}\big) = v(\widetilde{R}_1) + v(\widetilde{R}_2), \\ \text{and} \quad \widetilde{R}_1 \preceq \widetilde{R}_2 \implies v(\widetilde{R}_1) \le v(\widetilde{R}_2). \end{gathered} \tag{2.4}$$

Let us consider the equation

$$\rho(\widetilde{R}_1, \widetilde{R}_2) = v\big(\widetilde{\max}\{\widetilde{R}_1, \widetilde{R}_2\}\big) - v\big(\widetilde{\min}\{\widetilde{R}_1, \widetilde{R}_2\}\big). \tag{2.5}$$

Now we show that (2.5) represents the *metric* on $\Psi(X)$, i.e., it satisfies the following conditions:

(1) $\rho(\widetilde{R}_1, \widetilde{R}_2) = 0 \Longleftrightarrow \widetilde{R}_1 = \widetilde{R}_2$;
(2) $\rho(\widetilde{R}_1, \widetilde{R}_2) = \rho(\widetilde{R}_2, \widetilde{R}_1)$;
(3) $\rho(\widetilde{R}_1, \widetilde{R}) + \rho(\widetilde{R}, \widetilde{R}_2) \ge \rho(\widetilde{R}_1, \widetilde{R}_2)$, for all $\widetilde{R} \in \Psi(X)$.

Proof. Let

$$\rho(\widetilde{R}_1, \widetilde{R}_2) = 0 \implies v\big(\widetilde{\max}\{\widetilde{R}_1, \widetilde{R}_2\}\big) = v\big(\widetilde{\min}\{\widetilde{R}_1, \widetilde{R}_2\}\big).$$

By (2.2) and (2.4) we can conclude that

$$\widetilde{\max}\{\widetilde{R}_1, \widetilde{R}_2\} = \widetilde{\min}\{\widetilde{R}_1, \widetilde{R}_2\} \implies \widetilde{R}_1 = \widetilde{R}_2.$$

Now let

$$\begin{aligned} \widetilde{R}_1 = \widetilde{R}_2 \implies \widetilde{\max}\{\widetilde{R}_1, \widetilde{R}_2\} = \widetilde{\min}\{\widetilde{R}_1, \widetilde{R}_2\} \implies \\ \implies v\big(\widetilde{\max}\{\widetilde{R}_1, \widetilde{R}_2\}\big) = v\big(\widetilde{\min}\{\widetilde{R}_1, \widetilde{R}_2\}\big) \implies \rho(\widetilde{R}_1, \widetilde{R}_2) = 0. \end{aligned}$$

So, (1) is valid. We have

$$\begin{aligned} \rho(\widetilde{R}_1, \widetilde{R}_2) &= v\big(\widetilde{\max}\{\widetilde{R}_1, \widetilde{R}_2\}\big) - v\big(\widetilde{\min}\{\widetilde{R}_1, \widetilde{R}_2\}\big) \\ &= v\big(\widetilde{\max}\{\widetilde{R}_2, \widetilde{R}_1\}\big) - v\big(\widetilde{\min}\{\widetilde{R}_2, \widetilde{R}_1\}\big) = \rho(\widetilde{R}_2, \widetilde{R}_1), \end{aligned}$$

and (2) is also true. Finally,

$$
\begin{aligned}
\rho(\widetilde{R}_1,\widetilde{R}_2) &= v\big(\widetilde{\max}\{\widetilde{R}_1,\widetilde{R}_2\}\big) - v\big(\widetilde{\min}\{\widetilde{R}_1,\widetilde{R}_2\}\big)\\
&\overset{(2.4)}{\le} v\Big(\widetilde{\max}\Big\{\widetilde{\max}\{\widetilde{R}_1,\widetilde{R}\},\widetilde{\max}\{\widetilde{R},\widetilde{R}_2\}\Big\}\Big) - v\Big(\widetilde{\min}\Big\{\widetilde{\min}\{\widetilde{R}_1,\widetilde{R}\},\widetilde{\min}\{\widetilde{R},\widetilde{R}_2\}\Big\}\Big)\\
&\overset{(2.4)}{=} v\big(\widetilde{\max}\{\widetilde{R}_1,\widetilde{R}\}\big) + v\big(\widetilde{\max}\{\widetilde{R},\widetilde{R}_2\}\big) - v\Big(\widetilde{\min}\Big\{\widetilde{\max}\{\widetilde{R}_1,\widetilde{R}\},\widetilde{\max}\{\widetilde{R},\widetilde{R}_2\}\Big\}\Big)\\
&\quad - v\big(\widetilde{\min}\{\widetilde{R}_1,\widetilde{R}\}\big) - v\big(\widetilde{\min}\{\widetilde{R},\widetilde{R}_2\}\big) + v\Big(\widetilde{\max}\Big\{\widetilde{\min}\{\widetilde{R}_1,\widetilde{R}\},\widetilde{\min}\{\widetilde{R},\widetilde{R}_2\}\Big\}\Big)\\
&\overset{(2.3),(2.4)}{=} \rho(\widetilde{R}_1,\widetilde{R}) + \rho(\widetilde{R},\widetilde{R}_2) + v\Big(\widetilde{\min}\big\{\widetilde{R},\widetilde{\max}\{\widetilde{R}_1,\widetilde{R}_2\}\big\}\Big) - v\Big(\widetilde{\max}\big\{\widetilde{R},\widetilde{\min}\{\widetilde{R}_1,\widetilde{R}_2\}\big\}\Big)\\
&\overset{(2.4)}{\le} \rho(\widetilde{R}_1,\widetilde{R}) + \rho(\widetilde{R},\widetilde{R}_2),
\end{aligned}
$$

and (3) is proved. The proof is complete. □

Thus, (2.5) indeed is the metric on $\Psi(X)$ with isotone valuation v. Metric (2.5) is called the *metric space* of triangular fuzzy numbers.

Definition 2.2. In the metric space the triangular fuzzy number $\widetilde{R}^*$ is *the representative* of the finite collection of triangular fuzzy numbers $\{\widetilde{R}_j\}$, $j=\overline{1,m}$, $m=2,3,\dots$, if

$$
\sum_{j=1}^{m}\rho(\widetilde{R}^*,\widetilde{R}_j) \le \sum_{j=1}^{m}\rho(\widetilde{S},\widetilde{R}_j), \quad \forall \widetilde{S}\in\Psi(X). \tag{2.6}
$$

To simplify the further theoretical constructions we need to introduce the concept of *increasing shuffling* of a finite collection of triangular fuzzy numbers. We begin with an example.

Suppose we have a finite collection of triangular fuzzy numbers:

	a_j	b_j	c_j
$\widetilde{R}_1$	7	7.5	8
$\widetilde{R}_2$	6	6.1	7.7
$\widetilde{R}_3$	1	3	5
$\widetilde{R}_4$	7.6	7.9	8.1

Compare with it the following finite collection of triangular fuzzy numbers:

	a'_j	b'_j	c'_j
$\widetilde{R}'_1$	1	3	5
$\widetilde{R}'_2$	6	6.1	7.7
$\widetilde{R}'_3$	7	7.5	8
$\widetilde{R}'_4$	7.6	7.9	8.1

We see that the columns contain the same values, but they are increasingly ordered in the second table. By an increasing shuffling of the finite collection of triangular fuzzy numbers $\{\widetilde{R}_1,\widetilde{R}_2,\widetilde{R}_3,\widetilde{R}_4\}$ we mean the finite collection of triangular fuzzy numbers $\{\widetilde{R}'_1,\widetilde{R}'_2,\widetilde{R}'_3,\widetilde{R}'_4\}$. The strict definition of an increasing shuffling of the set of triangular fuzzy numbers will be given below, whereas now we have to answer one question, namely: whether the members of this shuffling are also triangular fuzzy numbers. The question can be rephrased as follows: Does the correlation $a'_j \le b'_j \le c'_j$, $j = \overline{1,m}$, $m = 2, 3, \ldots$, hold? The next proposition provides a positive answer.

Proposition 2.1. *Consider the finite collections of triangular fuzzy numbers* $\{\widetilde{R}_j\} = \{(a_j, b_j, c_j)\}$ *and* $\{\widetilde{R}'_j\} = \{(a'_j, b'_j, c'_j)\}$*, where the sets* $\{a_j\}$ *and* $\{a'_j\}$*,* $\{b_j\}$ *and* $\{b'_j\}$*,* $\{c_j\}$ *and* $\{c'_j\}$ *are pairwise equal and* $a'_1 \le a'_2 \le \cdots \le a'_m$*,* $b'_1 \le b'_2 \le \cdots \le b'_m$*,* $c'_1 \le c'_2 \le \cdots \le c'_m$*,* $j = \overline{1,m}$*,* $m = 2, 3, \ldots$*. Then* $a'_j \le b'_j \le c'_j$*, i.e.* $\{\widetilde{R}'_j\} \in \Psi(X)$*.*

Proof. We prove the proposition for the sets $\{a_j\}$, $\{a'_j\}$, and $\{b_j\}$, $\{b'_j\}$, $j = \overline{1,m}$, $m = 2, 3, \ldots$. The proof for the other sets of vertices of triangular fuzzy numbers is analogous. We proceed by induction.

- (i) $m = 2$. We have $\{a_1, b_1\}$, $\{a_2, b_2\}$, where $a_1 \le b_1$, $a_2 \le b_2$; $\{a'_j\} = \{a_j\}$, $\{b'_j\} = \{b_j\}$, $j \in \{1, 2\}$, where $a'_1 \le a'_2$, $b'_1 \le b'_2$. It is obvious that $a'_1 = \min\{a_1, a_2\} \le \min\{b_1, b_2\} = b'_1$ and $a'_2 = \max\{a_1, a_2\} \le \max\{b_1, b_2\} = b'_2$.
- (ii) Assume that $a'_j \le b'_j$ for all $j \in \{3, 4, \ldots, m-1\}$, $m = 4, 5, \ldots$.
- (iii) Consider a'_m and b'_m. In fact, $a'_m = \max\{a'_j, a_m\}$ and $b'_m = \max\{b'_j, b_m\}$. This and (ii) imply $a'_m \le b'_m$.

□

Now we are able to introduce a strict definition of increasing shuffling.

Definition 2.3. The finite collection of triangular fuzzy numbers $\{\widetilde{R}'_j\}$ is the increasing shuffling of the finite collection of triangular fuzzy numbers $\{\widetilde{R}_j\}$ if the finite sets $\{a_j\}$ and $\{a'_j\}$, $\{b_j\}$ and $\{b'_j\}$, $\{c_j\}$ and $\{c'_j\}$ are pairwise equal and $a'_1 \le a'_2 \le \cdots \le a'_m$, $b'_1 \le b'_2 \le \cdots \le b'_m$, $c'_1 \le c'_2 \le \cdots \le c'_m$, $j = \overline{1,m}$, $m = 2, 3, \ldots$.

By this definition and (2.5) it is obvious that the equality

$$\sum_{j=1}^{m} \rho(\widetilde{S}, \widetilde{R}_j) = \sum_{j=1}^{m} \rho(\widetilde{S}, \widetilde{R}'_j) \tag{2.7}$$

holds in the metric space for any $\widetilde{S} \in \Psi(X)$ and the finite collection of triangular fuzzy numbers $\{\widetilde{R}_j\}$, $j = \overline{1,m}$, $m = 2, 3, \ldots$. From (2.7) it follows that the *representatives of the finite collection of triangular fuzzy numbers and its increasing shuffling coincide.*

By Definition 2.3 and (2.1), the following proposition is valid.

Proposition 2.2. *The increasing shuffling is a finite collection of nested triangular fuzzy numbers*:

$$\widetilde{R}'_1 \preceq \widetilde{R}'_2 \preceq \cdots \preceq \widetilde{R}'_m, \quad m = 2, 3, \ldots.$$

It is easy to see that

$$\widetilde{R}_1 \preceq \widetilde{R}_2 \implies \rho(\widetilde{R}_1, \widetilde{R}_2) = v(\widetilde{R}_2) - v(\widetilde{R}_1). \tag{2.8}$$

Theorem 2.1. *In the metric space of triangular fuzzy numbers the representative* $\widetilde{R}^*$ *of the finite collection of triangular fuzzy numbers* $\{\widetilde{R}_j\}$*,* $j = \overline{1,m}$*,* $m = 2, 3, \ldots$ *is determined as follows*:

$$\widetilde{R}'_{m/2} \preceq \widetilde{R}^* \preceq \widetilde{R}'_{m/2+1} \quad \textit{if } m \textit{ is even};$$

$$\widetilde{R}^* = \widetilde{R}'_{(m+1)/2} \quad \textit{if } m \textit{ is odd}.$$

Proof. First, we show the validity of the correlation

$$\widetilde{R}'_p \preceq \widetilde{R}^* \preceq \widetilde{R}'_{p+1}, \quad p \in \{1, 2, \ldots, m-1\}. \tag{$*$}$$

Assume that $\widetilde{R}'_k \npreceq \widetilde{R}^*$ and $\widetilde{R}^* \npreceq \widetilde{R}'_k$, $k \in \{1, 2, \ldots, m\}$.

(a) $m = 2$. Consider

$$\rho(\widetilde{R}^*, \widetilde{R}'_j) - \rho(\widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_1\}, \widetilde{R}'_j) + \rho(\widetilde{R}^*) - \rho(\widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_2\}, \widetilde{R}'_j).$$

By (2.5), (2.4), and the properties of the isotone valuation v we have

$$\begin{aligned} &- 2v(\widetilde{R}^*) + 2v\big(\widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_j\}\big) + v\big(\widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_1\}\big) - v\big(\widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_2\}\big) \\ &\qquad\qquad \geq -2v(\widetilde{R}^*) + 2v\big(\widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_j\}\big) > 0. \end{aligned}$$

From this it follows that

$$\sum_{j=1}^{2} \rho(\widetilde{R}^*, \widetilde{R}'_j) - \sum_{j=1}^{2} \rho\big(\widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_k\}, \widetilde{R}'_j\big) > 0, \quad k = 1, 2,$$

a contradiction to (2.6). Thus, for $m = 2$ the above assumption is invalid.

(b) $m > 2$. Let $k \leq (m+1)/2$. Then for any $j \in \{1, 2, \ldots, k-1\}$, by (2.5), (2.4), and the properties of the isotone valuation v we have

$$\begin{aligned} &\rho(\widetilde{R}^*, \widetilde{R}'_j) - \rho\big(\widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_k\}, \widetilde{R}'_j\big) \\ &= 2v\big(\widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_j\}\big) - v\big(\widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_k\}\big) - v(\widetilde{R}^*) \geq v(\widetilde{R}^*) - v\big(\widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_k\}\big); \end{aligned}$$

this leads to

$$\rho(\widetilde{R}^*, \widetilde{R}'_j) - \rho\big(\widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_k\}, \widetilde{R}'_j\big) \geq -\rho\big(\widetilde{R}^*, \widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_k\}\big). \tag{$**$}$$

Further, for any $j \in \{k, k + \overline{1, m}\}$ we similarly obtain

$$\rho\big(\widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_j\}\big) - \rho\big(\widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_k\}, \widetilde{R}'_j\big) = \rho\big(\widetilde{R}^*, \widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_k\}\big). \tag{$***$}$$

By $(**)$ and $(***)$ we have

$$\begin{aligned} &\sum_{j=1}^{k-1} \rho(\widetilde{R}^*, \widetilde{R}'_j) - \sum_{j=1}^{k-1} \rho\big(\widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_k\}, \widetilde{R}'_j\big) \geq (1-k)\rho\big(\widetilde{R}^*, \widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_k\}\big), \\ &\sum_{j=k}^{m} \rho(\widetilde{R}^*, \widetilde{R}'_j) - \sum_{j=k}^{m} \rho\big(\widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_k\}, \widetilde{R}'_j\big) = (m-k+1)\rho\big(\widetilde{R}^*, \widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_k\}\big). \end{aligned}$$

The last two equations yield

$$\sum_{j=1}^{m} \rho(\widetilde{R}^*, \widetilde{R}'_j) - \sum_{j=1}^{m} \rho\big(\widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_k\}, \widetilde{R}'_j\big) \geq (m-2k+2)\rho\big(\widetilde{R}^*, \widetilde{\max}\{\widetilde{R}^*, \widetilde{R}'_k\}\big) > 0,$$

a contradiction to (2.6).

Now let $k > (m+1)/2$. Then, analogously to the above, we have

$$\sum_{j=1}^{m} \rho(\widetilde{R}^*, \widetilde{R}'_j) - \sum_{j=1}^{m} \rho\big(\widetilde{\min}\{\widetilde{R}^*, \widetilde{R}'_k\}, \widetilde{R}'_j\big) \geq (2k-m)\rho\big(\widetilde{R}^*, \widetilde{\min}\{\widetilde{R}^*, \widetilde{R}'_k\}\big) > 0,$$

also a contradiction to (2.6).

It remains to show that $\widetilde{R}'_1 \preceq \widetilde{R}^* \preceq \widetilde{R}'_m$. Let $\widetilde{R}^* \prec \widetilde{R}'_1$. It is not difficult to verify that

$$\rho(\widetilde{R}^*, \widetilde{R}'_j) - \rho(\widetilde{R}'_1, \widetilde{R}'_j) > 0$$

and, consequently,

$$\sum_{j=1}^{m} \rho(\widetilde{R}^*, \widetilde{R}'_j) - \sum_{j=1}^{m} \rho(\widetilde{R}'_1, \widetilde{R}'_j) > 0;$$

again a contradiction to (2.6). The proof for the case $\widetilde{R}^* \preceq \widetilde{R}'_m$ is similar. So, (∗) is proved. From this and (2.8) it follows that

$$\sum_{j=1}^{m} \rho(\widetilde{R}^*, \widetilde{R}'_j) = \sum_{j=1}^{m} \left|v(\widetilde{R}^*) - v(\widetilde{R}'_j)\right|.$$

As is known, the last expression attains a minimal value when $v(\widetilde{R}^*)$ is the median of the nondecreasing numerical sequence $\{v(\widetilde{R}'_j)\}$, $j = \overline{1,m}$, $m = 2, 3, \ldots$. Thus, if m is even then

$$v(\widetilde{R}'_{m/2}) \le v(\widetilde{R}^*) \le v(\widetilde{R}'_{m/2+1}),$$

and if m is odd, then

$$v(\widetilde{R}^*) = v(\widetilde{R}'_{(m+1)/2}).$$

Since we are dealing with a finite collection of nested triangular fuzzy numbers, we can conclude that

$$v(\widetilde{R}_k) \le v(\widetilde{R}_l) \implies \widetilde{R}_k \preceq \widetilde{R}_l, \qquad v(\widetilde{R}_k) = v(\widetilde{R}_l) \implies \widetilde{R}_k = \widetilde{R}_l.$$

Thus,

$$v(\widetilde{R}'_{m/2}) \le v(\widetilde{R}^*) \le v(\widetilde{R}'_{m/2+1}) \implies \widetilde{R}'_{m/2} \preceq \widetilde{R}^* \preceq \widetilde{R}'_{m/2+1}$$

if m is even, and

$$v(\widetilde{R}^*) = v(\widetilde{R}'_{(m+1)/2}) \implies \widetilde{R}^* = \widetilde{R}'_{(m+1)/2}$$

if m is odd. The proof is complete. □

To realize our approach, we need a special aggregation operator that satisfies certain requirements. Today, in fuzzy sets theory there are a lot of well-known fuzzy aggregation operators such as triangular norms, conorms, averaging with weight coefficients, and so on.

In the metric space of triangular fuzzy numbers the representative of a finite collection of triangular fuzzy numbers, as stated in Definition 2.2, is a new kind of fuzzy aggregation operator. By Theorem 2.1 the representative of the finite collection of triangular fuzzy numbers $\{\widetilde{R}_j\}$, $j = \overline{1,m}$, $m = 2, 3, \ldots$, yields $\widetilde{R}'_{m/2} \preceq \widetilde{R}^* \preceq \widetilde{R}'_{m/2+1}$ if m is even, and $\widetilde{R}^* = \widetilde{R}'_{(m+1)/2}$ if m is odd. Now we will introduce a fuzzy aggregation operator that always meets these requirements.

From Theorem 2.1 it follows that if m is even and $\widetilde{R}'_{m/2} \prec \widetilde{R}'_{m/2+1}$ then the representative can take an infinite set of values (if $\widetilde{R}'_{m/2} = \widetilde{R}'_{m/2+1}$, then we consider the interval $[\widetilde{R}'_{m/2-1}; \widetilde{R}'_{m/2+2}]$ and use the approach proposed below for odd m and, at the same time, $\widetilde{R}'_{m/2}$ will be taken into account twice and so on). Let us consider this case in detail.

Let m be even: $m = 2p$, $p = 1, 2, \ldots$. Then the representative may take a lot of values from the interval $[\widetilde{R}'_p; \widetilde{R}'_{p+1}]$. Let us determine such a value that takes into account the values of all other members of the increasing shuffling of the considered finite collection of triangular fuzzy numbers. Calculate the sum of distances between $\widetilde{R}'_p$ and all other members of the increasing shuffling before it. Do the same for $\widetilde{R}'_{p+1}$ and all the members of increasing shuffling after it. Since our argumentation is based on the metric approach, it is natural to assume that the representative should be located "closer" to that endpoint of the interval $[\widetilde{R}'_p; R'_{p+1}]$ beyond which the calculated sum has a minimal value. In other words, the representative should be located "closer" to that endpoint of interval $[\widetilde{R}'_p; \widetilde{R}'_{p+1}]$ beyond which the increasing stuffing's members are located "closer" to one another.

Thus, when these sums are equal to one another, we obtain that the representative is equal to the extended arithmetic mean of the triangular fuzzy numbers $\widetilde{R}'_p = (a_{p,i})$, $\widetilde{R}'_{p+1} = (a_{p+1,i})$:

$$\widetilde{R}^* = (a^* = (a_{p,i} + a_{p+1,i})/2), \quad i = \overline{1,3}.$$

If the sum of distances between the left hand of that interval and all the other members of the increasing shuffling located before this hand is equal to zero, then the representative is equal to $\widetilde{R}'_p$. If the sum of distances between the right hand of that interval and all the other members of increasing shuffling located after this hand is equal to zero, then the representative is equal to $\widetilde{R}'_{p+1}$. By these arguments we can determine the representative uniquely as follows:

$$\widetilde{R}^* = \begin{cases} \dfrac{a'_{p,i} + a'_{p+1,i}}{2} & \text{if} \quad \sum\limits_{j=1}^{p-1} \rho(\widetilde{R}'_j, \widetilde{R}'_p) = \sum\limits_{j=p+2}^{m} \rho(\widetilde{R}'_j, \widetilde{R}'_{p+1}), \\ a'_{p,i} + \left(\dfrac{\sum\limits_{i=1}^{p-1} \rho(\widetilde{R}'_j, \widetilde{R}'_p)}{\sum\limits_{j=1}^{p-1} \rho(\widetilde{R}'_j, \widetilde{R}'_p) + \sum\limits_{j=p+2}^{m} \rho(\widetilde{R}'_j, \widetilde{R}'_{p+1})} (a'_{p+1,i} - a'_{p,i}) \right) & \text{otherwise}, \end{cases} \quad i = \overline{1,3}. \tag{2.9}$$

Now let m be odd: $m = 2p + 1$, $p = 1, 2, \ldots$. In this case, by Theorem 2.1 the representative is equal to $\widetilde{R}'_{p+1}$. If we take this value off and $\widetilde{R}'_p \subset \widetilde{R}'_{p+2}$, then any other fixed triangular fuzzy number among an infinite number of triangular fuzzy numbers from the interval $[\widetilde{R}'_p; \widetilde{R}'_{p+2}]$ is also a representative. The question is how to determine the unique representative so as to take into account all increasing stuffing members of the given finite collection of triangular fuzzy numbers. For this we can use the above-mentioned approach with an additional "weight" of $\widetilde{R}'_{p+1}$. Calculate the sum of distances between $\widetilde{R}'_p$ and all other members of the increasing shuffling that are less than or equal to $\widetilde{R}'_{p+1}$. Do the same for $\widetilde{R}'_{p+2}$ and all other members of the increasing shuffling that are larger than or equal to $\widetilde{R}'_{p+1}$. As a result, we obtain the only representative:

$$\widetilde{R}^* = \begin{cases} \dfrac{a'_{p,i} + a'_{p+2,i}}{2} & \text{if} \quad \sum\limits_{j=1}^{p+1} \rho(\widetilde{R}'_j, \widetilde{R}'_p) = \sum\limits_{j=p+1}^{m} \rho(\widetilde{R}'_j, \widetilde{R}'_{p+2}), \\ a'_{p,i} + \left(\dfrac{\sum\limits_{i=1}^{p+1} \rho(\widetilde{R}'_j, \widetilde{R}'_p)}{\sum\limits_{j=1}^{p+1} \rho(\widetilde{R}'_j, \widetilde{R}'_p) + \sum\limits_{j=p+1}^{m} \rho(\widetilde{R}'_j, \widetilde{R}'_{p+2})} (a'_{p+2,i} - a'_{p,i}) \right) & \text{otherwise}, \end{cases} \quad i = \overline{1,3}. \tag{2.10}$$

A thorough examination of (2.9) and (2.10) gives a good opportunity to combine both formulas. One can be easily convinced that the following correlations are correct (here and in the sequel the symbol [ ] denotes the integer part of a number):

$$\sum_{j=1}^{p-1} \rho(\widetilde{R}'_j, \widetilde{R}'_p) = \sum_{j=1}^{p} \rho(\widetilde{R}'_j, \widetilde{R}'_p), \quad \sum_{j=p+2}^{m} \rho(\widetilde{R}'_j, \widetilde{R}'_{p+1}) = \sum_{j=p+1}^{m} \rho(\widetilde{R}'_j, \widetilde{R}'_{p+1}),$$
$$m = 2p \implies [m/2] = p, \quad [(m+1)/2] = p, \quad [(m+3)/2] = p+1;$$
$$m = 2p+1 \implies [m/2] = p; \quad [(m+1)/2] = p+1; \quad [(m+3)/2] = p+2.$$

Hence we obtain the general formula for a representative:

$$\widetilde{R}^* = \begin{cases} \dfrac{a'_{[m/2],i} + a'_{[(m+3)/2],i}}{2} & \text{if} \quad \sum\limits_{j=1}^{[(m+1)/2]} \rho(\widetilde{R}'_j, \widetilde{R}'_{[m/2]}) = \sum\limits_{j=[m/2]+1}^{m} \rho(\widetilde{R}'_j, \widetilde{R}'_{[(m+3)/2]}), \\ a'_{[m/2],i} + \left(\dfrac{\sum\limits_{j=1}^{[(m+1)/2]} \rho(\widetilde{R}'_j, \widetilde{R}'_{[m/2]})}{\sum\limits_{j=1}^{[(m+1)/2]} \rho(\widetilde{R}'_j, \widetilde{R}'_{[m/2]}) + \sum\limits_{j=[m/2]+1}^{m} \rho(\widetilde{R}'_j, \widetilde{R}'_{[(m+3)/2]})} \times (a'_{[(m+3)/2],i} - a'_{[m/2],i}) \right) & \text{otherwise,} \end{cases} \quad i = \overline{1,3}. \tag{2.11}$$

Remark 2.1. It is easy to show that the representative determined by (2.11) is a triangular fuzzy number.

3. Methodology for Fuzzy Aggregation of Expert Qualitative Opinions

Let us consider the situation where we have several alternatives of making management decisions or, in other words, several projects. The project database is either incomplete or totally absent and, moreover, the quantitative criteria of evaluation of the projects practically do not exist. Recall that people think in uncertain categories, usually in a vague "more or less" manner. In that case, in the process of questioning of experts we use linguistic variables. Classical fuzzy theory is a powerful tool to manage granularity, namely linguistic variables, introduced by Zadeh (see [13]): By a linguistic variable we mean a variable whose values are words or sentences in a natural or artificial language. For example, age is a linguistic variable, its values are linguistic rather than numerical, i.e., young, not young, very young, quite young, old, not very old and not very young, etc., rather than 20; 21; 22; 23; ..." (see [12]). In order to better understand the essence of a linguistic variable, we present one of its formal definitions.

Definition 3.1 (see [12])**.** A classical linguistic variable is a quintuple $(X; V; \mu; [0,1]; \leq)$, where X is a set (called the 'domain'), V is a set (of 'linguistic values'), $([0,1], \leq)$ is the real unit interval with its usual ordering, and μ is mapping $\mu : V \to F(X, [0,1])$ that represents each linguistic value v by a membership function $\mu_v := \mu(v)$ on X.

Let us now partly specify the components of the quintuple $(X; V; \mu; [0,1]; \leq)$. The universe is denoted by $X = [0, b] \subset \mathbb{R}$. Further we have to choose a scale of linguistic variables. There is a wealth of literature on qualitative scales of evaluations (see, e.g., [11]). In the general case, the set V has the form

$$V = \{v_1, v_2, \dots, v_N\}, \quad N = 2, 3, \dots. \tag{3.1}$$

Note that a qualitative scale may vary depending on the considered project. This happens when it is obvious that one or several components out of V are not needed and the scale $V' | V' \subset V$ will be used.

Thus a group of m, $m = 2, 3, \dots$, experts is convened and each of them is requested to fill in the following table by marking his/her evaluation with the asterisk:

Now we proceed to the stage of processing of expert evaluations, i.e., we must determine the mapping μ for the conversion of expert qualitative evaluations into quantitative indexes. Here we propose the following approach. Depending on the project data domain, where we have to make a decision, to each evaluation from the set $V = \{v_1, v_2, \dots, v_N\}$ we assign a fuzzy set in the form of a triangular

Table 3.1.

Evaluation/Expert No.	v_1	v_2	...	v_N
1				
2				
...				
m				

fuzzy number as follows:

$$\mu : V \to F(X, [0, 1]) = \widetilde{R}_j \in \Psi(X), \quad j = \overline{1, m}, \quad \widetilde{R}_1 \prec \widetilde{R}_2 \prec \cdots \prec \widetilde{R}_m. \tag{3.2}$$

Let us write (3.2) in the tabular form:

Table 3.2.

v_1	v_2	...	v_N
$\widetilde{R}(v_1)$	$\widetilde{R}(v_2)$		$\widetilde{R}(v_N)$

Therefore, in the process of evaluation of a certain project by combining the results of Tables 3.1 and 3.2 we have a finite collection of triangular fuzzy numbers. To arrive at the final result, the obtained fuzzy numbers must be aggregated. In the literature there exist a sufficient number of aggregation methods for the decision-making process, which are based on fuzzy set theory (see, e.g., [2, 3, 5]).

We propose a new approach for the aggregation of a finite number of triangular fuzzy numbers. The problem consists in processing these triangular fuzzy numbers so that a consensus could be found. In constructing any kind of aggregation method under group decision-making, the key task is to determine the well-justified weights of importance for each expert. Let us consider a finite collection of triangular fuzzy numbers obtained by conversion of expert quantitative estimates. To our mind, the representative of this collection, i.e., a triangular fuzzy number such that the sum of distances between it and all other members of the given finite collection is minimal, is of particular interest. The representative can be regarded as a kind of group consensus, but in that case the degrees of experts' importance are neglected. The representative is something like a standard for the members of the considered collection. As the weights of physical bodies are measured by comparing them with the Paris standard kilogram, it seems natural for us to determine experts' weights of importance depending on how close experts' estimates are to the representative.

Thus, the main idea of the proposed method reduces to the following. The weight of importance for each expert is determined by a function inversely proportional to the distance between his/her transformed estimate and the representative of the finite collection of all experts' transformed estimates, i.e., the smaller the distance between an expert's estimate and the representative, the larger the weight of his importance.

Let us give a formal description of the method. Let $\widetilde{R}_j$, $j \in \{1, 2, \ldots, m\}$, $m = 2, 3, \ldots$, be a triangular fuzzy number representing the jth expert's subjective estimate of the rating to an alternative under a given criterion. The transformed estimates of all experts form the finite collection of triangular fuzzy numbers $\{\widetilde{R}_j\}$. By Definition 2.3 and formula (2.11) we find the increasing shuffling $\{\widetilde{R}'_j\}$ and the representative $\widetilde{R}^*$ of this collection. Denote the jth expert's aggregation weight (weight of importance) and the final result of aggregation by ω_j and $\widetilde{R} = (\widetilde{a}, \widetilde{b}, \widetilde{c})$, respectively.

By the above reasoning, the weights and the final result of aggregation can be defined in the form

$$\omega_j = \frac{(\rho(\widetilde{R}^*, \widetilde{R}_j))^{-1}}{\sum\limits_{j=1}^{m} (\rho(\widetilde{R}^*, \widetilde{R}_j))^{-1}} \tag{3.3}$$

and

$$\widetilde{R} = \sum_{j=1}^{m} (\omega_j \odot \widetilde{R}_j). \tag{3.4}$$

Here $\odot$ is the fuzzy multiplication operator (see [8]). It is obvious that $\sum\limits_{j=1}^{m} \omega_j = 1$.

This approach looks plausible, but if at least one member of the finite collection of expert estimates coincides with the representative, then the function u_j discontinues. We investigate the cases where the function u_j becomes discontinuous. For the sake of simplicity, we use the notation:

$$\rho_j = \rho(\widetilde{R}_j, \widetilde{R}^*), \quad j = \overline{1, m}, \ m = 2, 3, \dots .$$

Let

$$\left\{\widetilde{R}_1 = (a_1, b_1, c_1), \ \widetilde{R}_2 = (a_2, b_2, c_2), \ \widetilde{R}_3 = (a_3, b_3, c_3)\right\}$$

be a finite collection of three triangular fuzzy numbers, and $\widetilde{R}^* = (a^*, b^*, c^*)$ be its representative. Let us consider the following three cases:

(i) Assume that exactly one member of this finite collection coincides with the representative, say, $\widetilde{R}_3 = \widetilde{R}^*$. Then the last equation can be rewritten as

$$a_3 = a^* \pm \varepsilon_1, \quad b_3 = b^* \pm \varepsilon_2, \quad c_3 = c^* \pm \varepsilon_3; \quad \varepsilon_k \to 0, \quad k = \overline{1, 3} \quad \text{are infinitesimals.}$$

This implies that $\rho_3 = \pm\delta \to 0$ is infinitesimal. Without loss of generality, here and in the sequel we will operate with positive infinitesimals.

Consider

$$\sum_{j=1}^{3} \omega_j = \frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\delta} = \frac{\delta\rho_2 + \delta\rho_1 + \rho_1\rho_2}{\delta\rho_1\rho_2} \implies$$

$$\implies \omega_1 = \frac{\delta\rho_2}{\delta\rho_2 + \delta\rho_1 + \rho_1\rho_2}, \quad \omega_2 = \frac{\delta\rho_1}{\delta\rho_2 + \delta\rho_1 + \rho_1\rho_2}, \quad \omega_3 = \frac{\rho_1\rho_2}{\delta\rho_2 + \delta\rho_1 + \rho_1\rho_2}.$$

We determine the first coordinate of the result of aggregation $\widetilde{R} = (\widetilde{a}, \widetilde{b}, \widetilde{c})$ by (3.4):

$$\widetilde{a} = \frac{a^*\rho_1\rho_2 + \varepsilon_1\rho_1\rho_2 + \delta\rho_2 a_1 + \delta\rho_1 a_2}{\rho_1\rho_2 + \delta\rho_2 + \delta\rho_1}.$$

It is easy to see that the last expression is continuous and tends to the limit a^*. Similarly, we obtain that

$$\widetilde{R} = (a^*, b^*, c^*) = \widetilde{R}^*.$$

(ii) Assume that two members of the finite collection coincide with the representative, say, $\widetilde{R}_1 = \widetilde{R}_2 = \widetilde{R}^*$. Then the last equation can be rewritten as

$$a_1 = a^* + \varepsilon_1, \quad a_2 = a^* + \varepsilon_2, \quad \dots, \quad c_1 = c^* + \varepsilon_5, \quad c_2 = c^* + \varepsilon_6; \quad \varepsilon_k \to 0, \quad k = \overline{1, 6}.$$

Hence $\rho_1 = \delta_1$, $\rho_2 = \delta_2$, $\delta_1, \delta_2 \to 0$.

Consider

$$\omega_1 = \frac{\delta_2\rho_3}{\delta_2\rho_3 + \delta_1\rho_3 + \delta_1\delta_2}, \quad \omega_2 = \frac{\delta_1\rho_3}{\delta_2\rho_3 + \delta_1\rho_3 + \delta_1\delta_2}, \quad \omega_3 = \frac{\delta_1\delta_2}{\delta_2\rho_3 + \delta_1\rho_3 + \delta_1\delta_2}.$$

By (3.4) we obtain

$$\widetilde{a} = \frac{a^*(\delta_2\rho_3 + \delta_1\rho_3) + \varepsilon_1\delta_2\rho_3 + \varepsilon_2\delta_1\rho_3 + \delta_1\delta_2 a_3}{(\delta_2\rho_3 + \delta_1\rho_3) + \delta_1\delta_2}.$$

It is easy to see that the last expression is continuous and tends to the limit a^*. Similarly, we obtain that

$$\widetilde{R} = (a^*, b^*, c^*) = \widetilde{R}^*.$$

(iii) Assume that all three members of the finite collection of triangular fuzzy numbers coincide with the representative:

$$\widetilde{R}_1 = \widetilde{R}_2 = \widetilde{R}_3 = \widetilde{R}^*.$$

Then the last equation can be rewritten as

$$a_1 = a^* + \varepsilon_1, \quad a_2 = a^* + \varepsilon_2, \quad a_3 = a^* + \varepsilon_3, \quad \ldots,$$
$$c_1 = c^* + \varepsilon_7, \quad c_2 = c^* + \varepsilon_8, \quad c_3 = c^* + \varepsilon_9;$$
$$\varepsilon_k \to 0, \quad k = \overline{1,9}.$$

Therefore,

$$\rho_1 = \delta_1, \quad \rho_2 = \delta_2, \quad \rho_3 = \delta_3, \quad \delta_1, \delta_2, \delta_3 \to 0.$$

Consider

$$\omega_1 = \frac{\delta_2\delta_3}{\delta_2\delta_3 + \delta_1\delta_3 + \delta_1\delta_2}, \quad \omega_2 = \frac{\delta_1\delta_3}{\delta_2\rho_3 + \delta_1\rho_3 + \delta_1\delta_2}, \quad \omega_3 = \frac{\delta_1\delta_2}{\delta_2\rho_3 + \delta_1\rho_3 + \delta_1\delta_2}.$$

By (3.4) we obtain

$$\widetilde{a} = \frac{a^*(\delta_2\delta_3 + \delta_1\delta_3 + \delta_1\delta_2) + \varepsilon_1\delta_2\delta_3 + \varepsilon_2\delta_1\delta_3 + \varepsilon_3\delta_1\delta_2}{\delta_2\delta_3 + \delta_1\delta_3 + \delta_1\delta_2}.$$

It is easy to see that the last expression is continuous and tends to the limit a^*. Similarly, we obtain that

$$\widetilde{R} = (a^*, b^*, c^*) = \widetilde{R}^*.$$

The above arguments can obviously be extended to any finite collection of triangular fuzzy numbers. Thus we have proved the validity of the following proposition.

Proposition 3.1. *For any finite collection of triangular fuzzy numbers* $\{\widetilde{R}_j\}$, $j = \overline{1,m}$, $m = 2, 3, \ldots,$ *the following holds*:

(a) $\widetilde{R} = \sum_{j=1}^{m} (\omega_j \odot \widetilde{R}_j)$ *is always continuous* (*here* ω_j *is given by* (3.3));

(b) *if there exists at least one* j *such that* $\rho(\widetilde{R}_j, \widetilde{R}^*) = 0$, *then* $\widetilde{R} = \widetilde{R}^*$.

Corollary 3.1. *For all* $t, j \in \{1, 2, \ldots, m\}$,

$$\widetilde{R}_t = \widetilde{R}_j \implies \widetilde{R} = \widetilde{R}^*.$$

Corollary 3.2. *If all estimates are identical, then* $\omega_j = 1/m$.

Proof. This assertion immediately follows from (3.4). □

Now we present the algorithm of our fuzzy aggregation approach.

Algorithm

Step 0. Initialization: There are the finite collection of triangular fuzzy numbers $\{\widetilde{R}_j\}$ and its increasing shuffling $\{\widetilde{R}'_j\}$, $j = \overline{1,m}$, $m = 2, 3, \ldots$. Denote the aggregation weight of the jth expert by ω_j and the final result by $\widetilde{R}$.

Step 1. Compute the representative $\widetilde{R}^*$ of $\{\widetilde{R}'_j\}$, $j = \overline{1,m}$, $m = 2, 3, \ldots$, by (2.11).

Step 2. Do Step 3 for $j = \overline{1,m}$.

Step 3. Compute $\Delta_j = \rho(\widetilde{R}^*, \widetilde{R}_j)$.

- If at least one $\Delta_j = 0$, then $\widetilde{R} = \widetilde{R}^*$;
- if $\Delta_j > 0$ for all j, then compute ω_j by (3.3) and obtain the final result by (3.4).

Upon obtaining the qualitative result of aggregation, its defuzzification into a linguistic variable must be carried out. This is done as follows (the notation is the same as in the above algorithm):

Step 4. Compute $\min \rho(\widetilde{R}, \widetilde{R}_j)$, $j = \overline{1,m}$.

- If the minimum is reached for only $\widetilde{R}_k$, $k \in \{1, 2, \ldots, m\}$ or several triangular fuzzy numbers that are equal to $\widetilde{R}_k$, then the resulting evaluation of group decision making is $v_k \in V$;
- if the minimum is reached for two different $\widetilde{R}_k$ and by (3.2) $\widetilde{R}_{k+1}$, $k \in \{1, 2, \ldots, m-1\}$, then the resulting evaluation of group decision making is equally close to both evaluations $v_k, v_{k+1} \in V$.

As a result, the manager (managers) receive a well-argumeted recommendation for making the final decision.

4. Illustrations

Let us give an example of the practical application of the introduced method. Assume that we have some new project (from an arbitrary area) and the formal method of evaluating decision-making alternatives is absent. Then the best way out is to involve experts in the process of group decision-making.

We introduce the following components of a quintuple $(X; V; \mu; [0,1]; \leq)$. The interval $X = [0, 10]$ is the universe. The scale of linguistic variables (3.1) is specified as

$$V = \Big\{ v_1 = \text{very bad (VB)},\ v_2 = \text{not very bad (NVB)},\\ v_3 = \text{problematic (Prb)},\ v_4 = \text{potentially beneficial (PB)},\ v_5 = \text{good (G)},\\ v_6 = \text{very good (VG)},\ v_7 = \text{to be realized immediately (TBRI)} \Big\}.$$

It is obvious that this scale of expert evaluations has a sufficiently wide evaluation range.

Here we use the metric based on the isotone evaluation $v(\widetilde{R}) = \sum_{i=1}^{3} a_i$, where $\widetilde{R} = (a_i)$. We have

$$\widetilde{\max}\{\widetilde{R}_1, \widetilde{R}_2\} = \big(\max\{a_1, b_1\}, \ldots, \max\{a_4, b_4\}\big), \quad \widetilde{\min}\{\widetilde{R}_1, \widetilde{R}_2\} = \big(\min\{a_1, b_1\}, \ldots, \min\{a_3, b_3\}\big).$$

It readily follows that

$$v(\widetilde{\max}\{\widetilde{R}_1, \widetilde{R}_2\}) + v(\widetilde{\min}\{\widetilde{R}_1, \widetilde{R}_2\}) = v(\widetilde{R}_1) + v(\widetilde{R}_2)$$

and

$$\widetilde{R}_1 \preceq \widetilde{R}_2 \implies v(\widetilde{R}_1) \leq v(\widetilde{R}_2).$$

So (2.4) is fulfilled. Hence the metric is determined as follows:

$$\rho(\widetilde{R}_1, \widetilde{R}_2) = v(\widetilde{\max}\{\widetilde{R}_1, \widetilde{R}_2\}) - v(\widetilde{\min}\{\widetilde{R}_1, \widetilde{R}_2\}) = \sum_{i=1}^{3} \max\{a_i, b_i\} - \sum_{i=1}^{3} \min\{a_i, b_i\} = \sum_{i=1}^{3} |a_i - b_i|.$$

Let the universe X be $[0;13]$. We determine the one-to-one correspondence between the sets of linguistic variables V and triangular fuzzy numbers (TFN) $\Psi(X)$ by representing the function in the tabular form:

Table 4.1.

Evaluation/TFN	VB	NVB	Prb	PB	G	VG	TBRI
$\widetilde{R}$	(0,0.1)	(1,2.5,4)	(3,4.5,6)	(5,6.5,8)	(7,8.5,10)	(9,10.5,12)	(11,13,13)

Assume that the group of five experts is organized to evaluate the project and each of them is requested to fill in the following table by marking his/her evaluation with an asterisk:

Table 4.2.

Evaluation/Expert No.	VB	NVB	Prb	PB	G	VG	TBRI
1					*		
2			*				
3				*			
4			*				
5				*			

As a result, using Tables 4.1 and 4.2 we obtain the following finite collection of triangular fuzzy numbers:

$$\widetilde{R}_1 = (7, 8.5, 10),\ \ \widetilde{R}_2 = (3, 4.5, 6),\ \ \widetilde{R}_3 = (5, 6.5, 8),\ \ \widetilde{R}_4 = (3, 4.5, 6),\ \ \widetilde{R}_5 = (5, 6.5, 8).$$

Now we perform calculations by our algorithm.

Step 0. Initialization:

$$\{\widetilde{R}_j\} = \big\{(7, 8.5, 10),\ (3, 4.5, 6),\ (5, 6.5, 8),\ (3, 4.5, 6),\ (5, 6.5, 8)\big\}$$

and its increasing shuffling

$$\{\widetilde{R}'_j\} = \big\{(3, 4.5, 6),\ (3, 4.5, 6),\ (5, 6.5, 8),\ (5, 6.5, 8),\ (7, 8.5, 10)\big\}, \quad j = \overline{1, 5}.$$

Denote the aggregation weight of the jth expert by ω_j and the final result by $\widetilde{R}$.

Step 1. By (2.11) compute the representative $\widetilde{R}^* = (4, 5.5, 7)$.

Step 2. Do Step 3 for $j = \overline{1, 5}$.

Step 3. Compute $\Delta_j = \rho(\widetilde{R}^*, \widetilde{R}_j)$: $\Delta_1 = 3$, $\Delta_2 = 3$, $\Delta_3 = 3$, $\Delta_4 = 3$, and $\Delta_5 = 9$. Since all $\Delta_j > 0$, by (3.1) we obtain $\omega_1 = \omega_2 = \omega_3 = \omega_4 = 3/13$, $\omega_5 = 1/13$, and by (3.2) we come to the final result

$$\widetilde{R} = (4.23,\ 5.731,\ 7.23).$$

Now we perform the defuzzification of the obtained result into a linguistic variable. This is done as explained above.

Compute $\min \rho(\widetilde{R}, \widetilde{R}_j)$, $j = \overline{1,5}$: $\rho(\widetilde{R}, \widetilde{R}_1) = 8.309$, $\rho(\widetilde{R}, \widetilde{R}_2) = 3.691$, $\rho(\widetilde{R}, \widetilde{R}_3) = \rho(\widetilde{R}, \widetilde{R}_5) = 2.309$.

We obtain that the minimum is reached for TNF $\widetilde{R}_3$ and $\widetilde{R}_5$, both of which correspond to the linguistic variable PB (potentially beneficial). As a result we conclude that the aggregated evaluation of the considered project by the quantitative scale is the closest one to the linguistic variable PB (potentially beneficial). In our opinion, this is well-argumented evidence in favor of making the final decision on the project.

5. Conclusion

This paper is the second part of the proposed methodology for the solution of frequently encountered control problems in situations, in which there is no previous experience and the initial information is incomplete and/or uncertain and/or weakly structured. In Part 1 (see [10]), we consider an approach to the processing of quantitative expert evaluations in the process of group decision-making under uncertainty. In Part 2, we propose an approach to processing qualitative expert evaluations in the process of group decision-making. The approach is based on the use of triangular fuzzy numbers.

In group decision-making the opinions of experts are expressed by linguistic variables like *very bad*, *not very bad*, *problematic*, *good*, and so on. The technique of conversion of expert quantitative opinions to triangular fuzzy numbers is considered. A simple method of expressing expert opinions by triangular fuzzy numbers is introduced.

The proposed method is based on the general metric that embraces the whole class of metrics on the set of triangular fuzzy numbers. A new approach is worked out to determine the degrees of experts' importance depending on the closeness of the expert evaluations to the representative of a finite collection of all triangular fuzzy evaluations. A special fuzzy aggregation operator is constructed for the realization of the approach. Also, a simple technique is proposed for the defuzzification of quantitative results into linguistic variables.

The proposed methodology is discussed in full detail and its algorithm is worked out. The advantage of the algorithm is its simplicity—for small and medium problems it can be realized using an ordinary calculator.

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