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ON PERIODIC MOTION AND BIFURCATIONS IN THREE-DIMENSIONAL NONLINEAR SYSTEMS

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We present geometric conditions for the existence of a closed trajectory with symmetry in three-dimensional nonlinear systems. A generator with quadratic nonlinearity and a Chua circuit are considered as examples.

1. Preliminary Results. Statement of the Problem

The oscillations and stability of nonlinear multidimensional systems are used in the problems of mechanics [1] and radiophysics [2]. The problem of existence of periodic solutions in three-dimensional autonomous systems is studied in [3] with the use of the principle of torus formulated earlier. In the present paper, we consider three-dimensional systems with certain symmetry conditions. This simplifies the solution of the problem, which is reduced to the problem of existence of closed integral curves. The dynamics of three-dimensional nonlinear systems is connected with bifurcation processes and the appearance of both periodic motions and strange attractors (see [2, 4–6] and the references therein).

The solution of the posed problem is based on the following results:

(i) a procedure of detection of the bifurcation processes;

(ii) a principle of symmetry for two-dimensional systems;

(iii) a principle of comparison used to confirm the instability of solutions of the original system in the neighborhood of the origin.

The bifurcations leading to changes in the qualitative behavior of the system can be studied by using the variational equations [6]. In the present paper, the variational equations differ from the known equations by the dependence of the coefficients of equations not on time but on the partial solutions of the system of differential equations [7].

Consider a system

$$\frac{dx}{dt} = F(x, p), \quad x(t) \in R^n, \quad p \in R^m, \tag{1}$$

where $n = 3$, $F(x, p)$ is a smooth function and R^m is the space of parameters. We introduce a small deviation in the neighborhood of partial solutions $\bar{x}_i$, $i = 1, 2, \ldots, n$, namely, $\delta x_i = x_i(t) - \bar{x}_i(t)$, and consider δx_i as new coordinates. The linear system corresponding to system (1) in the coordinates δx_i

$$\frac{d\delta x}{dt} = \mathcal{A}(\bar{x})\delta x, \quad \delta x \in R^n,$$

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where $\mathcal{A}(\bar{x}) = \partial F/\partial x|_{x=\bar{x}}$, is called a *variational system.* By analyzing the roots of the characteristic equations of the matrix $\mathcal{A}(\bar{x})$, we can construct, e.g., the separatrices and obtain estimates for the parameters of an orbitally stable system [7].

In analytically given systems of ordinary differential equations of the form $dx/dt = \mathcal{A}(t)x$, we introduce characteristic exponents of the nontrivial solutions as follows:

$$\Lambda_j = \overline{\lim}_{t\to\infty}[t^{-1}\ln\|x_j(t)\|], \quad j = 1, 2, \ldots, n,$$

where $x_j(t)$ is the j th fundamental solution of the system of differential equations and $\|\cdot\|$ is the Euclidean norm. The numbers Λ_j are called generalized characteristic exponents of system (1). For the variational system used to describe the evolution of perturbations δx near a partial solution $\bar{x}(t)$ of the nonlinear system (1), the family Λ_j is called the collection of characteristic Lyapunov exponents of the partial solution $\bar{x}(t)$ (or the phase trajectory).

The existence of periodic motions in two-dimensional systems is established with the use of the symmetry conditions and presented in [8]. The symmetry principle is generalized to the principle of skew symmetry in [7]. As preliminary results, we present some information about the symmetry principle [8]. We write a two-dimensional system in the form

$$\frac{dx_1}{dt} = F_1(x), \quad \frac{dx_2}{dt} = F_2(x),$$

where $x_1, x_2 \in R$, $F_1 \in C(R^2, R)$, $F_2 \in C(R^2, R)$, and $F_i(0,0) = 0$, $i = 1, 2$. The geometric symmetry principle, which can be regarded as basic in deducing the conditions of closure of the phase trajectory relative to the center, can be formulated as follows [8]:

In a two-dimensional system, a symmetry of the trajectory exists if the function $F_1(x)$ is even in x_1 and the function $F_2(x)$ is odd in x_1, i.e.,

$$F_1(-x_1, x_2) = F_1(x_1, x_2) \quad \text{and} \quad F_2(-x_1, x_2) = -F_2(x_1, x_2).$$

This statement is based on the fact that the Ox_2-axis in the plane Ox_1x_2 is the axis of symmetry and any integral curve to the left of the Ox_2-axis is the mirror reflection of a curve to the right of this axis. According to the symmetry principle, we can conclude that *the symmetry of the trajectory exists in the system if the function $F_2(x)$ is even with respect to x_2 and the function $F_1(x)$ is odd with respect to x_2*, i.e.,

$$F_2(x_1, -x_2) = F_2(x_1, x_2) \quad \text{and} \quad F_1(x_1, -x_2) = -F_1(x_1, x_2).$$

It suffices to assume that an integral curve starting on the Ox_1-axis returns to the Ox_1-axis after its extension. Here, Ox_1 plays the role of the axis of symmetry.

The conditions of skew symmetry for a two-dimensional system [7] take the form

$$F_1(-x_1, x_2) = -F_1(x_1, -x_2) \quad \text{and} \quad F_2(-x_1, x_2) = -F_2(x_1, -x_2). \tag{2}$$

By using the conditions of skew symmetry, one can establish the existence of a closed integral curve for a nonlinear oscillator with unstable singular point (unstable focus). In the presence of stable focus, the trajectory with skew symmetry comes to the singular point. In both cases, conditions (2) are satisfied. If a two-dimensional nonlinear system of the form (1) has a saddle at the origin and two skew-symmetric singular stable points, then conditions (2) guarantee the existence of two branches of skew-symmetric trajectories starting from the origin. A more detailed application of the principle of skew symmetry is considered in the Chua problem described in what follows.

The theorems on symmetry and skew symmetry for three-dimensional systems are proved under the assumption of instability of the analyzed three-dimensional system. This result can be obtained by using, e.g., the method of comparison [9–11]. According to the method of comparison, we write the equations of comparison (Ważewski-type equations) with the property of quasimonotonicity. The main sources and results of the investigations of stability of monotone systems can be found in [9, 10] (see also the survey [11]).

The key statement of the method of comparison can be formulated as follows: if, for the analyzed system, there exists a Lyapunov function satisfying the appropriate conditions, then various dynamical properties of the original system follow from the corresponding dynamical properties of the system of comparison [11].

Consider a system

$$\frac{d\vartheta_j}{dt} = q_j(\vartheta_1, \ldots, \vartheta_k), \quad j = 1, \ldots, k, \tag{3}$$

under the following assumptions:

1. System (3) is a Ważewski system, i.e., the components of the vector function $q(\vartheta)$ are quasimonotonically increasing functions. For a function $q(\vartheta)$ to be quasimonotonically increasing, it is necessary and sufficient that the conditions $\partial q_j / \partial \vartheta_i \geq 0$ be satisfied for $j \neq i$.

2. The right-hand side of system (3) is continuous and the solution of the Cauchy problem is locally unique for any $\vartheta_0 \in \mathcal{R}^k$.

The theorem on instability of the Ważewski system in the cone $\mathcal{K}$ (see [9, 11]). If Assumptions 1 and 2 are true for the Ważewski system, then there exists a sequence $\vartheta_m \in \mathcal{K},\ \vartheta_m \to 0$ as $m \to \infty,$ such that the inequalities

$$q_j(\vartheta_m) \geq 0, \quad j = 1, \ldots, k, \tag{4}$$

are true for any m and, in addition, the inequality is strict for at least one $j,$ and moreover, there exists a neighborhood of the origin V such that the vector field is not equal to zero in the set $\mathcal{K}_{\vartheta_m} \bigcap V,$ then the trivial solution of system (3) is unstable in the cone.

We restrict ourselves to the application of the theorem of comparison under Assumptions 1 and 2 in the case where inequality (4) is true. To establish the existence of periodic solutions, it is necessary that the trajectory starting from the origin be closed due to certain properties of symmetry.

2. Theorem on the Symmetry Principle for Three-Dimensional Systems

According to the symmetry principle for three-dimensional systems, we determine a coordinate plane such that the analyzed three-dimensional integral curve is projected onto this plane in the form of a closed symmetric curve. In the other two coordinate planes, the process is stable and may have a symmetry.

Consider a system of three nonlinear differential equations

$$\frac{dx}{dt} = -xz + y, \quad \frac{dy}{dt} = -x, \quad \frac{dz}{dt} = -b\,(z + m + f(x)), \tag{5}$$

where m and b are positive parameters. Assume that the function $f(x)$ is defined, continuous, and satisfies the condition of uniqueness of solutions at every point x. Let $f(x)$ be a function twice continuously differentiable in the neighborhood of the origin. Assume that a closed trajectory lies on the surface and has certain symmetry properties. It is possible to find a plane onto which the trajectory is projected in the form of a closed curve. The closed trajectory has certain projections onto the other two planes. The last two projections onto the coordinate planes may have symmetry but not necessarily closures. One more assumption is connected with the instability of system (5) in the neighborhood of zero. Since the geometric symmetry and stability of two-dimensional projections are possible, we can represent the above-mentioned results in the form of the following theorem:

Theorem 1. *Consider system (5) under the following assumptions:*

(i) system (5) has an unstable solution in the neighborhood of the origin;

(ii) the motion of system (5) in the plane Oxz *is described by the system of equations*

$$\frac{dx}{dt} = -xz, \quad \frac{dz}{dt} = -b(z + m + f(x)), \tag{6}$$

which has two stable equilibrium positions.

Then a closed integral curve exists in the three-dimensional system (5).

Proof. The motion specified by Eqs. (5) in the plane Oxy is described by the system of equations

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x, \tag{7}$$

which has a symmetric closed trajectory: a singular point (center); the trajectory in the plane is symmetric about the Ox- and Oy-axes. The motion of Eqs. (5) in the plane Oyz is described by the system

$$\frac{dy}{dt} = 0, \quad \frac{dz}{dt} = -b(z + m). \tag{8}$$

The singular point of system (8) has the exponents of the characteristic equation $\lambda_1 = 0$ and $\lambda_2 = -b$. Assume that the initial conditions of system (5) disturb the formation of the closed symmetric trajectory of system (7). Then system (6) is perturbed and we observe the formation of trajectories approaching the singular points with coordinates $(\pm x_0, z_0)$. The oscillations of system (5) take the form of a steady-state process characterized by an integral curve with symmetry in the space $Oxyz$. Moreover, the projection of system (8) onto the plane Oyz does not affect this steady-state process.

The theorem is proved.

The proposed theorem has a geometric character. We consider the mechanism of formation of a periodic three-dimensional curve with symmetry. In analyzing the motions in coordinate planes, the main problem is to find a plane Oxy in which we get a closed curve symmetric about two axes. The other two coordinate planes stabilize the qualitative picture and symmetry in the case of stability of the singular points of systems (6) and (8).

Application of Theorem 1. *On the Closure of the Trajectory of Generator with Quadratic Nonlinearity.* Consider a generator with quadratic inertial nonlinearity in the form of the following dimensionless system presented in [2]:

$$\frac{dX}{dt} = mX - XZ + Y, \quad \frac{dY}{dt} = -X, \quad \frac{dZ}{dt} = -b(Z - X^2). \tag{9}$$

System (9) has one singular point $O(0,0,0)$. We introduce small deviations δX, δY, and δZ from the partial solutions $\bar{X}(t)$, $\bar{Y}(t)$, and $\bar{Z}(t)$ of system (9) and write the variational equations

$$\frac{d\delta X}{dt} = (m - \bar{Z})\delta X + \delta Y - \bar{X}\delta Z, \quad \frac{d\delta Y}{dt} = -\delta X, \quad \frac{d\delta Z}{dt} = -b(\delta Z - 2\bar{X}\delta X). \tag{10}$$

The characteristic equation of system (10) has the form

$$\lambda^3 + \lambda^2(b - m + \bar{Z}) + \lambda(b(-m + \bar{Z} + 2\bar{X}^2) + 1) + b = 0. \tag{11}$$

The roots corresponding to the singular point $O(0,0,0)$ of system (9) are determined from the equation

$$\lambda^3 + \lambda^2(b - m) + \lambda(-bm + 1) + b = 0$$

or

$$(\lambda + b)(\lambda^2 - \lambda m + 1) = 0.$$

They are equal to $\lambda_{1,2} = m/2 \pm \sqrt{(m/2)^2 - 1}$ and $\lambda_3 = -b$. The point O is a saddle-focus for $(m/2)^2 < 1$.

We show that there exist points on the OZ-axis at which the characteristic equation (10) is split into two equations. At the point $A(X = 0, Y = 0, Z = m + b)$, the characteristic equation has the form

$$(\lambda + b)(\lambda^2 + b\lambda + 1) = 0.$$

The characteristic exponents are the following: $\lambda_{1,2} = -b/2 \pm \sqrt{(b/2)^2 - 1}$ and $\lambda_3 = -b$. At the point $C(X = 0, Y = 0, Z = m)$, the characteristic equation has the form

$$(\lambda + b)(\lambda^2 + 1) = 0$$

and the characteristic exponents at this point are the following: $\lambda_{1,2} = \pm i$ and $\lambda_3 = -b$. At the point $D(X = 0, Y = 0, Z = -m)$, the characteristic equation has the form

$$(\lambda + b)(\lambda^2 - 2m\lambda + 1) = 0$$

and the characteristic exponents at this point are the following: $\lambda_{1,2} = m \pm \sqrt{m^2 - 1}$ and $\lambda_3 = -b$.

Thus, at the point C, the real parts of two complex conjugate roots change sign. The characteristic equation (11) contains the partial solutions $\bar{X}$ and $\bar{Z}$. The plane OXZ is a coordinate plane that defines bifurcation processes of system (9).

To prove the instability of system (9), we use the method of comparison. We introduce the functions $V_1 = X^2/2$, $V_2 = Y^2/2$, and $V_3 = X^2/2 + Z^2/2$. By virtue of system (9), the derivatives of the functions V_j, $j = 1, 2, 3$, admit the estimates

$$\frac{dV_1}{dt} = mX^2 - X^2 Z + XY \le mX^2 + \frac{Y^2}{2} + \frac{X^2}{2} + \frac{Z^2}{2} + \frac{X^4}{2},$$

$$\frac{dV_2}{dt} = -XY \le \frac{X^2}{2} + \frac{Y^2}{2},$$

$$\frac{dV_3}{dt} = bX^2Z + mX^2 - X^2Z + XY - bZ^2 \le \left(m + \frac{b}{2}\right)\frac{X^2}{2} + \frac{Y^2}{2} + \frac{(1-b)(X^2+Z^2)}{2} + \frac{(1+b)X^4}{2}.$$

Here, the derivatives of two variables are replaced by the sum

$$\pm bX^2Z \le \frac{bX^4}{2} + \frac{bZ^2}{2}, \quad \pm XY \le \frac{X^2}{2} + \frac{Y^2}{2}, \quad \pm X^2Z \le \frac{X^4}{2} + \frac{Z^2}{2}.$$

We write the system of comparison

$$\frac{d\vartheta_1}{dt} = 2m\vartheta_1 + \vartheta_2 + \vartheta_3 + 2\vartheta_1^2,$$
$$\frac{d\vartheta_2}{dt} = \vartheta_1 + \vartheta_2, \tag{12}$$
$$\frac{d\vartheta_3}{dt} = (2m+b)\vartheta_1 + \vartheta_2 + (1-b)\vartheta_3 + 2(1+b)\vartheta_1^2.$$

Under the condition $b \le 1$, the system of comparison (12) shows that a trajectory moves from zero. We give the following values of parameters: $(m, b) = (1; 0.2)$.

We shift the origin of coordinates to the point C and introduce the new coordinate system $Cxyz$, where $x = X$, $y = Y$, and $z = Z - m$. Then system (9) takes the form of system (5)

$$\frac{dx}{dt} = -xz + y, \quad \frac{dy}{dt} = -x, \quad \frac{dz}{dt} = -b(z + m - x^2). \tag{13}$$

Consider the motion of the trajectory of system (13) on the plane Cxz defined by the system of equations

$$\frac{dx}{dt} = -xz, \quad \frac{dz}{dt} = -b(z + m - x^2). \tag{14}$$

Let us determine singular points of system (14). We set $z = 0$. System (14) has two singular points with coordinates $x = \pm\sqrt{m}$, $z = 0$. The singular points $C_1(\sqrt{m}, 0)$ and $C_2(-\sqrt{m}, 0)$ have the characteristic exponents $\lambda_{1,2} = -\frac{b}{2}(1 \pm \sqrt{1 - 8m/b})$. The appearance of a closed trajectory of system (13) is caused by an initial perturbation with respect to the variable x leading to a perturbation of the variable y. Since all conditions of Theorem 1 are satisfied, there exists a closed curve of system (13).

The three-dimensional image of the trajectory of system (13) and its projection onto the plane Cxy are shown in Figs. 1a and b, respectively, for the initial perturbation $x(0) = 0.01$.

Consider the second procedure of finding the singular point of Eq. (14). We set $x = 0$. Then the singular point has the coordinates $x = 0$, $z = -m$. In the case of initial perturbation solely of the variable z, a closed trajectory of system (5) does not appear because the projection of system (13) onto the plane Cxy is a point due to the absence of perturbation of the system with respect to the variables x and y.

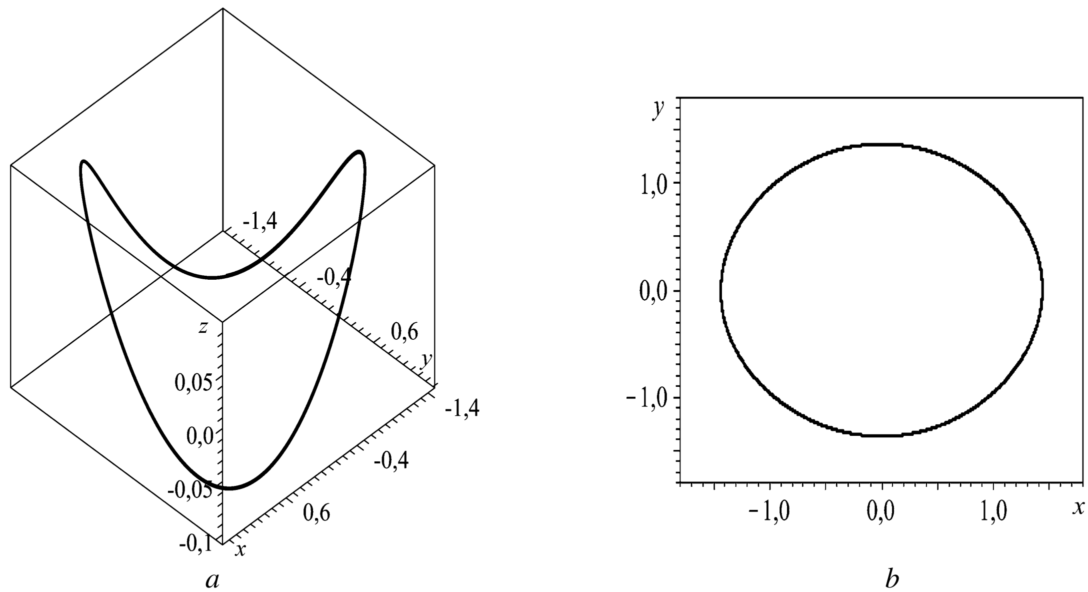


Fig. 1

3. Systems with Skew Symmetry

For a plane (two-dimensional) system, the principle of skew symmetry is discussed in [7]. A nonlinear system of the form (1),

$$\frac{dx_1}{dt} = F_1(x_1, x_2), \qquad \frac{dx_2}{dt} = F_2(x_1, x_2)$$

has a skew-symmetric trajectory in the plane Ox_1x_2 if conditions (2) are satisfied.

For the three-dimensional system (1), we make the following assumptions:

Assumption 1. *System (1) has three singular points. The singular point $O(0,0,0)$ is a saddle-focus with zero saddle value.*

Assumption 2. *The right-hand side of system (1) satisfies the skew-symmetry conditions (10) in three coordinate planes Ox_1x_2, Ox_1x_3, and Ox_2x_3.*

Assumption 3. *In one coordinate plane, a system of the form (1) has a circular dissipative curve; in the other two planes, the system has nonzero stable singular points.*

Theorem 2. *Let Propositions 1–3 be true for the differential system (1). Then, in the neighborhood of three singular points of system (1), there exists a closed integral curve.*

Proof. The physical aspect of the proof is given in [12]. It is shown in [12] that if the trajectory of a conservative system satisfies the symmetry condition [skew-symmetry condition of the form (2)], then there exists a closed trajectory. Conditions of the form (2) must be satisfied for two-dimensional systems on each coordinate plane. On one coordinate plane, a trajectory is circular. For the appearance of a limit cycle, a condition for leaving the neighborhood of zero by the trajectory must be satisfied. This is realized due to the presence of stable singular points on coordinate planes. The saddle-focus point $O(0,0,0)$ with zero saddle value takes part in the formation of skew-symmetry of the trajectory. Conditions of the form (2) lead not only to closure but also to skew-symmetry of projections.

The theorem is proved.

Application of Theorem 2. *On the Closure of Trajectory and Bifurcation Processes in the Chua System.* Consider the system of Chua nonlinear differential equations [6, 13, 14]

$$\frac{dx}{dt} = \alpha(ax - bx^3 + y), \quad \frac{dy}{dt} = x - y + z, \quad \frac{dz}{dt} = -\beta y, \tag{15}$$

where a, b, α, and β are positive parameters. We prove the existence of a skew-symmetric limit cycle in system (15). System (15) has three equilibrium states: a singular point $O(0,0,0)$ and singular points $A(x_A = \sqrt{a/b}$, $y_A = 0$, $z_A = -\sqrt{a/b}\,)$ and $B(x_B = -\sqrt{a/b}$, $y_B = 0$, $z_B = \sqrt{a/b}\,)$. Introducing small deviations δx, δy, and δz from the partial solutions $\bar{x}$, $\bar{y}$, and $\bar{z}$ of system (15), we write the variational equations

$$\frac{d\delta x}{dt} = \alpha(a\delta x - 3b\bar{x}^2\delta x + \delta y), \quad \frac{d\delta y}{dt} = \delta x - \delta y + \delta z, \quad \frac{d\delta z}{dt} = -\beta\delta y.$$

The variational system has the characteristic equation

$$\lambda^3 + \lambda^2(1 + \alpha(-a + 3b\bar{x}^2)) + \lambda(\beta - \alpha(1 + a - 3b\bar{x}^2)) + \alpha\beta(-a + 3b\bar{x}^2) = 0. \tag{16}$$

The characteristic equation (16) depends only on the partial solution $\bar{x}$. The characteristic exponents of the point O are defined by the equation

$$\lambda^3 + \lambda^2(1 - \alpha a) + \lambda(\beta - \alpha(1 + a)) - \alpha\beta a = 0.$$

Consider the motion relative to the singular point O. On the plane Oxy, the trajectory of system (15) has a curve defined by the equations

$$\frac{dx}{dt} = \alpha(ax - bx^3 + y), \quad \frac{dy}{dt} = x - y. \tag{17}$$

System (17) has the following singular points:

$$O_1(0,0), \quad E(\sqrt{(a+1)/b}, \sqrt{(a+1)/b}), \quad F(-\sqrt{(a+1)/b}, -\sqrt{(a+1)/b}).$$

For system (17), the variational equations

$$\frac{d\delta x}{dt} = \alpha(a\delta x - 3b\bar{x}^2\delta x + \delta y), \quad \frac{d\delta y}{dt} = \delta x - \delta y$$

have the characteristic equation

$$(\lambda + 1)(\lambda - \alpha(a - 3b\bar{x}^2)) - \alpha = 0. \tag{18}$$

The singular point O_1 has the characteristic exponents

$$\lambda_{1,2} = \frac{-1 + \alpha a}{2} \pm \sqrt{\left(\frac{-1 + \alpha a}{2}\right)^2 + \alpha(a + 1)}.$$

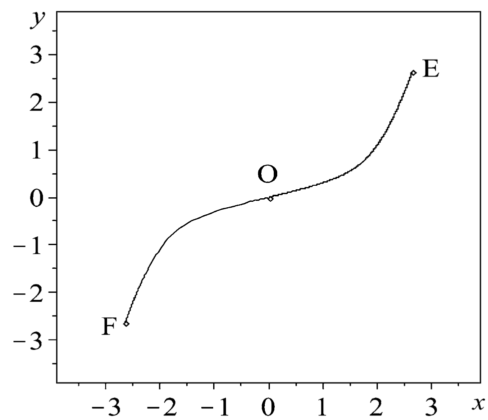


Fig. 2

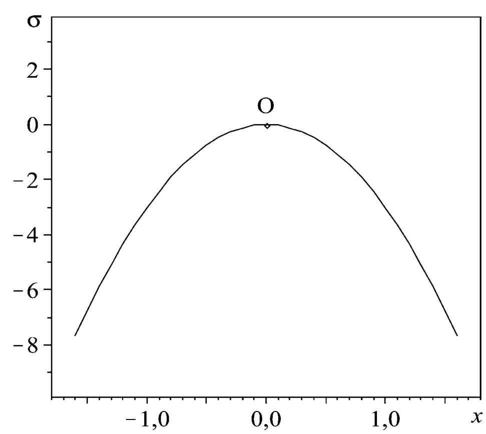


Fig. 3

We choose parameters so that

$$\alpha a = 1. \tag{19}$$

In this case, $\lambda_{1,2} = \pm\sqrt{1+\alpha}$ and the point O_1 is a saddle. By substituting the coordinate x of the points E and F into the characteristic equation (18), we obtain the characteristic exponents of the points E and F

$$\lambda_{1,2} = \frac{-3(1+\alpha)}{2} \pm \sqrt{\frac{9(1+\alpha)^2}{4} - 2(1+\alpha)}. \tag{20}$$

In the plane Oyz, the motion of an image point is described by the equations

$$\frac{dy}{dt} = -y + z, \quad \frac{dz}{dt} = -\beta y. \tag{21}$$

System (21) describes a linear dissipative oscillator

$$\ddot{z} + \dot{z} + \beta z = 0$$

and satisfies conditions (2). The singular point O_2 of system (21) is a stable focus. In the plane Oxz, the motion of the image point is described by the equations

$$\frac{dx}{dt} = \alpha(ax - bx^3), \quad \frac{dz}{dt} = 0.$$

The singular point O_3 has the characteristic exponents $\lambda_1 = \alpha a$ and $\lambda_2 = 0$. The singular points G and H have the coordinates $x = \pm\sqrt{a/b}$, $z = 0$. The points G and H have the characteristic exponents $\lambda_1 < 0$ and $\lambda_2 = 0$. The equations formally satisfy conditions (2).

We take the following values of parameters:

$$(a, b, \alpha, \beta) = (1/6, 1/6, 6, 7). \tag{22}$$

In choosing the values of parameters (22), we take into account the following conditions:

(a) condition (19);

(b) the singular points E and F of system (17) are stable nodes [according to relation(20)]; the points E and F are depicted in Fig. 2;

(c) according to the characteristic equation (16), at the saddle-focus point O, the saddle value is equal to zero, $\sigma = 2\mathrm{Re}\,\lambda_{1,2} + \lambda_3 = 0$.

The initial values are specified according to the following estimates:

$$|x(0)| > |x_A|, \quad |y(0)| \geq 0, \quad |z(0)| > |z_A|. \tag{23}$$

Under these conditions, the curve is closed relative to the singular points O, A, and B. The choice of the initial conditions (23) can be numerically corrected with regard for inequalities (23).

Consider the range of values of the parameter β,

$$7 < \beta < 10.1. \tag{24}$$

The other parameters are specified by (22). The bifurcation process in system (15) is controlled by a single variable x. The curve $\sigma(x)$ plotted according to the characteristic equation (16) specifies the dependence of the saddle value on the coordinate $\sigma(x)$. Thus, at the saddle-focus points A and B, the saddle value is negative, i.e.,

$$\sigma = 2\,\mathrm{Re}\,\lambda_{1,2} + \lambda_3 < 0$$

within the range (24) of the parameter β. The saddle value is negative at all points except the point O (see Fig. 3; $\beta = 8.7$).

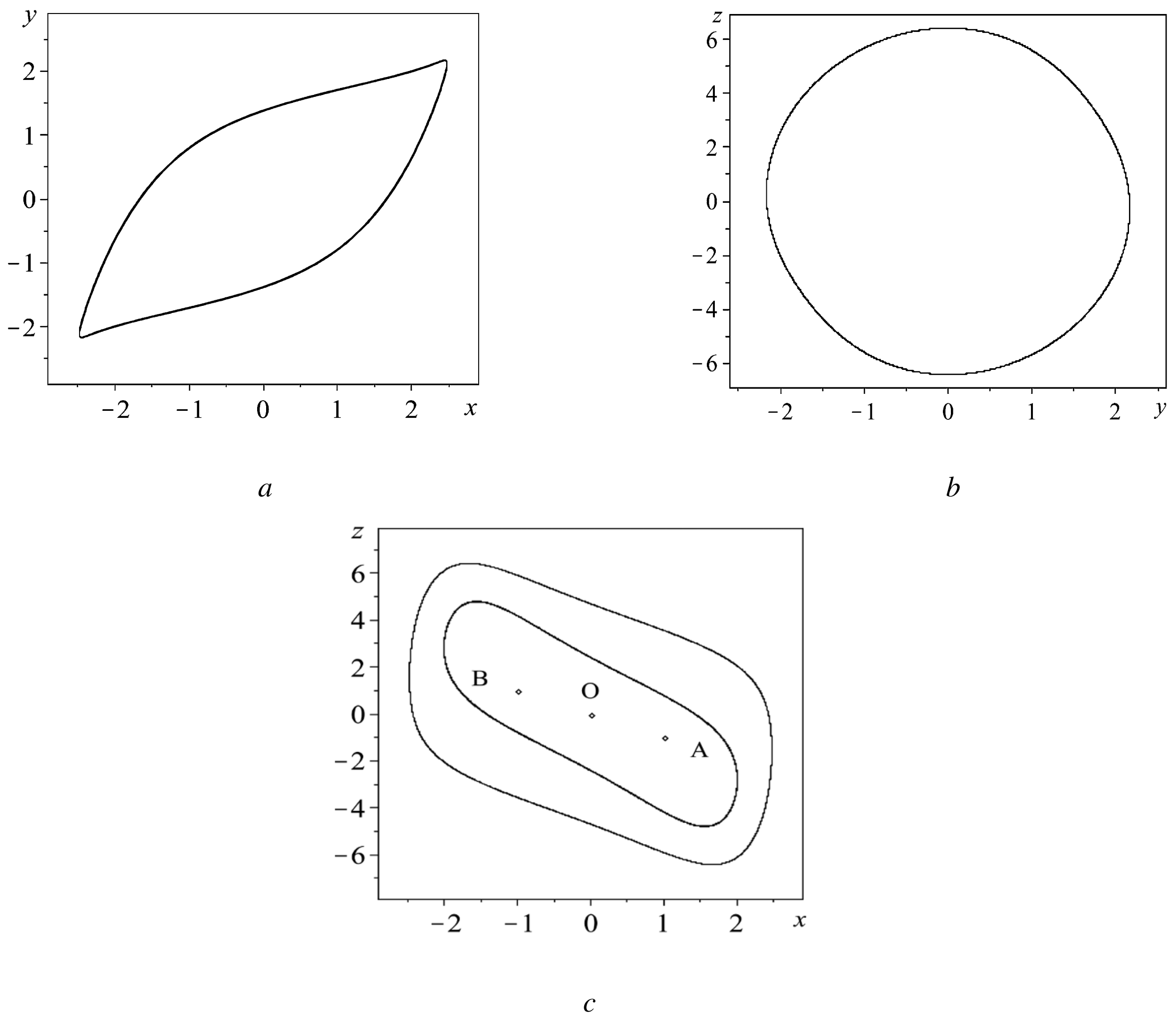


Fig. 4

Assume that the initial perturbations satisfy conditions of the form (23). The solutions of the Chua system form closed skew-symmetric integral curves within the range of values (24) of the parameter β. In Fig. 4, we present three closed curves of system (15) in the projections onto the coordinate planes. In the plane section xz, the outer closed curve is plotted for $\beta = 7$ and the initial perturbations $x(0) = -1.7$, $y(0) = 0.2$, and $z(0) = -1.7$. The inner curve is plotted for $\beta = 10.1$ and the initial perturbations $x(0) = -2.9$, $y(0) = 0.8$, and $z(0) = -2.9$. At the saddle-focus point O, the saddle value is equal to zero, i.e., $\sigma = 2\,\mathrm{Re}\,\lambda_{1,2} + \lambda_3 = 0$ within the range of values (24) of the parameter β. The values of the other parameters are given by relation (22).

We now consider the solutions of the Chua system caused by the singular points A and B. The points A and B are skew symmetric. We associate the point A with a coordinate system $Avyw$ and write equations of motion in new coordinates,

$$\frac{dv}{dt} = \alpha\left(-2av - bv^2\left(3\sqrt{\frac{a}{b}} + v\right) + y\right), \quad \frac{dy}{dt} = v - y + w, \quad \frac{dw}{dt} = -\beta y, \tag{25}$$

where $v = x - \sqrt{a/b}$ and $w = z + \sqrt{a/b}$. The points A and B may form closed curves excluding the point O. *The parameter β must guarantee that the trajectories closed with respect to A and B are disjoint.* Consider the process of motion relative to one singular point. We introduce small deviations δv, δy, and δw from the

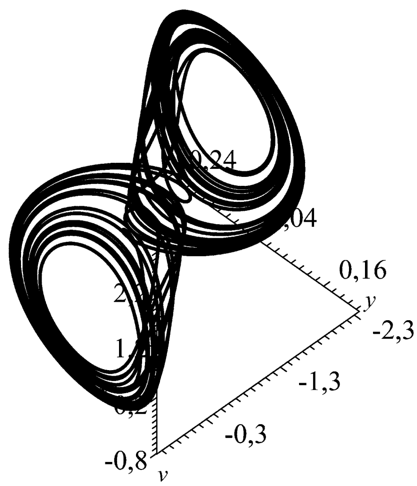


Fig. 5

partial solutions $\bar{v}$, $\bar{y}$, and $\bar{w}$ of system (25) and write the following variational equations:

$$\frac{d\delta v}{dt} = \alpha\left(-2a\delta v - 6b\sqrt{\frac{a}{b}}\bar{v}\delta v - 3b\bar{v}^2\delta v + \delta y\right), \quad \frac{d\delta y}{dt} = \delta v - \delta y + \delta w, \quad \frac{d\delta w}{dt} = -\beta\delta y. \tag{26}$$

System (26) has the characteristic equation

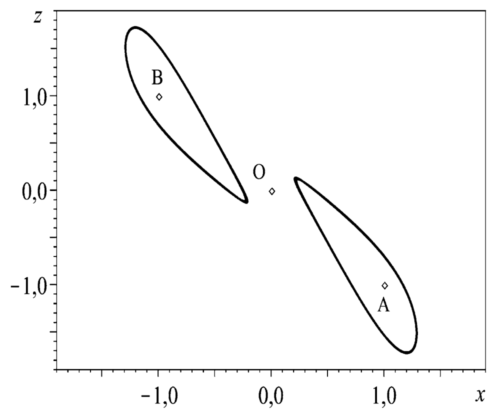
$$\lambda^3 + \lambda^2\left(1 + \alpha\left(2a + 6b\sqrt{\frac{a}{b}}\bar{v} + 3b\bar{v}^2\right)\right) + \lambda\left(\alpha\left(2a + 6b\sqrt{\frac{a}{b}}\bar{v} + 3b\bar{v}^2 - 1\right) + \beta\right) + \alpha\beta\left(2a + 6b\sqrt{\frac{a}{b}}\bar{v} + 3b\bar{v}^2\right) = 0.$$

In the coordinate system $Avyw$, the singular point O has the coordinates $v_O = -\sqrt{a/b}$, $y_O = 0$, $w_O = \sqrt{a/b}$. For the initial perturbations

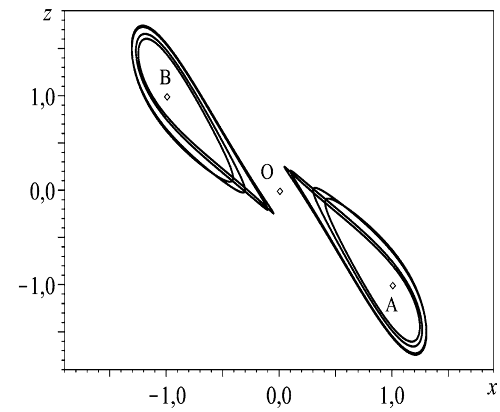
$$|v(0)| < \sqrt{a/b}, \quad |y(0)| \geq 0, \quad |w_O(0)| < \sqrt{a/b}, \tag{27}$$

the motion under the influence of the singular point A (or B) becomes predominant in the system. The estimates of the initial conditions (23) can be numerically corrected. We introduce initial perturbations of the form (27) taking into account the influence of the singular points A and B for the values of the parameter β (24).

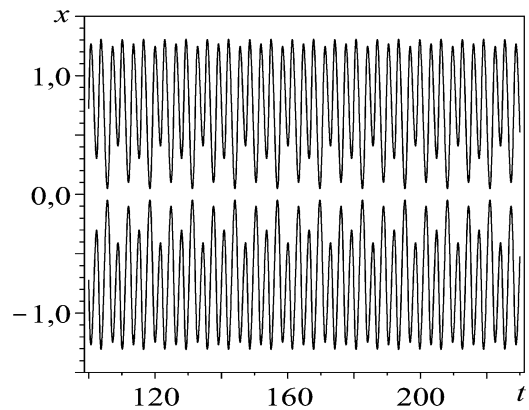
For $\beta < 8.3$ from range (24), the Chua system possesses an integral curve in the neighborhood of the point A passing into the neighborhood of the point B. The transition from the neighborhood of the point A into the neighborhood of the point B means orbital instability. In the motion of the image point in the neighborhood of the point A (or B), all points of the trajectory are saddle-focus points with negative saddle values. As the image point passes into the neighborhood of the point B, the signature of the spectrum of characteristic Lyapunov exponents becomes positive. For $\beta = 8.3$, a fragment of this process for the initial perturbations $v(0) = 0.2$, $y(0) = 0$, and $w(0) = 0.2$ is shown in Fig. 5.



a



b



c

Fig. 6

A subsequent increase in the parameter β is accompanied by the appearance of two limit cycles. The transition to the limit cycles passes through a bifurcation of multiple increase in the period. The two limit cycles in the coordinate system $Oxyz$ for the initial values $x(0) = 0.2$, $y(0) = 0$, and $z(0) = -0.2$ (a cycle relative to the point A) and $x(0) = -0.2$, $y(0) = 0$, $z(0) = 0.2$ (a cycle relative the point B) and the parameter $\beta = 9$ are depicted in Fig. 6a. The two limit cycles with multiple (quadruple) period ($\beta = 8.7$) in the coordinate plane Oxz are presented in Fig. 6b. Their time realizations $x(t)$ are shown in Fig. 6c.

In radiophysics, the term "multistability" indicates the coexistence of several attractors in the phase space. This is caused by the initial perturbations. In the Chua system, multistability is caused by the influence of a certain singular point on the behavior of the trajectory. This influence is determined by the estimates of different initial conditions (23) and (27). Within the range $8.3 < \beta < 9$, the limit cycles relative to the points A and B suffer bifurcations of multiple increase in the period. In Fig. 6, the trajectory of the limit cycle hits a point where $x \approx 0$. The curve abruptly changes the direction of motion. The phenomenon of a multiple increase in the period is caused by the *nonuniformity of motion of the image point*. This occurs in the immediate vicinity of the point O. In this case, the saddle value is close to zero.

The formation of two limit cycles without symmetry (see Fig. 6) is connected with the geometric theorem according to which all points of the trajectory are attractive saddle-focus points with negative saddle values.

4. Conclusions

The present paper is devoted to the classification of physical objects generating multidimensional attractors. We prove two theorems on the existence of limit cycles with symmetry in three-dimensional systems.

The second theorem is connected with the skew symmetry of projections onto the coordinate planes generating the closure of the three-dimensional trajectory. As an example, we consider the mathematical Chua model. We establish the conditions under which the Chua system generates periodic signals. In the case of a single limit cycle (see Fig. 4), the skew-symmetry principle is adapted for three-dimensional systems. This adaptation is connected with Theorem 2. The presented values of parameters (22) form a point in the space of parameters. In a small neighborhood of this point, there exists a set of points also associated with the limit cycles. We establish the cause of the appearance of multiple periods of the limit cycles. The mechanism of multistability is described.

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