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ON THE CLASS NUMBERS OF ALGEBRAIC NUMBER FIELDS

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Let K be a number field of degree n over $\mathbb{Q}$ and let d, h, and R be the absolute values of the discriminant, class number, and regulator of K, respectively. It is known that if K contains no quadratic subfield, then

$$h\,R \gg \frac{d^{1/2}}{\log d},$$

where the implied constant depends only on n. In Theorem 1, this lower estimate is improved for pure cubic fields.

Consider the family $\mathcal{K}_n$, where $K \in \mathcal{K}_n$ if K is a totally real number field of degree n whose normal closure has the symmetric group S_n as its Galois group. In Theorem 2, it is proved that for a fixed $n \geq 2$, there are infinitely many $K \in \mathcal{K}_n$ with

$$h \gg d^{1/2}(\log\log d)^{n-1}/(\log d)^n,$$

where the implied constant depends only on n.

This somewhat improves the analogous result $h \gg d^{1/2}/(\log d)^n$ of W. Duke [MR 1966783 (2004g:11103)]. *Bibliography: 16 titles.*

1

Let K be an algebraic number field of degree $n(K) = [K : \mathbb{Q}]$, $n(K) =: n \geq 2$. The study of $h(K) =: h$, the class number of the field K, is based on investigating the Dedekind zeta function of the field K,

$$\zeta_K(s) = \sum_{\mathfrak{a}} (N\mathfrak{a})^{-s} \quad (\sigma > 1),$$

where the summation is carried out over all integral nonzero ideals $\mathfrak{a}$ in K; $s = \sigma + it$. It is known that $\zeta_K(s)$ is a meromorphic function over the entire complex plane with a single simple pole at $s = 1$ with residue, say, $\varkappa_K$. The function $\zeta_K(s)$ satisfies a Riemann type functional equation $\ll s \to 1 - s \gg$. Every field K has r_1 real and $2r_2$ imaginary conjugate fields (consequently, $r_1 + 2r_2 = n$).

An important role is played by the Dedekind formula

$$\varkappa_K = 2^{r_1}(2\pi)^{r_2}\frac{h(K)R(K)}{w(K)|d(K)|^{1/2}},$$

where $d(K) =: d$ and $R(K) =: R$ signify the discriminant and regulator of the field K, respectively; $w(K) =: w$ is the number of roots of 1 contained in K. The question on upper and lower estimates of the residue $\varkappa_K$ arises. An upper estimate is established comparatively easily and gives an inequality with an effective constant. Landau [1] proved that

$$hR \ll |d|^{1/2} \log^{n-1} |d|, \tag{1.1}$$

where the implicit constant depends only on n. Later Siegel [2] and Lavrik [3] calculated this constant.

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Present the remark of Lavrik: For $|d| \geq 5$,

$$hR < w|d|^{1/2} \log^{n-1} |d|.$$

From (1.1), as Landau showed, it follows that

$$h \underset{n}{\ll} |d|^{1/2} \log^{n-1} |d|. \tag{1.2}$$

It is considerably harder to obtain a lower estimate for $\varkappa_K$. Consider the sequences of normal extensions K of the field $\mathbb{Q}$ such that $n/\log|d| \to 0$. Then, by the Brauer–Siegel theorem [4],

$$\varkappa_K \gg |d|^{-\varepsilon}, \tag{1.3}$$

where $\varepsilon > 0$ is an arbitrary fixed number, and the implicit constant depends on ε and n; an explicit form of this dependence is so far unknown.

From this, in particular, it follows that for all fields K of a given degree $n \geq 2$ with sufficiently large $|d|$, we have

$$hR > |d|^{\frac{1}{2}-\varepsilon}. \tag{1.4}$$

It is known that the lower bound (1.3) is closely related to the (hypothetical) existence of an exceptional zero β_0 (also called the Landau–Siegel zero) for $\zeta_K(s)$, i.e., a real zero in the interval

$$1 - [c(n) \log|d(K)|]^{-1} \leq \beta_0 < 1.$$

If the function $\zeta_K(s)$ has no exceptional zero (according to Heilbronn [5] and Stark [6], this holds, in particular, for the fields K containing no quadratic subfields), then, for such fields K,

$$h\,R \gg \frac{|d|^{1/2}}{\log|d|}, \tag{1.5}$$

where the implicit constant depends only on n.

Inequalities (1.1), (1.3)–(1.5) can be sharpened under some additional conditions. Assume that for every irreducible character χ of the group Gal $(\widehat{K}/\mathbb{Q})$, where $\widehat{K}/\mathbb{Q}$ is the Galois closure of the field $K/\mathbb{Q}$, the Artin L-function $L(s,\chi)$ is entire (the Artin hypothesis) and satisfies the generalized Riemann hypothesis (GRH). Then

$$\frac{|d(K)|^{1/2}}{\log\log|d(K)|} \underset{n}{\ll} h(K)R(K) \underset{n}{\ll} |d(K)|^{1/2} (\log\log|d(K)|)^{n-1}.$$

In order to obtain information on the class number $h(K)$, estimates of the regulator $R(K)$ are needed. Significant results on the value of $R(K)$ were obtained by Remak [7, 8]. One of his estimates is as follows: If $R(K)$ is not a totally imaginary quadratic extension of a totally real field, then

$$R(K) \underset{n}{\gg} \log|d(K)|.$$

Consequently, for such fields K, from (1.1) it follows that

$$h \underset{n}{\ll} |d|^{1/2} (\log|d|)^{n-2}. \tag{1.6}$$

In [9], an infinite sequence of fields K_m ($m = 1, 2, \dots$) of a given degree $n \geq 2$ with a small regulator and, consequently, by (1.4), with an extremally large class number

$$h(K_m) > |d(K_m)|^{\frac{1}{2}-\varepsilon}, \tag{1.7}$$

where $\varepsilon > 0$ is an arbitrary fixed number, was constructed. The result is unconditional and can be transferred to any given signature.

Stronger results of the same type are presented in §3 of the present paper; one of them is due to the author (see Theorem 2).

Here, it is appropriate to make several remarks regarding the real quadratic fields $\mathbb{Q}(\sqrt{d})$. Let χ be the primitive quadratic character modulo d and let $L(s,\chi)$ be the corresponding Dirichlet L-function. Then $L(1,\chi) = hRd^{-1/2}$. Since $L(1,\chi) \ll \log d$ and $R > \left(\frac{1}{2} + o(1)\right)\log d$, then $h \ll \sqrt{d}$ (the old Landau estimate). Under the assumption of the GRH for the Dirichlet L-functions, Littlewood [10] proved that

$$L(1,\chi) < (2e^{\gamma} + o(1)) \log\log d.$$

Consequently, assuming the GRH, we have

$$h < (4e^{\gamma} + o(1)) d^{1/2} (\log d)^{-1} \log\log d.$$

Montgomery and Weinberger [11] showed that the following theorem is unconditionally valid: There is an absolute constant $c > 0$ such that

$$h > c d^{1/2} (\log d)^{-1} \log\log d \tag{1.8}$$

for infinitely many real quadratic fields $\mathbb{Q}(\sqrt{d})$.

Unconditionally, it is very difficult to improve inequalities (1.1), (1.3)–(1.5) in the general case. Some advances are possible for cubic fields (see §2); in particular, in §2 Theorem 1 of the author is proved, which is connected with pure cubic fields.

2

Consider a pure cubic field $K = \mathbb{Q}(\sqrt[3]{m})$, where m is cube-free, $m \neq \pm 1$; we have $m = ab^2$, where a and b are square-free and coprime integers. It is known that if $a^2 \not\equiv b^2 \pmod 9$, then $d(K) = -27a^2b^2$; if $a^2 \equiv b^2 \pmod 9$, then $d(K) = -3a^2b^2$.

For pure cubic fields, the Landau results (1.1), (1.2), and (1.6) were improved by Cohn [12]. Restrict ourselves to the case of positive a and b. Cohn used the fact that in the case of a pure cubic field K, the residue $\varkappa_K$ reduces to a finite expression involving the Dedekind eta function. This implies the inequality

$$h\,R \ll |d|^{1/2} \log|d| \cdot \log\log|d|, \tag{2.1}$$

improving inequality (1.1) in the particular case in question. This implies the following improvement of (1.6):

$$h \ll |d|^{1/2} \log\log|d|.$$

Pass to estimating hR from below in the case of a pure cubic field K. In accordance with what has been said above, $\zeta_K(s)$ has no exceptional zero; therefore, we are going to sharpen the estimate (1.5).

Theorem 1. *If K is a pure cubic field, then*

$$h\,R \gg \frac{|d|^{1/2}}{(\log|d|)^{1/2} \cdot (\log\log|d|)^{1/2}},$$

where the implicit constant is absolute.

Proof. It is known that for $\sigma > 1$, the Dedekid zeta function of any field K factorizes in the Euler product

$$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - N(\mathfrak{p})^{-s}\right)^{-1} = \prod_{p} \prod_{\mathfrak{p}|p} \left(1 - N(\mathfrak{p})^{-s}\right)^{-1} \quad (\sigma > 1), \tag{2.2}$$

where p in the outer product runs over all prime rational numbers, whereas $\mathfrak{p}$ in the inner product runs over all prime ideals of the field K above p.

Vinogradov [13] approximated $\varkappa_K$ by a segment of the Euler product and obtained, for the field K, the following analog of the classical Mertens formula (which we state in the particular case): For a field K for which the Dedekind zeta function has no exceptional zero, the asymptotic formula

$$hR = \frac{w|d|^{1/2}}{2^{r_1}(2\pi)^{r_2}} \cdot \frac{e^{-\gamma}}{\log D} \cdot \prod_{\substack{\mathfrak{p} \\ N(\mathfrak{p}) \le D}} \left(1 - N(\mathfrak{p})^{-1}\right)^{-1} \left(1 + \frac{\theta}{\log D}\right) \tag{2.3}$$

is valid, where $\log D \ge c(n) \log|d| \cdot \log\log|d|$; $c(n)$ is a suitable positive constant depending only on n; $\theta \underset{n}{\ll} 1$.

Pass to the proof of Theorem 1 itself. Apply the asymptotic formula (2.3) in the case of the pure cubic field $K = \mathbb{Q}(\sqrt[3]{m})$. We need the following special case of the law of decomposition of prime numbers p in the field K (see [14]): If $(p, |d|) = 1$ and $p \equiv 2 \pmod 3$, then, in K,

$$p = \mathfrak{p}_1\mathfrak{p}_2,$$

where $\mathfrak{p}_1$ is a prime ideal of the field K of degree 1, and $\mathfrak{p}_2$ is a prime ideal of the field K of degree 2.

If $p \big| |d|$ and $p \ne 3$, then $p = \mathfrak{p}_1^3$, where $\mathfrak{p}_1$ is a prime ideal of the field K of degree 1.

Set $\log D = c \log|d| \log\log|d|$, where $c > 0$ is a suitable absolute constant. Taking into account (2.2), we have

$$\prod_{\substack{\mathfrak{p} \\ N(\mathfrak{p}) \le D}} \left(1 - N(\mathfrak{p})^{-1}\right)^{-1} \ge \prod_{\substack{p \le D \\ p \equiv 2 \pmod 3 \\ (p,|d|)=1}} \left(1 - \frac{1}{p}\right)^{-1} \gg (\log|d|)^{1/2} (\log\log|d|)^{1/2}.$$

Formula (2.3) for the pure cubic field K, together with the latter estimate, proves Theorem 1. □

Remark 1. Duke [15] presented a new result for $n = 3$: The upper bound (3.1) is attained unconditionally by infinitely many Abelian cubic fields.

3

First, present the Duke results [16], improving (1.7) in an important particular case. We will assume that $K \in \mathcal{K}_n$ if K is a totally real field of algebraic numbers of degree n whose normal closure $\widehat{K}$ has the full symmetric group S_n as its Galois group Gal $(\widehat{K}/\mathbb{Q})$. For such fields,

$$\zeta_K(s) = \zeta(s) L(s, \chi),$$

where the character χ of the Galois representation is irreducible and has degree $n - 1$ and conductor d. In addition,

$$h = \frac{d^{1/2}}{2^{n-1} R} L(1, \chi).$$

The Artin hypothesis and GRH yield

$$L(1, \chi) \ll (\log\log d)^{n-1}.$$

Remak [7] proved that if K contains no nontrivial subfields (this is true for $K \in \mathcal{K}_n$), then

$$R \gg (\log d)^{n-1}.$$

Therefore, assuming the Artin hypothesis and GRH, for totally real fields K containing no nontrivial subfields and, in particular, for $K \in \mathcal{K}_n$, we have

$$h \ll d^{1/2} (\log\log d / \log d)^{n-1}; \tag{3.1}$$

here, the implicit constant depends only on n.

The question on the sharpness of the estimate (3.1) arises. From the Montgomery and Weinberger result (1.8) it follows that for $n = 2$, the estimate (3.1) cannot be improved except for the value of the constant. A similar fact was shown by Duke in [16] for $n \geq 3$, but only assuming the Artin hypothesis and GRH. His result is stated in the following way [16, Theorem 1]: Fix an $n \geq 2$ and assume that every Artin L-function is an entire function and satisfies the GRH. Then there exists a constant $c > 0$, depending only on n, such that there are fields $K \in \mathcal{K}_n$ with arbitrarily large discriminant d for which $h > c\, d^{1/2}(\log\log d/\log d)^{n-1}$.

The best unconditional result was obtained in [16]: There are infinitely many fields $K \in \mathcal{K}_n$ with

$$h \gg d^{1/2}(\log d)^{-n}; \tag{3.2}$$

here, the implicit constant depends only on n.

This improves the estimate (1.7).

The estimate (3.2) is improved in Theorem 2 below.

Theorem 2. *Fix an $n \geq 2$. There are infinitely many $K \in \mathcal{K}_n$ with*

$$h \gg d^{1/2}(\log\log d)^{n-1}/(\log d)^n;$$

here, the implicit constant depends only on n.

Proof. We rely on the following important result by Duke [16, Proposition 4]:

Fix an $n \geq 2$. There is a constant $c > 0$, depending only on n, such that there are fields $K \in \mathcal{K}_n$ with arbitrarily large discriminant d for which every prime number p with $c \leq p \leq \log d$ splits completely in K and $R \leq c(\log d)^{n-1}$, where R is the regulator of K.

Note that as K the fields obtained by adjoining to $\mathbb{Q}$ a root of the polynomial

$$f(x,t) = (x-t)(x-2^2t)(x-3^2t)\dots(x-n^2t) - t$$

for a suitable integral value of t are taken.

Write (2.3) in the particular case of a field $K \in \mathcal{K}_n$:

$$h\,R = \frac{d^{1/2}}{2^{n-1}} \cdot \frac{e^{-\gamma}}{\log D} \cdot \prod_{\substack{\mathfrak{p} \\ N(\mathfrak{p}) \leq D}} \left(1 - N(\mathfrak{p})^{-1}\right)^{-1}\left(1 + \frac{\theta}{\log D}\right),$$

where $\log D = c(n)\log d \cdot \log\log d$ with a suitable constant $c(n) > 0$ depending only on n, and $|\theta| \ll_n 1$. Now as K we take the fields occurring in the above-stated Duke proposition. We have

$$\begin{aligned} h\,R &\gg_n d^{1/2}(\log d \cdot \log\log d)^{-1} \prod_{\substack{\mathfrak{p} \\ N(\mathfrak{p}) \leq \log d}} \left(1 - N(\mathfrak{p})^{-1}\right)^{-1} \\ &\gg d^{1/2}(\log d \cdot \log\log d)^{-1} \prod_{p \leq \log d} \left(1 - \frac{1}{p}\right)^{-n} \\ &\asymp d^{1/2}(\log d)^{-1}(\log\log d)^{-1}(\log\log d)^n. \end{aligned}$$

For the fields K considered, we have

$$R \ll_n (\log d)^{n-1},$$

which completes the proof of the theorem. ∎

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