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REPRESENTATIONS AND INEQUALITIES FOR GENERALIZED HYPERGEOMETRIC FUNCTIONS

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An integral representation for the generalized hypergeometric function unifying known representations via generalized Stieltjes, Laplace, and cosine Fourier transforms is found. Using positivity conditions for the weight in this representation, various new facts regarding generalized hypergeometric functions, including complete monotonicity, log-convexity in upper parameters, monotonicity of ratios, and new proofs of Luke's bounds are established. In addition, two-sided inequalities for the Bessel type hypergeometric functions are derived with the use of their series representations. Bibliography: 22 titles.

1. Introduction

Standard notation $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$ for the real, complex and positive integer numbers, respectively, is adopted. $\mathbb{N}_0$ denotes $\mathbb{N} \cup \{0\}$. In the previous works [9, 11], we obtained some representations, inequalities, monotonicity and other properties for the Gauss type generalized hypergeometric function ${}_{q+1}F_q$. The latter is the $p = q + 1$ special case of the function [3, 15]

$$ {}_pF_q\left(\begin{matrix}A\\B\end{matrix}\middle|\, z\right) = {}_pF_q\left(A; B; z\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n n!} z^n, \tag{1} $$

where $A = (a_1, a_2, \ldots, a_p)$ and $B = (b_1, b_2, \ldots, b_q)$, $b_j \notin -\mathbb{N}_0$, are parameter vectors, and $(a)_n$ denotes the rising factorial, defined by $(a)_0 = 1$, $(a)_n = a(a+1)\cdots(a+n-1)$, $n \geq 1$. The series in (1) converges in the entire complex z-plane if $p \leq q$ and inside the unit disk if $p = q+1$. In the latter case, the sum can be extended to a function holomorphic in the cut plane $\mathbb{C}\backslash[1, \infty)$. The main tool employed in [9,11] in investigating the function ${}_{q+1}F_q$ is the generalized Stieltjes transform (see (3) below) of a measure with density expressed by the G-function of Meijer, cf. [9, Theorem 2]. Such a representation appeared earlier in [15, Theorem 4.2.11]. We suggested more relaxed conditions on the parameters and studied the nonnegativity of the representing measure. This leads to the monotonicity of the ratios, two-sided bounds, mapping properties, and other results for the Gauss type hypergeometric functions ${}_{q+1}F_q$.

Another line of research pursued in [6,7,12] hinges on the series representation (1) and yields, among other things, a number of properties of the Kummer type hypergeometric functions ${}_qF_q$, including logarithmic concavity or convexity in parameters, inequalities for logarithmic derivatives, and bounds for the Turánians. In this note, an integral representation for the general hypergeometric function ${}_pF_q$ is introduced. As particular cases, it includes representations by the generalized Stieltjes, Laplace, and cosine Fourier transforms. Starting with this representation, we will obtain new properties of the Gauss type functions ${}_{q+1}F_q$, the Kummer type functions ${}_qF_q$, and the Bessel type functions ${}_{q-1}F_q$, including conditions for complete monotonicity, monotonicity of ratios, and log-convexity in upper parameters. Moreover, we furnish new proofs for Luke's inequalities from [16], allowing their extension to a wider parameter range. Finally, we discover new bounds for the Bessel type hypergeometric functions ${}_pF_q$ with $p < q$ of a positive argument.

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2. Representations for ${}_pF_q$ and their implications

Let $0 \le m \le q$ and $0 \le n \le p$ be integers, and let $A \in \mathbb{C}^p$ and $B \in \mathbb{C}^q$ be such that $a_i - b_j - 1 \notin \mathbb{N}_0$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$. We will heavily use Meijer's G-function [3, Sec. 16.17], defined by the contour integral

$$G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} A \\ B \end{matrix}\right) := \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(b_1+s)\cdots\Gamma(b_m+s)\Gamma(1-a_1-s)\cdots\Gamma(1-a_n-s)z^{-s}}{\Gamma(a_{n+1}+s)\cdots\Gamma(a_p+s)\Gamma(1-b_{m+1}-s)\cdots\Gamma(1-b_q-s)} ds. \tag{2}$$

The contour $\mathcal{L}$ begins and ends at infinity and separates the poles of the integrand of the form $-b_j - k$, $k \in \mathbb{N}_0$, leaving them on the left, from the poles of the form $-a_j + k + 1$, $k \in \mathbb{N}_0$, leaving them on the right. Under the above conditions, such a contour always exists and can be chosen to make the integral in (2) convergent. More details regarding the choice of $\mathcal{L}$ and conditions for the convergence in (2) can be found in [3], [14, Chap. 1 and 2], and [20, Chap. 8].

We will abbreviate $\prod_{i=1}^{p} \Gamma(a_i)$ to $\Gamma(A)$ and $\prod_{i=1}^{p} (a_i)_n$ to $(A)_n$ throughout the paper. Expressions like $A + \alpha$, where $\alpha \in \mathbb{C}$, and $\mathrm{Re}(A) > 0$ will be understood elementwise. The key role in the investigations carried out in [9, 11] is played by the generalized Stieltjes transform representation

$${}_{q+1}F_q\left(\begin{matrix} \sigma, A \\ B \end{matrix} \,\middle|\, -z\right) = \frac{\Gamma(B)}{\Gamma(A)} \int_0^1 (1+zt)^{-\sigma} G^{q,0}_{q,q}\left(t \,\middle|\, \begin{matrix} B \\ A \end{matrix}\right) \frac{dt}{t}, \tag{3}$$

which is readily proved by termwise integration. Note that both the generalized Stieltjes kernel $(1+zt)^{-\sigma} = {}_1F_0(\sigma; -; -zt)$ and the Laplace kernel $e^{-zt} = {}_0F_0(-; -; -zt)$ are particular cases of a more general hypergeometric kernel. This simple observation leads us to the following theorem.

Theorem 1. *Assume that* $p_1 \ge 0$, $p_2 \ge 1$, $q_1, q_2 \ge 0$, $p_2 \ge q_2$, $p = p_1 + p_2$, $q = q_1 + q_2$, *and* $p \le q + 1$ *are integers (these conditions imply that* $p_1 \le q_1 + 1$*). Write* $A_1 = (a_1, \dots, a_{p_1})$, $A_2 = (a_{p_1+1}, \dots, a_p)$, $B_1 = (b_1, \dots, b_{q_1})$, *and* $B_2 = (b_{q_1+1}, \dots, b_q)$ *for complex parameter vectors satisfying* $\mathrm{Re}(A_2) > 0$. *Then*

$${}_pF_q(A_1, A_2; B_1, B_2; -z) = \frac{\Gamma(B_2)}{\Gamma(A_2)} \int_0^\infty {}_{p_1}F_{q_1}(A_1; B_1; -zt) G^{p_2,0}_{q_2,p_2}\left(t \,\middle|\, \begin{matrix} B_2 \\ A_2 \end{matrix}\right) \frac{dt}{t}. \tag{4}$$

This formula is valid for $z \in \mathbb{C}$ *if* $p_1 \le q_1$ *or* $z \in \mathbb{C}\backslash(-\infty, -1]$ *if* $p_1 = q_1 + 1$*; if* $p_2 = q_2$*, then the additional assumption* $\mathrm{Re}(\psi_2) > 0$*, where* $\psi_2 = \sum_{i=p_1+1}^{p} (b_i - a_i)$*, must be adopted (in this case, the* G*-function in* (4) *vanishes for* $t > 1$*). If* $p_2 = q_2$ *and* $\psi_2 = 0$*, then*

$${}_pF_q(A_1, A_2; B_1, B_2; -z) = \frac{\Gamma(B_2)}{\Gamma(A_2)} \left\{ {}_{p_1}F_{q_1}(A_1; B_1; -z) + \int_0^1 {}_{p_1}F_{q_1}(A_1; B_1; -zt) G^{p_2,0}_{q_2,p_2}\left(t \,\middle|\, \begin{matrix} B_2 \\ A_2 \end{matrix}\right) \frac{dt}{t} \right\}, \tag{5}$$

where $z \in \mathbb{C}$ *if* $p_1 \le q_1$ *or* $z \in \mathbb{C}\backslash(-\infty, -1]$ *if* $p_1 = q_1 + 1$.

Proof. Once the correctness of termwise integration has been justified, in order to establish (4), it suffices to write the kernel ${}_{p_1}F_{q_1}$ as the series (1) and integrate it term by term. In order to demonstrate the convergence of the integral in (4) and justify the exchange of summation and integration, we resort to the asymptotic relation

$$G^{p_2,0}_{q_2,p_2}\left(x\middle|\begin{matrix}B\\A\end{matrix}\right)=O\left(x^a\ln^{m-1}(x)\right)\quad\text{as}\ \ x\to 0,\tag{6}$$

where $a=\min(\mathrm{Re}(a_1),\ldots,\mathrm{Re}(a_p))$, and the minimum is taken over those a_i for which $a_i-b_j\notin\mathbb{N}_0$ for all $j=1,\ldots,q_2$. The positive integer m is the maximal multiplicity among the numbers a_i for which the minimum is attained. This formula follows from [14, Corollary 1.12.1] or [13, Eq. (11)]. It proves the convergence in (4) around zero. Near infinity, for $p_2>q_2$ we have

$$G^{p_2,0}_{q_2,p_2}\left(x\middle|\begin{matrix}B\\A\end{matrix}\right)=\frac{(2\pi)^{\frac{1}{2}(\mu-1)}}{\sqrt{\mu}}x^{(1-\alpha)/\mu}e^{-\mu x^{1/\mu}}\left[1+O(x^{-1/\mu})\right]\quad\text{as}\quad x\to\infty,\tag{7}$$

where $\mu=p_2-q_2$ and $\alpha=\sum\limits_{i=q_1+1}^{q}b_i-\sum\limits_{i=p_1+1}^{p}a_i+\frac{1}{2}(p_2-q_2+1)$. This formula is a particular case of the formula on page 289 in [4], which is implied by formula (7.8) of the same paper. If $p_2=q_2$ and $\mathrm{Re}(\psi_2)>0$, then, according to [20, 8.2.59],

$$G^{p_2,0}_{q_2,p_2}\left(x\middle|\begin{matrix}B_2\\A_2\end{matrix}\right)=O((1-x)^{\mathrm{Re}(\psi_2)-1})\quad\text{as}\quad x\uparrow 1$$

and, according to [9, Lemma 1] (also see the proof of Theorem 2 below),

$$G^{p_2,0}_{q_2,p_2}\left(x\middle|\begin{matrix}B_2\\A_2\end{matrix}\right)=0\quad\text{for}\quad x>1.$$

This shows the convergence in (4) around unity for $p_2=q_2$. Finally, (5) follows from [8, Theorem 1]. □

Remark. The condition $p_2\geq q_2$ is necessary in the above theorem because, for $p_2<q_2$,

$$G^{p_2,0}_{q_2,p_2}\left(x\middle|\begin{matrix}B_2\\A_2\end{matrix}\right)=0\quad\text{for all}\quad x\in\mathbb{R}.$$

This condition shows that for $p<q$, the function ${}_pF_q$ cannot be represented by the Laplace or generalized Stieltjes transform. The most "extreme" representation we can obtain in this case is

$${}_pF_q(A;B;-z)=\frac{\Gamma(B_2)}{\Gamma(A)}\int\limits_0^1{}_0F_{q-p}(-;B_1;-zt)G^{p,0}_{p,p}\left(t\middle|\begin{matrix}B_2\\A\end{matrix}\right)\frac{dt}{t},$$

where essentially the kernel ${}_0F_m$ is the Bessel function if $m=1$ or the so-called hyper-Bessel function if $m>1$ (see [15]). Note that this kernel cannot be represented by Theorem 1 because of the condition $p_2\geq 1$. It is sometimes desirable, however, to have a representation with a kernel independent of the parameters of the function being represented. This can easily be achieved by introducing artificial parameters $\alpha_j>0$ to obtain

$${}_pF_q(A;B;-z)=\frac{\Gamma(B)}{\Gamma(A)\prod\limits_{i=1}^{q-p}\Gamma(\alpha_i)}\int\limits_0^1{}_0F_{q-p}(-;\alpha_1,\ldots,\alpha_{q-p};-zt)G^{q,0}_{q,q}\left(t\middle|\begin{matrix}B\\A,\alpha_1,\ldots,\alpha_{q-p}\end{matrix}\right)\frac{dt}{t}.\tag{8}$$

We must require $\sum b_i>\sum a_i+\sum\alpha_i$ for the convergence of the above integral. In particular, by choosing $\alpha_i=i/(q-p+1)$, we obtain the kernel in terms of the so-called generalized cosine,

$$\cos_n(z)=\sum_{j=0}^{\infty}\frac{(-1)^jz^{nj}}{(nj)!}={}_0F_{n-1}(-;1/n,2/n,\ldots,(n-1)/n;-(z/n)^n).$$

The representation with such a kernel was originally suggested by Kiryakova in [15]. An important particular case $p = q - 1$ leads to the standard cosine kernel, as indicated in the corollary below. Before stating it, define the parametric excess by

$$\psi = \sum_{k=1}^{q} b_k - \sum_{k=1}^{p} a_k. \tag{9}$$

Corollary 1. *Let* $\mathrm{Re}(A) > 0$ *elementwise. Then*

$$ {}_{q+1}F_q\left(\begin{matrix}A\\B\end{matrix}\middle| -z\right) = \frac{\Gamma(B)}{\Gamma(A)} \int_0^\infty e^{-zt} G^{q+1,0}_{q,q+1}\left(t\,\middle|\begin{matrix}B\\A\end{matrix}\right)\frac{dt}{t}. \tag{10}$$

In addition, if $\mathrm{Re}(\psi) > 0$*, then*

$$ {}_{q}F_q\left(\begin{matrix}A\\B\end{matrix}\middle| -z\right) = \frac{\Gamma(B)}{\Gamma(A)} \int_0^1 e^{-zt} G^{q,0}_{q,q}\left(t\,\middle|\begin{matrix}B\\A\end{matrix}\right)\frac{dt}{t}. \tag{11}$$

If $\mathrm{Re}(\psi) > 1/2$*, then*

$$ {}_{q-1}F_q\left(\begin{matrix}A\\B\end{matrix}\middle| -z\right) = \frac{\Gamma(B)}{\sqrt{\pi}\Gamma(A)} \int_0^1 \cos(2\sqrt{zt}) G^{q,0}_{q,q}\left(t\,\middle|\begin{matrix}B\\A,1/2\end{matrix}\right)\frac{dt}{t}. \tag{12}$$

If $\psi = 0$*, then* (11) *takes the form*

$$ {}_{q}F_q\left(\begin{matrix}A\\B\end{matrix}\middle| -z\right) = \frac{\Gamma(B)}{\Gamma(A)}\left\{e^{-z} + \int_0^1 e^{-zt} G^{q,0}_{q,q}\left(t\,\middle|\begin{matrix}B\\A\end{matrix}\right)\frac{dt}{t}\right\}.$$

If $\psi = 1/2$*, then* (12) *takes the form*

$$ {}_{q-1}F_q\left(\begin{matrix}A\\B\end{matrix}\middle| -z\right) = \frac{\Gamma(B)}{\sqrt{\pi}\Gamma(A)}\left\{\cos(2\sqrt{z}) + \int_0^1 \cos(2\sqrt{zt}) G^{q,0}_{q,q}\left(t\,\middle|\begin{matrix}B\\A,1/2\end{matrix}\right)\frac{dt}{t}\right\}.$$

Application of the integral representations (3), (4), (5), (8) (10), (11), and (12) to investigation of properties of the generalized hypergeometric function ${}_pF_q$ depends heavily on the positivity of the representing measures, expressed here in terms of Meijer's G-function. Sufficient conditions for the positivity are furnished in the next theorem.

Theorem 2. *Assume that* $A, B \in \mathbb{R}^q$ *are such that*

$$v(t) = \sum_{j=1}^{q} (t^{a_j} - t^{b_j}) \geq 0 \quad on \quad (0, 1]. \tag{13}$$

Then

$$G^{q,0}_{q,q}\left(t\,\middle|\begin{matrix}B\\A\end{matrix}\right) \geq 0 \quad on \quad (0, 1). \tag{14}$$

Before giving a proof of this theorem, we recall that a nonnegative function f defined on $(0, \infty)$ is said to be completely monotone if it has derivatives of all orders and $(-1)^n f^{(n)}(x) \geq 0$ for $n \in \mathbb{N}_0$ and $x > 0$ [22, Definition 1.3]. This inequality is known to be strict unless f is a constant. By the celebrated Bernstein theorem, a function is completely monotone if and only if it is the Laplace transform of a nonnegative measure [22, Theorem 1.4]. A positive function f is said to be logarithmically completely monotone if $-(\log f)'$ is completely monotone [22, Definition 5.8]. The class of logarithmically completely monotone functions is a proper subset of the class of completely monotone functions. Their importance stems from the fact that they represent the Laplace transforms of infinitely divisible probability distributions, see [22, Theorem 5.9] and [21, Sec. 51].

Proof of Theorem 2. First note that

$$G^{q,0}_{q,q}\left(t\,\middle|\,\begin{matrix}B\\A\end{matrix}\right)=0$$

for $t>1$ and all (complex) values of A and B. This follows from the fact that for $t>1$, choosing the right loop to be the contour of integration in (2) yields a convergent integral, see [14, Theorem 1.1]. On the other hand, there are no poles of the integrand inside this contour, whence the above equality follows by Cauchy's theorem. This explains the restriction $t\in(0,1)$ in the theorem statement. Further, by virtue of the formula

$$t^{\alpha}G^{q,0}_{q,q}\left(t\,\middle|\,\begin{matrix}B\\A\end{matrix}\right)=G^{q,0}_{q,q}\left(t\,\middle|\,\begin{matrix}B+\alpha\\A+\alpha\end{matrix}\right)$$

(see [20, 8.2.2.15] or [3, 16.19.2]), we can restrict our attention to the case $A,B>0$. Indeed, adding a sufficiently large α to A and B neither alters the sign of Meijer's G in (14) nor the sign of $v(t)$ in (13). Adopting the assumption $A,B>0$, we are in a position to apply [5, Lemma 2.1], whose particular case (essentially contained already in [1, Theorem 10]) states that the ratio $x\to\Gamma(A+x)/\Gamma(B+x)$ is logarithmically completely monotone if and only if condition (13) is fulfilled. Hence, under (13), this function also is completely monotone. If $\psi>0$, then

$$\frac{\Gamma(A+x)}{\Gamma(B+x)}=\int\limits_0^{\infty}e^{-xt}G^{q,0}_{q,q}\left(e^{-t}\,\middle|\,\begin{matrix}B\\A\end{matrix}\right)dt,$$

and the representing measure must be nonnegative by Berstein's theorem. This measure is unique according to [22, Proposition 1.2]. Nonnegativity is extended to $\psi=0$ by continuity. If $\psi<0$, then $v(t)$ cannot be nonnegative on $(0,1]$ because $v(1)=0$ and $v'(1)=-\psi$. □

Condition (13) is probably also necessary for (14), at least if $\psi\geq 0$. However, this condition is very difficult to verify. Some sufficient conditions are known to hold for inequality (13). In order to cite the corresponding results, we need to introduce the following terminology [17, Definition A.2]. It is said that a real vector $B=(b_1,\dots,b_q)$ is weakly supermajorized by a real vector $A=(a_1,\dots,a_q)$ (symbolized as $B\prec^W A$) if

$$\begin{gathered}0<a_1\leq a_2\leq\cdots\leq a_q,\qquad 0<b_1\leq b_2\leq\cdots\leq b_q,\\ \sum_{i=1}^{k}a_i\leq\sum_{i=1}^{k}b_i\quad\text{for}\quad k=1,2\dots,q.\end{gathered}\tag{15}$$

If, in addition, $\psi(=\sum\limits_{i=1}^{q}(b_i-a_i))=0$, then B is said to be majorized by A, or $B\prec A$.

It will be convenient to assume that A and B (or A_i and B_i when they appear) are ordered ascending whenever they are real. From a theorem of Tomić (see [17, Proposition 4.B.2]) it immediately follows hat $v(t)\geq 0$ if $B\prec^W A$. In the present context, this fact was first used by Alzer [1, Theorem 10]. For the particular situation $q=2^n$, $n=0,1,\dots$, Grinshpan and Ismail [5, Theorems 1.1,1.2] derived two different sets of conditions sufficient for (13) to be valid.

By combining the nonnegativity of the G-function with representations (3) and (11), we obtain some sufficient conditions for the generalized hypergeometric functions to be completely monotone or logarithmically completely monotone.

Theorem 3. *Let $v(t) \geq 0$ on $(0,1]$ and let $\sigma > 0$. Then the functions*

$$x \to {}_{q+1}F_q\left(\begin{matrix}\sigma, A\\ B\end{matrix}\middle| -x\right) \quad \text{and} \quad x \to {}_qF_q\left(\begin{matrix}A\\ B\end{matrix}\middle| -x\right)$$

are completely monotone on $(0,\infty)$. In particular, this holds if $B \prec^W A$.

Theorem 4. *Let $\sigma > 0$ and let $v(t) \geq 0$ on $(0,1]$ (in particular, this holds if $B \prec^W A$). Then the function*

$$x \to x^{-\sigma}{}_{q+1}F_q\left(\begin{matrix}\sigma, A\\ B\end{matrix}\middle| -\frac{1}{x}\right)$$

is completely monotone on $(0,\infty)$. If $0 < \sigma \leq 1$, then it is logarithmically completely monotone.

Proof. By factoring the generalized Stieltjes transform (3) into repeated Laplace transforms ans using [10, Theorem 8], we obtain

$$\begin{aligned} x^{-\sigma}{}_{q+1}F_q(\sigma, A; B; -1/x) &= \frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-ux}u^{\sigma-1}\int_0^1 e^{-ut}d\rho(t)\,du \\ &= \frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-ux}u^{\sigma-1}{}_qF_q(A; B; -u)\,du, \end{aligned}$$

where

$$d\rho(t) = \frac{\Gamma(B)}{\Gamma(A)}G^{q,0}_{q,q}\left(t\,\middle|\begin{matrix}B\\ A\end{matrix}\right)\frac{dt}{t}$$

is nonnegative by Theorem 2. This implies complete monotonicity. Further, according to [21, Theorem 51.4], a probability distribution is infinitely divisible if it has log-convex density. The function $u^{\sigma-1}\int_0^1 e^{-ut}d\rho(t)$ is log-convex for $0 < \sigma \leq 1$ because both factors are log-convex (the second factor is log-convex by complete monotonicity). Thus, the function in the statement of the theorem is the Laplace transform of an infinitely divisible distribution, whence it is logarithmically completely monotone by [1, Proposition on p. 387] or [22, Theorem 5.9]. □

By applying the methods of proofs from [9, 11] to representations (4) and (5), it is straightforward to establish the two propositions below (cf. Theorems 4 and 7 from [9]). The symbol A_1' will denote A_1 without its maximal element.

Theorem 5. *Keeping the notation and constraints of Theorem 1, assume, in addition, that $A_1, B_1 > 0$, $p_2 = q_2$, and $\sum_{j=p_1+1}^{p}(t^{a_j} - t^{b_j}) \geq 0$ (or $B_2 \prec^W A_2$). Then, for every fixed $\mu > 0$, the function*

$$x \to \frac{{}_pF_q\left(\begin{matrix}A_1, A_2+\mu\\ B_1, B_2+\mu\end{matrix}\middle| -x\right)}{{}_pF_q\left(\begin{matrix}A_1, A_2\\ B_1, B_2\end{matrix}\middle| -x\right)}$$

is monotone decreasing on $(-\infty, 0)$ if $p \leq q$ or on $(-1, 0)$ if $p = q+1$. If $p = q$ and $\sum_{j=1}^{p_1}(t^{a_j} - t^{b_j}) \geq 0$ (or $B_1 \prec^W A_1$), then the above quotient decreases on the entire real line. If $p = q+1$ and $\sum_{j=1}^{q_1}(t^{a_j} - t^{b_j}) \geq 0$ (or $B_1 \prec^W A_1'$), then the above quotient decreases on $(-1, \infty)$.

Theorem 6. *Keeping the notation and constraints of Theorem* 1*, assume, in addition, that* $A_1, B_1 > 0$*,* $p_2 = q_2$*, and* $\sum_{j=p_1+1}^{p}(t^{a_j} - t^{b_j}) \geq 0$ *(or* $B_2 \prec^W A_2$*). Then the function*

$$\mu \to {}_pF_q\left(\begin{matrix}A_1, A_2+\mu\\B_1, B_2+\mu\end{matrix}\middle| -x\right)$$

is log-convex on $(0,\infty)$ *for every fixed* $x \in (-\infty, 0)$ *if* $p \leq q$ *or* $x \in (-1, 0)$ *if* $p = q+1$*. If* $p = q$ *and* $\sum_{j=1}^{p_1}(t^{a_j} - t^{b_j}) \geq 0$ *(or* $B_1 \prec^W A_1$*), then log-convexity holds for every real* x*, whereas for* $p = q+1$ *and* $\sum_{j=1}^{q_1}(t^{a_j} - t^{b_j}) \geq 0$ *(or* $B_1 \prec^W A_1'$*) log-convexity holds for every fixed* $x \in (-1, \infty)$*.*

Remark. It is readily seen that the conditions $B_1 \prec^W A_1$ and $B_2 \prec^W A_2$ imply $B \prec^W A$ (for these relations to make sense, one must assume that $p_1 = q_1$ and $p_2 = q_2$). For this reason, Theorems 5 and 6 are the strongest in an informal sense for $p_1 = q_1 = 0$, i.e., for the functions

$$x \to {}_qF_q\left(\begin{matrix}A+\mu\\B+\mu\end{matrix}\middle| -x\right)\Big/{}_qF_q\left(\begin{matrix}A\\B\end{matrix}\middle| -x\right) \text{ and } \mu \to {}_qF_q\left(\begin{matrix}A+\mu\\B+\mu\end{matrix}\middle| -x\right).$$

3. Inequalities for the Kummer and Gauss type functions

In Theorem 16 of his paper [16], Luke gave two-sided bounds for the function ${}_qF_q(A;B;x)$ under the restrictions $b_i \geq a_i > 0$, $i = 1, 2, \ldots, q$. He indicated that these bounds are "easily proved" without providing such proofs. In this section, we supply two different proofs of Luke's inequalities, one valid for positive values of the argument x and the other valid for all real x. In this way, we substantially relax Luke's conditions. For negative argument values, our conditions are given in terms of the nonnegativity of $v(t)$ or the weak majorization $B \prec^W A$. For positive argument values, the conditions can be weakened further and are given in terms of the elementary symmetric polynomials, defined by

$$e_k(x_1, \ldots, x_q) = \sum_{1 \leq j_1 < j_2 \cdots < j_k \leq q} x_{j_1} x_{j_2} \cdots x_{j_k}, \quad k = 1, 2, \ldots, q.$$

Theorem 7. *Assume that*

$$\frac{e_q(b_1, \ldots, b_q)}{e_q(a_1, \ldots, a_q)} \geq \frac{e_{q-1}(b_1, \ldots, b_q)}{e_{q-1}(a_1, \ldots, a_q)} \geq \cdots \geq \frac{e_1(b_1, \ldots, b_q)}{e_1(a_1, \ldots, a_q)} \geq 1 \tag{16}$$

and that every elementary symmetric polynomial above is nonnegative. Then

$$e^{f_1 x} \leq {}_qF_q(A;B;x) \leq 1 - f_1 + f_1 e^x \quad \textit{for} \quad x \geq 0, \tag{17}$$

where $f_1 = \prod_{i=1}^{q}(a_i/b_i)$*. Moreover, the upper bound holds true if every fraction in* (16) *is merely not less than* 1*.*

Remark. Note that conditions (16) are strictly weaker than $B \prec^W A$, as we demonstrated in [9, Lemma 3].

Proof. By $f_n = \prod_{i=1}^{q}[(a_i)_n/(b_i)_n]$ denote the coefficient at $x^n/n!$ in the power series expansion (1) of ${}_qF_q(A;B;x)$. Then the conditions

$$e_i(b_1, \ldots, b_q) \geq e_i(a_1, \ldots, a_q), \quad i = 1, \ldots, q,$$

(i.e., every fraction in (16) is not less than 1) imply that

$$\frac{f_{n+1}}{f_n} = R(n) = \prod_{i=1}^{q} \frac{a_i + n}{b_i + n} \le 1$$

because $e_{q-i}(a_1, \dots, a_q)$ $(e_{q-i}(b_1, \dots, b_q))$ is the coefficient at n^i in the polynomial in the numerator (denominator) of $R(n)$. Thus, $f_{n+1} \le f_n$, whence $f_n \le f_1$ for $n = 1, 2, \dots$. Consequently, for $x \ge 0$, we obtain

$${}_qF_q(A; B; x) = 1 + \sum_{n=1}^{\infty} f_n \frac{x^n}{n!} = 1 + f_1 \sum_{n=1}^{\infty} \frac{f_n}{f_1} \frac{x^n}{n!} \le 1 + f_1 \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 - f_1 + f_1 e^x.$$

Further, under conditions (16), the function $R(x)$ defined above is increasing according to [12, Lemma 2]. This leads to the following inequalities ($k \ge 0$):

$$R(0) = \prod_{i=1}^{q} \frac{a_i}{b_i} \le \prod_{i=1}^{q} \frac{a_i + k}{b_i + k} = R(k) \Rightarrow (f_1)^n = \prod_{i=1}^{q} \frac{(a_i)^n}{(b_i)^n} \le \prod_{i=1}^{q} \frac{(a_i)_n}{(b_i)_n} = f_n, \quad n = 1, 2, \dots .$$

Consequently,

$${}_qF_q(A; B; x) = 1 + \sum_{n=1}^{\infty} f_n \frac{x^n}{n!} \ge 1 + \sum_{n=1}^{\infty} (f_1)^n \frac{x^n}{n!} = e^{f_1 x},$$

which completes the proof. □

Remark. Inequalities (17) can be refined to the bounds

$$1 + \frac{f_1^2}{f_2}(e^{(f_2/f_1)x} - 1) \le {}_qF_q(A; B; x) \le 1 - f_2 + (f_1 - f_2)x + f_2 e^x, \tag{18}$$

valid for $x \ge 0$ under the assumptions of Theorem 7. Indeed, the upper bound is obtained by writing

$$\begin{aligned} {}_qF_q(A; B; x) &= 1 + f_1 x + f_2 \sum_{n=2}^{\infty} \frac{f_n}{f_2} \frac{x^n}{n!} \le 1 + f_1 x + f_2 \sum_{n=2}^{\infty} \frac{x^n}{n!} \\ &= 1 - f_2 + (f_1 - f_2)x + f_2 e^x, \end{aligned}$$

where we have used the fact that $f_{n+1} \le f_n$ for $n = 2, 3, \dots$, provided that every fraction in (16) is not less than 1. In order to prove the lower bound, we note that under conditions (16), we have $(f_2/f_1)^{n-1} \le f_n/f_1$ for $n = 2, 3, \dots$ by the increase of $R(x)$. Then

$$\begin{aligned} {}_qF_q(A; B; x) &= 1 + f_1 x + f_1 \sum_{n=2}^{\infty} \frac{f_n}{f_1} \frac{x^n}{n!} \\ &\ge 1 + f_1 x + f_1 \sum_{n=2}^{\infty} \left(\frac{f_2}{f_1}\right)^{n-1} \frac{x^n}{n!} = 1 + \frac{f_1^2}{f_2}(e^{(f_2/f_1)x} - 1). \end{aligned}$$

A similar trick can be applied to separate further terms.

Corollary 2. *Let $\sigma > 0$ and let the hypotheses of Theorem 7 be satisfied. Then, for $0 \le x < 1$,*

$$\frac{1}{(1 - f_1 x)^{\sigma}} \le {}_{q+1}F_q(\sigma, A; B; x) \le 1 - f_1 + \frac{f_1}{(1 - x)^{\sigma}}$$

and

$$1 - \frac{f_1^2}{f_2} + \frac{f_1^2}{f_2(1 - f_2 x/f_1)^{\sigma}} \le {}_{q+1}F_q(\sigma, A; B; x) \le 1 - f_2 + \sigma(f_1 - f_2)x + \frac{f_2}{(1 - x)^{\sigma}}.$$

Proof. Following Luke's idea from [16], write the bounds (17) for ${}_qF_q(A;B;t)$, multiply by $e^{-ty}t^{\sigma-1}$, and integrate using the relation

$$\int_0^\infty e^{-ty}t^{\sigma-1}{}_qF_q(A;B;t)\,dt = y^{-\sigma}\Gamma(\sigma){}_{q+1}F_q(\sigma,A;B;1/y).$$

In order to obtain the first inequality, it remains to write $x = 1/y$ in the resulting inequality and simplify the latter. The second inequality is proved by applying the same trick to (18). $\square$

Theorem 8. *Assume that* $A, B > 0$ *and* $\sum_{j=1}^q (t^{a_j} - t^{b_j}) \geq 0$ *(or* $B \prec^W A$*). Then, for all real* x*,*

$$e^{-f_1x} \leq {}_qF_q(A;B;-x) \leq 1 - f_1 + f_1e^{-x}.$$

Proof. In accordance with the integral form of Jensen's inequality [18, Chap. I, Eq. (7.15)],

$$\varphi\left(\int_a^b f(s)d\mu(s)\Big/\int_a^b d\mu(s)\right) \leq \int_a^b \varphi(f(s))d\mu(s)\Big/\int_a^b d\mu(s) \tag{19}$$

if φ is convex and f is integrable with respect to a nonnegative measure μ. Put $\varphi_x(y) = e^{-xy}$, $f(s) = s$, and

$$d\mu(s) = \frac{\Gamma(B)}{\Gamma(A)}G^{q,0}_{q,q}\left(s\,\middle|\,\begin{matrix}B\\A\end{matrix}\right)\frac{ds}{s}.$$

Then

$$\int_0^1 d\mu(s) = 1, \quad \int_0^1 f(s)d\mu(s) = \prod_{i=1}^q \frac{a_i}{b_i} = f_1,$$

$$\int_0^1 \varphi_x(f(s))d\mu(s) = {}_qF_q(A;B;-x).$$

The latter relation is (11) represented in a different form. This proves the lower bound. In order to demonstrate the upper bound, we will apply the converse Jensen inequality, due to Lah and Ribarić, which reads as follows. Set

$$A(g) = \int_m^M g(s)d\tau(s)\Big/\int_m^M d\tau(s)\ ,$$

where τ is a nonnegative measure and g is a continuous function. If $-\infty < m < M < \infty$ and φ is convex on $[m, M]$, then, according to [19, Theorem 3.37],

$$(M - m)A(\varphi(g)) \leq (M - A(g))\varphi(m) + (A(g) - m)\varphi(M).$$

Setting $\varphi_x(t) = e^{-xt}$, $d\tau(s) = d\mu(s)$, $g(s) = s$, and $[m, M] = [0, 1]$, we arrive at the upper bound of the theorem. $\square$

Corollary 3. *Let* $\sigma > 0$ *and let the hypotheses of Theorem* 8 *be satisfied. Then, for* $x \geq 0$*,*

$$\frac{1}{(1+f_1x)^\sigma} \leq {}_{q+1}F_q(\sigma,A;B;-x) \leq 1 - f_1 + \frac{f_1}{(1+x)^\sigma}.$$

Proof. Multiply inequality (17) written for ${}_qF_q(A;B;-xt)$ by $e^{-t}t^{\sigma-1}$ and integrate using the formula

$$\int_0^\infty e^{-t}t^{\sigma-1}{}_qF_q(A;B;-xt)\,dt=\Gamma(\sigma){}_{q+1}F_q(\sigma,A;B;-x). \qquad \square$$

4. Inequalities for the Bessel type functions

First, we will find an upper bound in the general situation $p<q$. As above, the symbol f_n will denote the coefficient at $x^n/n!$ in the series representation (1), i.e.,

$$f_n=\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{i=1}^{q}(b_i)_n}=\frac{(A)_n}{(B)_n}\quad\text{for}\quad n=0,1,\dots.$$

Theorem 9. *Let $p<q$. If*

$$e_{q-i}(b_1,\dots,b_q)\ge e_{p-i}(a_1,\dots,a_p),\quad i=0,1,\dots,p, \tag{20}$$

then, for $x\ge 0$,

$${}_pF_q(A;B;x)\le 1-f_1+f_1e^x.$$

If

$$\frac{e_q(b_1,\dots,b_q)}{e_p(a_1,\dots,a_p)}\le\frac{e_{q-1}(b_1,\dots,b_q)}{e_{p-1}(a_1,\dots,a_p)}\le\dots\le\frac{e_{q-p+1}(b_1,\dots,b_q)}{e_1(a_1,\dots,a_p)}\le e_{q-p}(b_1,\dots,b_q), \tag{21}$$

then, for $x\ge 0$,

$${}_pF_q(A;B;x)\le e^{f_1x}.$$

Proof. The proof of the first upper bound repeats that of the upper bound (17) in Theorem 7. In order to demonstrate the second bound, note that for $p<q$, the function

$$R(x)=\frac{\prod_{i=1}^{p}(a_i+x)}{\prod_{i=1}^{q}(b_i+x)}$$

is decreasing under conditions (21) according to [12, p. 394], which implies that

$$f_n=\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{i=1}^{q}(b_i)_n}\le\frac{\prod_{i=1}^{p}(a_i)^n}{\prod_{i=1}^{q}(b_i)^n}=(f_1)^n.$$

Hence

$${}_pF_q(A;B;x)=1+\sum_{n=1}^{\infty}f_n\frac{x^n}{n!}\le 1+\sum_{n=1}^{\infty}(f_1)^n\frac{x^n}{n!}=e^{f_1x}. \qquad \square$$

According to the asymptotic formula [3, 16.11.8],

$${}_{q-1}F_q\left(\begin{matrix}A\\B\end{matrix}\middle|\,x\right)=\frac{\Gamma(b_1)\cdots\Gamma(b_q)}{2\sqrt{\pi}\Gamma(a_1)\cdots\Gamma(a_{q-1})}x^{\nu}e^{2\sqrt{x}}\left(1+\frac{d_1}{\sqrt{x}}+O(x^{-1})\right)\quad\text{as}\quad x\to+\infty,$$

where $\nu=\frac{1}{2}\sum_{i=1}^{q-1}a_i-\frac{1}{2}\sum_{i=1}^{q}b_i+1/4$. Hence the upper bounds of Theorem 9 are very wrong in asymptotic order. In the most important case $p=q-1$, we can do much better.

Theorem 10. *Let $A, B > 0$ (understood elementwise). Then, for $x \geq 0$,*

$$e^{\sqrt{4x+c^2}-c}\left(\frac{1}{2}+\frac{1}{2c}\sqrt{4x+c^2}\right)^{-c} \leq {}_{q-1}F_q(A;B;x), \tag{22}$$

where $c > 0$ is given by

$$c = \max_{i\in\{1,2,\ldots,q\}}\left[\frac{e_i(b_1,b_2,\ldots,b_q)-e_i(a_1,a_2,\ldots,a_p)}{e_{i-1}(a_1,a_2,\ldots,a_p)}\right], \tag{23}$$

$p = q-1$, and $e_q(a_1,a_2,\ldots,a_p) = 0$.

Proof. If we could find a number c such that

$$f_n = \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n} \geq \frac{1}{(c)_n} \quad \text{for} \quad n = 1,2,\ldots, \tag{24}$$

then, for $x \geq 0$ ($p = q-1$),

$${}_pF_q(A;B;x) = 1+\sum_{n=1}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{x^n}{n!} \geq 1+\sum_{n=1}^{\infty}\frac{1}{(c)_n}\frac{x^n}{n!} = {}_0F_1(-;c;x). \tag{25}$$

Further, we can use some known lower bounds for the function ${}_0F_1(-;c;x)$ (which is equal to the modified Bessel function I_{c-1} up to a simple multiplier) in deriving lower bounds for ${}_pF_q(A;B;x)$ in terms of elementary functions. For (24) to hold, it suffices to satisfy $f_1c \geq 1$ and

$$\frac{f_{n+1}(c)_{n+1}}{f_n(c)_n} = \frac{(a_1+n)\cdots(a_p+n)(c+n)}{(b_1+n)\cdots(b_q+n)} \geq 1, \quad n = 1,2,\ldots.$$

In turn, the above inequality holds if (recall that $q = p+1$)

$$e_i(a_1,a_2,\ldots,a_p,c) \geq e_i(b_1,b_2,\ldots,b_q), \quad i = 1,2,\ldots,q,$$

or

$$e_i(a_1,a_2,\ldots,a_p)+ce_{i-1}(a_1,a_2,\ldots,a_p) \geq e_i(b_1,b_2,\ldots,b_q), \quad i = 1,2,\ldots,q.$$

For these q inequalities to be satisfied, we must set

$$c = \max_{i\in\{1,2,\ldots,q\}}\left[\frac{e_i(b_1,b_2,\ldots,b_q)-e_i(a_1,a_2,\ldots,a_p)}{e_{i-1}(a_1,a_2,\ldots,a_p)}\right].$$

Here, $e_0 = 1$ and $e_q(a_1,a_2,\ldots,a_p) = 0$. Owing to the latter relation, we obtain $c > 0$ for any positive arrays A and B. Hence the problem reduces to finding good bounds for ${}_0F_1(-;c;x)$ for $x, c > 0$. Numerically best bounds, contained in [2, Eq. (11)], are in terms of the ratio $I_{\nu+1}/I_\nu$ of the modified Bessel functions

$$I_\nu(x) = (x/2)^\nu[\Gamma(\nu+1)]^{-1}{}_0F_1(-;\nu+1;x^2/4).$$

Written in terms of the logarithmic derivative of ${}_0F_1(-;c;x)$, these bounds read as

$$\frac{2}{\sqrt{4x+c^2}+c} \leq \frac{{}_0F_1'(-;c;x)}{{}_0F_1(-;c;x)} = \frac{{}_0F_1(-;c+1;x)/c}{{}_0F_1(-;c;x)} \leq \frac{2}{\sqrt{4x+(c+1)^2}+c-1},$$

where the derivative formula ${}_0F_1'(-;c;x) = {}_0F_1(-;c+1;x)/c$ has been used. Using the evaluation

$$\int_0^x \frac{2dt}{a+\sqrt{4tq+b^2}} = \frac{1}{q}\sqrt{4xq+b^2}-\frac{a}{q}\ln\frac{a+\sqrt{4xq+b^2}}{a+b}-\frac{b}{q},$$

we can integrate the above inequalities to obtain

$$\sqrt{4x+c^2} - c\log\frac{c+\sqrt{4x+c^2}}{2c} - c \le \log({}_0F_1(-;c;x))$$
$$\le \sqrt{4x+(c+1)^2} - (c-1)\log\frac{c-1+\sqrt{4x+(c+1)^2}}{2c} - (c+1).$$

Taking exponentials yields

$$e^{\sqrt{4x+c^2}-c}\left(\frac{1}{2}+\frac{1}{2c}\sqrt{4x+c^2}\right)^{-c} \le {}_0F_1(-;c;x)$$
$$\le e^{\sqrt{4x+(c+1)^2}-c-1}\left(\frac{c-1}{2c}+\frac{1}{2c}\sqrt{4x+(c+1)^2}\right)^{1-c}. \tag{26}$$

Combining the lower bound in (26) with (25) completes the proof of the theorem. □

Theorem 11. *Let $A, B > 0$ (understood elementwise) and let d given by*

$$d = \min_{i\in\{1,2,\dots,q\}}\left[\frac{e_i(b_1,b_2,\dots,b_q)-e_i(a_1,a_2,\dots,a_p)}{e_{i-1}(a_1,a_2,\dots,a_p)}\right] \tag{27}$$

be positive. Here, $p = q-1$, $e_0 = 1$, and $e_q(a_1,a_2,\dots,a_p) = 0$. Then, for $x \ge 0$,

$${}_{q-1}F_q(A;B;x) \le e^{\sqrt{4x+(d+1)^2}-d-1}\left(\frac{d-1}{2d}+\frac{1}{2d}\sqrt{4x+(d+1)^2}\right)^{1-d}. \tag{28}$$

Proof. If we could find a certain d such that

$$f_n(d)_n = \frac{(a_1)_n\cdots(a_p)_n(d)_n}{(b_1)_n\cdots(b_q)_n} \le 1 \quad \text{for} \quad n = 1,2,\dots,$$

then we would have

$${}_pF_q(A;B;x) = 1 + \sum_{n=1}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{x^n}{n!}$$
$$\le 1 + \sum_{n=1}^{\infty}\frac{1}{(d)_n}\frac{x^n}{n!} = {}_0F_1(-;d;x).$$

Application of the upper bound from (26) to the above inequality would prove (28). In order to find such a d, it suffices to satisfy $f_1 d \le 1$ and

$$\frac{f_{n+1}(d)_{n+1}}{f_n(d)_n} = \frac{(a_1+n)\cdots(a_p+n)(d+n)}{(b_1+n)\cdots(b_q+n)} \le 1.$$

In turn, the above inequality holds if (recall that $q = p+1$)

$$e_i(a_1,a_2,\dots,a_p,d) \le e_i(b_1,b_2,\dots,b_q), \quad i = 1,2,\dots,q,$$

or

$$e_i(a_1,a_2,\dots,a_p) + de_{i-1}(a_1,a_2,\dots,a_p) \le e_i(b_1,b_2,\dots,b_q), \quad i = 1,2,\dots,q.$$

For these q inequalities to be satisfied, we must choose d in accordance with (27). □

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