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SOME INEQUALITIES FOR TRIGONOMETRIC POLYNOMIALS AND FOURIER COEFFICIENTS

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The Bernstein inequalities for trigonometric polynomials are generalized. For sums of Fourier coefficients, upper bounds with certain constants in terms of quantities characterizing structural properties of functions are obtained. Bibliography: 9 *titles.*

Everywhere below, $\mathbb{R}$, $\mathbb{Z}_+$, and $\mathbb{N}$ are the sets of reals, nonnegative integers, and positive integers, respectively; $\|f\| = \max_{x\in\mathbb{R}} |f(x)|$; all the functions are real-valued, and H_n is the set of trigonometric polynomials of order not exceeding n. By L_p, where $1 \le p < \infty$, we denote the set of 2π-periodic measurable functions such that $\|f\|_p = \left(\int\limits_{-\pi}^{\pi} |f|^p\right)^{1/p} < \infty$. For $f \in L_1$ we set

$$a_k(f) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\cos kx\,dx, \quad b_k(f) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\sin kx\,dx,$$

$$\rho_k(f) = \sqrt{a_k^2(f) + b_k^2(f)}.$$

The following inequality, due to S. N. Bernshtein, is well known.

Theorem A (see [1, p. 47; 2])**.** *Let $n \in \mathbb{N}$, $T \in H_n$, and let a point $x_0 \in \mathbb{R}$ be such that $T(x_0) = \|T\|$. Then, for $t \in \left[-\frac{\pi}{n}, \frac{\pi}{n}\right]$,*

$$T(x_0 + t) \ge \|T\| \cos nt. \tag{1}$$

For polynomials of the form $T(x) = a\cos nx + b\sin nx$ inequality (1) *turns into an equality.*

In Sec. 1 of the present paper, some generalizations of inequality (1) are obtained.
In Sec. 2, for sums of the form

$$\sum_{k=n}^{\infty} k^{\alpha}\rho_k(f) \tag{2}$$

some upper bounds in terms of quantities characterizing the structural properties of the function f are established.

Sums of the form (2) were considered by several authors (for instance, see [1, pp. 647–648; 3]). The methods used in the present paper enable one to obtain the established inequalities with explicit constants.

1. Inequalities for trigonometric polynomials

Let a function f be given on $\mathbb{R}$ and let it be integrable on every finite interval; let $h > 0$, and let $r - 1 \in \mathbb{N}$.

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The Steklov function of the first order for f with step h is the function $S_{h,1}(f)$ defined by the formula

$$S_{h,1}(f,x) = \frac{1}{h}\int_{-h/2}^{h/2} f(x+t)\,dt.$$

The Steklov function of order r for the function f with step h is the function

$$S_{h,r}(f,x) = S_{h,1}(S_{h,r-1}(f),x).$$

For $r \in \mathbb{N}$, set

$$\psi_r(t) = \begin{cases} \frac{1}{(r-1)!} \sum\limits_{0\le k<|t|+\frac{r}{2}} (-1)^k C_r^k \left(|t| + \frac{r}{2} - k\right)^{r-1} & \text{if } |t| \le r/2, \\ 0 & \text{otherwise;} \end{cases}$$

$$\psi_{h,r}(t) = \frac{1}{h}\psi_r\left(\frac{t}{h}\right).$$

By $\delta_t^r(f,x)$ denote the central difference of order r of the function f with step t at a point x:

$$\delta_t^r(f,x) = \sum_{m=0}^{r} (-1)^m C_r^m f(x + rt/2 - mt).$$

If $b < a$, then we set $\sum\limits_a^b = 0$.

Theorem 1. *Let $n \in \mathbb{N}$, $T \in H_n$, $t \in \left[-\frac{\pi}{n}, \frac{\pi}{n}\right]$, $k \in \mathbb{Z}_+$, and let a point x_0 be such that $T^{(2k)}(x_0) = \|T^{(2k)}\|$. Then*

$$T(x_0+t) - \sum_{l=0}^{2k-1} \frac{T^{(l)}(x_0)}{l!} t^l \ge \frac{(-1)^k}{n^{2k}} \left(\cos nt - \sum_{l=0}^{k-1} \frac{(-1)^l}{(2l)!}(nt)^{2l}\right) \|T^{(2k)}\|. \tag{3}$$

For polynomials of the form $T(x) = a\cos nx + b\sin nx$ inequality (3) *turns into an equality.*

Proof. For $k = 0$ inequalities (1) and (3) coincide. Assume that $k \in \mathbb{N}$.

By applying inequality (1) to the polynomial $T^{(2k)}$, we obtain

$$T^{(2k)}(x_0 + tu) \ge \|T^{(2k)}\| \cos ntu, \tag{4}$$

where $u \in [0,1]$. Multiplying (4) by $\dfrac{t^{2k}(1-u)^{2k-1}}{(2k-1)!}$ and integrating with respect to u, we have

$$\frac{t^{2k}}{(2k-1)!}\int_0^1 (1-u)^{2k-1} T^{(2k)}(x_0+tu)\,du \ge \|T^{(2k)}\| \frac{t^{2k}}{(2k-1)!}\int_0^1 (1-u)^{2k-1}\cos ntu\,du.$$

It remains to apply the Taylor formula:

$$T(x_0+t) - \sum_{l=0}^{2k-1} \frac{T^{(l)}(x_0)}{l!} t^l = \frac{t^{2k}}{(2k-1)!}\int_0^1 (1-u)^{2k-1} T^{(2k)}(x_0+tu)\,du,$$

$$\cos nt - \sum_{l=0}^{k-1} \frac{(-1)^l}{(2l)!}(nt)^{2l} = \frac{(-1)^k n^{2k} t^{2k}}{(2k-1)!}\int_0^1 (1-u)^{2k-1}\cos ntu\,du.$$

Straightforward computations yield that for

$$T(x) = a\cos nx + b\sin nx = \sqrt{a^2+b^2}\cos(nx+\varphi)$$

and

$$x_0 = -\frac{\varphi}{n} + \frac{\pi\left(1+(-1)^{k+1}\right)}{2n}$$

inequality (3) with $|t| \le \frac{\pi}{n}$ becomes an equality. □

Corollary 1. *Let $n \in \mathbb{N}$, $T \in H_n$, $t \in \left[-\frac{\pi}{n}, \frac{\pi}{n}\right]$, $k \in \mathbb{N}$, and let a point x_0 be such that $|T^{(2k)}(x_0)| = \|T^{(2k)}\|$. Then*

$$\left|T(x_0+t) - \sum_{l=0}^{2k-1} \frac{T^{(l)}(x_0)}{l!} t^l\right| \ge \frac{1}{n^{2k}} \left|\cos nt - \sum_{l=0}^{k-1} \frac{(-1)^l}{(2l)!}(nt)^{2l}\right| \|T^{(2k)}\|. \tag{5}$$

For polynomials of the form $T(x) = a\cos nx + b\sin nx$ inequality (5) turns into an equality.

Proof. If $T^{(2k)}(x_0) = \|T^{(2k)}\|$, then, using the relation

$$(-1)^k \left(\cos x - \sum_{l=0}^{k-1} \frac{(-1)^l}{(2l)!} x^{2l}\right) \ge 0 \qquad (x \in \mathbb{R}),$$

we ascertain that inequalities (5) and (3) coincide.

If $T^{(2k)}(x_0) = -\|T^{(2k)}\|$, then the result established is applied to $-T$. □

Theorem 2. *Let $n \in \mathbb{N}$, $T \in H_n$, $k \in \mathbb{Z}_+$, and let a point x_0 be such that $T^{(2k+1)}(x_0) = \|T^{(2k+1)}\|$.*

If $t \in \left[0, \frac{\pi}{n}\right]$, then

$$T(x_0+t) - \sum_{l=0}^{2k} \frac{T^{(l)}(x_0)}{l!} t^l \ge \frac{(-1)^k}{n^{2k+1}} \left(\sin nt - \sum_{l=0}^{k-1} \frac{(-1)^l}{(2l+1)!}(nt)^{2l+1}\right) \|T^{(2k+1)}\|, \tag{6}$$

if $t \in \left[-\frac{\pi}{n}, 0\right]$, then

$$T(x_0+t) - \sum_{l=0}^{2k} \frac{T^{(l)}(x_0)}{l!} t^l \le \frac{(-1)^k}{n^{2k+1}} \left(\sin nt - \sum_{l=0}^{k-1} \frac{(-1)^l}{(2l+1)!}(nt)^{2l+1}\right) \|T^{(2k+1)}\|. \tag{7}$$

For polynomials of the form $T(x) = a\cos nx + b\sin nx$ inequalities (6) and (7) turn into equalities.

Proof. If $u \in [0,1]$, then, applying inequality (1) to the polynomial $T^{(2k+1)}$, we have

$$T^{(2k+1)}(x_0+tu) \ge \|T^{(2k+1)}\| \cos ntu. \tag{8}$$

Let $t \in \left[0, \frac{\pi}{n}\right]$. On multiplying inequality (8) by $\frac{t^{2k+1}(1-u)^{2k}}{(2k)!}$ and integrating with respect to u, we obtain

$$\frac{t^{2k+1}}{(2k)!} \int_0^1 T^{(2k+1)}(x_0+tu)(1-u)^{2k}\,du \ge \|T^{(2k+1)}\| \frac{t^{2k+1}}{(2k)!} \int_0^1 (1-u)^{2k} \cos ntu\,du. \tag{9}$$

Combining (9) with the Taylor expansions

$$T(x_0+t) - \sum_{l=0}^{2k} \frac{T^{(l)}(x_0)}{l!} t^l = \frac{t^{2k+1}}{(2k)!} \int_0^1 (1-u)^{2k} T^{(2k+1)}(x_0+tu)\,du$$

and

$$\sin nt - \sum_{l=0}^{k-1} \frac{(-1)^l}{(2l+1)!}(nt)^{2l+1} = \frac{(-1)^k n^{2k+1} t^{2k+1}}{(2k)!} \int_0^1 (1-u)^{2k} \cos ntu \, du,$$

we arrive at (6).

Straightforward computations show that for

$$T(x) = a\cos nx + b \sin nx = \sqrt{a^2+b^2}\cos(nx+\varphi)$$

and

$$x_0 = -\frac{\varphi}{n} + (-1)^{k+1}\frac{\pi}{2n}$$

inequality (6) with $|t| \le \frac{\pi}{n}$ becomes an equality.

If $t \in \left[-\frac{\pi}{n}, 0\right]$, then (7) is proved in a similar fashion, the only difference being that the sign of inequality (9) is changed for the opposite one. □

Corollary 2. *Let $n \in \mathbb{N}$, $T \in H_n$, $t \in \left[-\frac{\pi}{n}, \frac{\pi}{n}\right]$, $k \in \mathbb{Z}_+$, and let a point x_0 be such that $|T^{(2k+1)}(x_0)| = \|T^{(2k+1)}\|$. Then*

$$\left| T(x_0+t) - \sum_{l=0}^{2k} \frac{T^{(l)}(x_0)}{l!} t^l \right| \ge \frac{1}{n^{2k+1}} \left| \sin nt - \sum_{l=0}^{k-1} \frac{(-1)^l}{(2l+1)!}(nt)^{2l+1} \right| \|T^{(2k+1)}\|. \qquad (10)$$

For polynomials of the form $T(x) = a\cos nx + b\sin nx$ inequality (10) *turns into an equality.*

Proof. Let $T^{(2k+1)}(x_0) = \|T^{(2k+1)}\|$, $t \in \left[0, \frac{\pi}{n}\right]$. For $k = 0$, inequality (10) stems from (6) and the inequality $\sin nt \ge 0$. For $k \in \mathbb{N}$, (10) follows from (6) and the inequality

$$(-1)^k \left(\sin x - \sum_{l=0}^{k-1} \frac{(-1)^l}{(2l+1)!} x^{2l+1} \right) \ge 0 \qquad (x \ge 0).$$

If $t \in \left[-\frac{\pi}{n}, 0\right]$, then for $k = 0$ inequality (10) is implied by (7) and the relation

$$(-1)^k \left(\sin x - \sum_{l=0}^{k-1} \frac{(-1)^l}{(2l+1)!} x^{2l+1} \right) \le 0 \qquad (x \le 0).$$

In the case where $T^{(2k+1)}(x_0) = -\|T^{(2k+1)}\|$, the above-proved assertion should be applied to the polynomial $-T$. □

Corollary 3. *Let $n \in \mathbb{N}$, $T \in H_n$, $m \in \mathbb{Z}_+$, $t \in \left[0, \frac{\pi}{n}\right]$, and let a point x_0 be such that $T^{(m)}(x_0) = \|T^{(m)}\|$.*

If $m = 2k$ ($k \in \mathbb{Z}_+$), then

$$T(x_0+t) + T(x_0-t) - 2\sum_{l=0}^{k-1} \frac{T^{(2l)}(x_0)}{(2l)!} t^{2l} \ge \frac{2(-1)^k}{n^{2k}} \left(\cos nt - \sum_{l=0}^{k-1} \frac{(-1)^l}{(2l)!}(nt)^{2l} \right) \|T^{(2k)}\|; \quad (11)$$

if $m = 2k+1$ ($k \in \mathbb{Z}_+$), then

$$T(x_0+t) - T(x_0-t) - 2\sum_{l=0}^{k-1} \frac{T^{(2l+1)}(x_0)}{(2l+1)!} t^{2l+1}$$

$$\ge \frac{2(-1)^k}{n^{2k+1}} \left(\sin nt - \sum_{l=0}^{k-1} \frac{(-1)^l}{(2l+1)!}(nt)^{2l+1} \right) \|T^{(2k+1)}\|. \quad (12)$$

For polynomials of the form $T(x) = a\cos nx + b\sin nx$ inequalities (11) *and* (12) *turn into equalities.*

Proof. Applying (3), we have

$$T(x_0 - t) - \sum_{l=0}^{2k-1} \frac{T^{(l)}(x_0)}{l!}(-1)^l t^l \geq \frac{(-1)^k}{n^{2k}} \left(\cos nt - \sum_{l=0}^{k-1} \frac{(-1)^l}{(2l)!}(nt)^{2l} \right) \|T^{(2k)}\|. \qquad (13)$$

In order to prove (11), suffice it to sum (3) and (13).

Similarly, using (7), we derive

$$-T(x_0 - t) + \sum_{l=0}^{2k} \frac{T^{(l)}(x_0)}{l!}(-1)^l t^l \geq \frac{(-1)^k}{n^{2k+1}} \left(\sin nt - \sum_{l=0}^{k-1} \frac{(-1)^l}{(2l+1)!}(nt)^{2l+1} \right) \|T^{(2k+1)}\|.$$

Summing the inequality obtained with (6), we come to (12). □

Remark 1. For $m = 1$, inequality (12) is presented in [2].

Corollary 4. *Let $n \in \mathbb{N}$, $T \in H_n$, $m \in \mathbb{N}$, $t \in \left[0, \frac{\pi}{n}\right]$, and let a point x_0 be such that $|T^{(m)}(x_0)| = \|T^{(m)}\|$.*

If $m = 2k$ $(k \in \mathbb{N})$, then

$$\left| T(x_0 + t) + T(x_0 - t) - 2\sum_{l=0}^{k-1} \frac{T^{(2l)}(x_0)}{(2l)!} t^{2l} \right| \geq \frac{2}{n^{2k}} \left| \cos nt - \sum_{l=0}^{k-1} \frac{(-1)^l}{(2l)!}(nt)^{2l} \right| \|T^{(2k)}\|; \qquad (14)$$

if $m = 2k + 1$ $(k \in \mathbb{Z}_+)$, then

$$\left| T(x_0 + t) - T(x_0 - t) - 2\sum_{l=0}^{k-1} \frac{T^{(2l+1)}(x_0)}{(2l+1)!} t^{2l+1} \right|$$
$$\geq \frac{2}{n^{2k+1}} \left| \sin nt - \sum_{l=0}^{k-1} \frac{(-1)^l}{(2l+1)!}(nt)^{2l+1} \right| \|T^{(2k+1)}\|. \qquad (15)$$

For polynomials of the form $T(x) = a\cos nx + b\sin nx$ inequalities (14) *and* (15) *turn into equalities.*

Inequalities (14) and (15) are proved similarly to inequalities (5) and (10), respectively.

Remark 2. For $m = 1$, inequality (15) was presented in [4, p. 227].

Corollary 5. *Let $n \in \mathbb{N}$, $T \in H_n$, $k \in \mathbb{Z}_+$, $h \in \left(0, \frac{\pi}{n}\right]$, and let a point x_0 be such that $T^{(2k)}(x_0) = \|T^{(2k)}\|$. Then*

$$S_{2h,1}(T, x_0) - \sum_{l=0}^{k-1} \frac{T^{(2l)}(x_0)}{(2l+1)!} h^{2l} \geq \frac{(-1)^k}{n^{2k+1}h} \left(\sin nh - \sum_{l=0}^{k-1} \frac{(-1)^l}{(2l+1)!}(nh)^{2l+1} \right) \|T^{(2k)}\| \qquad (16)$$

and

$$S_{h,2}(T, x_0) - 2\sum_{l=0}^{k-1} \frac{T^{(2l)}(x_0)}{(2l+2)!} h^{2l} \geq \frac{2(-1)^{k+1}}{n^{2k+2}h^2} \left(\cos nh - \sum_{l=0}^{k} \frac{(-1)^l}{(2l)!}(nh)^{2l} \right) \|T^{(2k)}\|. \qquad (17)$$

For polynomials of the form $T(x) = a\cos nx + b\sin nx$ inequalities (16) *and* (17) *turn into equalities.*

Proof. It is known (see [5, p. 100]) that

$$S_{h,r}(f, x) = \int_{-\frac{rh}{2}}^{\frac{rh}{2}} f(x + t)\psi_{h,r}(t)\, dt. \qquad (18)$$

In order to prove (16), it is sufficient to multiply (3) by $\psi_{2h,1}$, integrate from $-h$ to h, and use (18) and the following relations:

$$\int_{-h}^{h} t^{2l+1}\psi_{2h,1}(t)\,dt = 0, \qquad \int_{-h}^{h} t^{2l}\psi_{2h,1}(t)\,dt = \frac{h^{2l}}{2l+1} \qquad (l \in \mathbb{Z}_+),$$

$$\int_{-h}^{h} \psi_{2h,1}(t)\cos nt\,dt = \frac{\sin nh}{nh}.$$

Since

$$\int_{-h}^{h} t^{2l+1}\psi_{h,2}(t)\,dt = 0, \quad \int_{-h}^{h} t^{2l}\psi_{h,2}(t)\,dt = \frac{2h^{2l}}{(2l+1)(2l+2)} \qquad (l \in \mathbb{Z}_+),$$

$$\int_{-h}^{h} \psi_{h,2}(t)\cos nt\,dt = \frac{2(1-\cos nh)}{(nh)^2},$$

multiplying (3) by $\psi_{h,2}$, integrating from $-h$ to h, and taking into account (18), we have

$$S_{h,2}(T,x_0) - 2\sum_{l=0}^{k-1}\frac{T^{(2l)}(x_0)}{(2l+2)!}h^{2l} \geq \frac{2(-1)^k}{n^{2k}}\left(\frac{1-\cos nh}{(nh)^2} - \sum_{l=0}^{k-1}\frac{(-1)^l}{(2l+2)!}(nh)^{2l}\right)\|T^{(2k)}\|.$$

Upon nondifficult transformations of the right-hand side of the latter inequality, we obtain (17). $\square$

Corollary 6. *Let $n \in \mathbb{N}$, $T \in H_n$, $k \in \mathbb{N}$, $h \in \left(0, \frac{\pi}{n}\right]$, and let a point x_0 be such that $|T^{(2k)}(x_0)| = \|T^{(2k)}\|$. Then*

$$\left|S_{2h,1}(T,x_0) - \sum_{l=0}^{k-1}\frac{T^{(2l)}(x_0)}{(2l+1)!}h^{2l}\right| \geq \frac{1}{n^{2k+1}h}\left|\sin nh - \sum_{l=0}^{k-1}\frac{(-1)^l}{(2l+1)!}(nh)^{2l+1}\right|\|T^{(2k)}\| \qquad (19)$$

and

$$\left|S_{h,2}(T,x_0) - 2\sum_{l=0}^{k-1}\frac{T^{(2l)}(x_0)}{(2l+2)!}h^{2l}\right| \geq \frac{2}{n^{2k+2}h^2}\left|\cos nh - \sum_{l=0}^{k}\frac{(-1)^l}{(2l)!}(nh)^{2l}\right|\|T^{(2k)}\|. \qquad (20)$$

For polynomials of the form $T(x) = a\cos nx + b\sin nx$ inequalities (19) and (20) turn into equalities.

Theorem 3. *Let $n, m \in \mathbb{N}$, $b, h > 0$, $\frac{mhb}{2} \leq \frac{\pi}{n}$, $T \in H_n$, let a point $x_0 \in \mathbb{R}$ be such that $|T^{(m)}(x_0)| = \|T^{(m)}\|$, and let Φ be a nonnegative function integrable on $[0, b]$. Then*

$$\left|\int_0^b \delta_{th}^m(T,x_0)\Phi(t)\,dt\right| \geq \frac{2}{h}\|T^{(m)}\| \int_0^{\frac{mhb}{2}}\left(\int_{\frac{2u}{m}}^{hb} t^{m-1}\psi_m\left(\frac{u}{t}\right)\Phi\left(\frac{t}{h}\right)\,dt\right)\cos nu\,du. \qquad (21)$$

For polynomials of the form $T(x) = \alpha\cos nx + \beta\sin nx$ inequality (21) turns into an equality.

Proof. Let $T^{(m)}(x_0) = \|T^{(m)}\|$ (in the case where $T^{(m)}(x_0) = -\|T^{(m)}\|$, one should consider $-T$).

Set

$$A=\int_0^b \delta_{th}^m(T,x_0)\Phi(t)\,dt.$$

By using (18) and the relation

$$S_{h,r}^{(r)}(f,x)=\frac{1}{h^r}\delta_h^r(f,x),$$

we obtain

$$\begin{aligned}\delta_t^m(T,x)&=t^{m-1}\int_{-\frac{mt}{2}}^{\frac{mt}{2}} T^{(m)}(x+u)\psi_m\Big(\frac{u}{t}\Big)\,du\\ &=t^{m-1}\int_0^{\frac{mt}{2}}\Big(T^{(m)}(x+u)+T^{(m)}(x-u)\Big)\,\psi_m\Big(\frac{u}{t}\Big)\,du.\end{aligned}$$

Therefore,

$$\begin{aligned}A&=\frac{1}{h}\int_0^{hb}\delta_t^m(T,x_0)\Phi\Big(\frac{t}{h}\Big)\,dt\\ &=\frac{1}{h}\int_0^{hb}t^{m-1}\left(\int_0^{\frac{mt}{2}}(T^{(m)}(x_0+u)+T^{(m)}(x_0-u))\psi_m\Big(\frac{u}{t}\Big)du\right)\Phi\Big(\frac{t}{h}\Big)dt.\end{aligned}$$

Changing the order of integration, we have

$$A=\frac{1}{h}\int_0^{\frac{mbh}{2}}\Big(T^{(m)}(x_0+u)+T^{(m)}(x_0-u)\Big)\left(\int_{\frac{2u}{m}}^{hb}t^{m-1}\psi_m\left(\frac{u}{t}\right)\Phi\left(\frac{t}{h}\right)\,dt\right)\,du.$$

It remains to use inequality (1).
Obviously, for

$$T(x)=\alpha\cos nx+\beta\sin nx=\sqrt{\alpha^2+\beta^2}\cos{(nx+\varphi)}$$

and

$$x_0=-\frac{\varphi}{n}+\frac{\pi\left(1+(-1)^{m+1}\right)}{4n}$$

inequality (21) with $|t|\le\frac{\pi}{n}$ becomes an equality. □

Now we derive a number of corollaries, taking the Steklov kernels as Φ. Denote

$$C(m,b,r,h,n)=\frac{2}{h}\int_0^{\frac{mhb}{2}}\left(\int_{\frac{2u}{m}}^{hb}t^{m-1}\psi_m\left(\frac{u}{t}\right)\psi_r\left(\frac{t}{h}\right)\,dt\right)\cos nu\,du.$$

Corollary 7. *Let $n \in \mathbb{N}$, $0 < h \le \frac{4\pi}{n}$, $T \in H_n$, and let a point $x_0 \in \mathbb{R}$ be such that $|T'(x_0)| = \|T'\|$. Then*

$$\|T'\| \le \frac{n^2 h}{4(1-\cos\frac{nh}{4})} \left| \int_0^{\frac{1}{2}} \delta^1_{th}(T, x_0)\, dt \right|.$$

Proof. As is known (see, e.g., [6, p. 332]),

$$\int x^l \cos nx\, dx = \sum_{p=0}^{l} p!\, C_l^p \frac{x^{l-p}}{n^{p+1}} \sin\left(nx + p\frac{\pi}{2}\right) + C \quad (C \in \mathbb{R}). \tag{22}$$

Consequently,

$$I_1(c) = \int_0^c u \cos nu\, du = \frac{nc \sin nc + \cos nc}{n^2} - \frac{1}{n^2}. \tag{23}$$

Thus,

$$\begin{aligned} C\left(1, \frac{1}{2}, 1, h, n\right) &= \frac{2}{h} \int_0^{\frac{h}{4}} \left(\int_{2u}^{\frac{h}{2}} dt \right) \cos nu\, du = \frac{2}{h} \int_0^{\frac{h}{4}} \left(\frac{h}{2} - 2u \right) \cos nu\, du \\ &= \frac{2}{h} \left(\frac{h}{2n} \sin\frac{nh}{4} - 2I_1\left(\frac{h}{4}\right) \right) = \frac{4}{n^2 h} \left(1 - \cos\frac{nh}{4} \right). \end{aligned}$$

The proof is completed by applying Theorem 3 with $m = 1$, $b = \frac{1}{2}$, and $\Phi = \psi_1$. ☐

Corollary 8. *Let $n \in \mathbb{N}$, $T \in H_n$, and let a point $x_0 \in \mathbb{R}$ be such that $|T'(x_0)| = \|T'\|$. Then*

$$\|T'\| \le \frac{\pi n}{2} \left| \int_0^{\frac{1}{2}} \delta^1_{\frac{4\pi t}{n}}(T, x_0)\, dt \right|.$$

Proof. Set $h = \frac{4\pi}{n}$ in Corollary 7. ☐

Corollary 9. *Let $n \in \mathbb{N}$, $T \in H_n$, and let a point $x_0 \in \mathbb{R}$ be such that $|T'(x_0)| = \|T'\|$. Then*

$$\|T'\| \le \frac{\pi n}{2} \left| \int_0^{\frac{1}{2}} \delta^1_{\frac{2\pi t}{n}}(T, x_0)\, dt \right|.$$

Proof. Set $h = \frac{2\pi}{n}$ in Corollary 7. ☐

Corollary 10. *Let $n \in \mathbb{N}$, $0 < h \le \frac{2\pi}{n}$, $T \in H_n$, and let a point $x_0 \in \mathbb{R}$ be such that $|T'(x_0)| = \|T'\|$. Then*

$$\|T'\| \le \frac{n^3 h^2}{4(nh - 2\sin\frac{nh}{2})} \left| \int_0^1 \delta^1_{th}(T, x_0)(1-t)\, dt \right|.$$

Proof. In view of relation (22), we have

$$I_2(c) = \int_0^c u^2 \cos nu\, du = \frac{(n^2c^2 - 2)\sin nc + 2nc\cos nc}{n^3}. \tag{24}$$

Thus, with regard to (23) and (24), we obtain

$$
\begin{aligned}
C\left(1,1,2,h,n\right) &= \frac{2}{h}\int\limits_0^{\frac{h}{2}}\left(\int_{2u}^{h}\left(1-\frac{t}{h}\right)\,dt\right)\cos nu\,du = \frac{1}{h^2}\int\limits_0^{\frac{h}{2}}(h-2u)^2\cos nu\,du \\
&= \frac{1}{h^2}\left(h^2\int\limits_0^{\frac{h}{2}}\cos nu\,du - 4hI_1\left(\frac{h}{2}\right)+4I_2\left(\frac{h}{2}\right)\right) = \frac{4(nh-2\sin\frac{nh}{2})}{n^3h^2}.
\end{aligned}
\tag{25}
$$

The proof is completed by applying Theorem 3 with $m=1$, $b=1$, and $\Phi=\psi_2$. □

Corollary 11. *Let $n\in\mathbb{N}$, $T\in H_n$, and let a point $x_0\in\mathbb{R}$ be such that $|T'(x_0)|=\|T'\|$. Then*

$$
\|T'\| \le \frac{\pi n}{2}\left|\int\limits_0^1 \delta^1_{\frac{2\pi t}{n}}(T,x_0)(1-t)\,dt\right|.
$$

Proof. Set $h=\frac{2\pi}{n}$ in Corollary 10. □

Corollary 12. *Let $n\in\mathbb{N}$, $0<h\le\frac{2\pi}{n}$, $T\in H_n$, and let a point $x_0\in\mathbb{R}$ be such that $|T''(x_0)|=\|T''\|$. Then*

$$
\|T''\| \le \frac{n^3h}{nh-2\sin\frac{nh}{2}}\left|\int\limits_0^{\frac{1}{2}} \delta^2_{th}(T,x_0)\,dt\right|.
$$

Proof. Similarly to (25) , we have

$$
\begin{aligned}
C\left(2,\frac{1}{2},1,h,n\right) &= \frac{2}{h}\int\limits_0^{\frac{h}{2}}\left(\int_{u}^{\frac{h}{2}} t\left(1-\frac{u}{t}\right)\,dt\right)\cos nu\,du \\
&= \frac{1}{4h}\int\limits_0^{\frac{h}{2}}(h-2u)^2\cos nu\,du = \frac{nh-2\sin\frac{nh}{2}}{n^3h}.
\end{aligned}
$$

It remains to apply Theorem 3 with $m=2$, $b=\frac{1}{2}$, and $\Phi=\psi_1$. □

Corollary 13. *Let $n\in\mathbb{N}$, $T\in H_n$, and let a point $x_0\in\mathbb{R}$ be such that $|T''(x_0)|=\|T''\|$. Then*

$$
\|T''\| \le n^2\left|\int\limits_0^{\frac{1}{2}} \delta^2_{\frac{2\pi t}{n}}(T,x_0)\,dt\right|.
$$

Proof. Set $h=\frac{2\pi}{n}$ in Corollary 12. □

Corollary 14. *Let $n\in\mathbb{N}$, $0<h\le\frac{\pi}{n}$, $T\in H_n$, and let a point $x_0\in\mathbb{R}$ be such that $|T''(x_0)|=\|T''\|$. Then*

$$
\|T''\| \le \frac{n^4h^2}{n^2h^2+2\cos nh-2}\left|\int\limits_0^1 \delta^2_{th}(T,x_0)(1-t)\,dt\right|.
$$

Proof. Using relation (22), we derive

$$I_3(c)=\int_0^c u^3\cos nu du=\frac{(n^3c^3-6nc)\sin nc+(3n^2c^2-6)\cos nc}{n^4}+\frac{6}{n^4}. \tag{26}$$

Thus, with account for (23), (24), and (26), we obtain

$$C\,(2,1,2,h,n)=\frac{2}{h}\int_0^h\left(\int_u^h t\left(1-\frac{u}{t}\right)\left(1-\frac{t}{h}\right)\,dt\right)\cos nu\,du=\frac{1}{3h^2}\int_0^h (h-u)^3\cos nu\,du$$

$$=\frac{1}{3h^2}\left(h^3\int_0^h \cos nu\,du-3h^2I_1(h)+3hI_2(h)-I_3(h)\right)=\frac{2\cos nh+n^2h^2-2}{n^4h^2}.$$

It remains to apply Theorem 3 with $m=2$, $b=1$, and $\Phi=\psi_2$. □

Corollary 15. *Let $n\in\mathbb{N}$, $T\in H_n$, and let a point $x_0\in\mathbb{R}$ be such that $|T''(x_0)|=\|T''\|$. Then*

$$\|T''\|\le\frac{\pi^2n^2}{\pi^2-4}\left|\int_0^1 \delta^2_{\frac{\pi t}{n}}(T,x_0)(1-t)\,dt\right|.$$

Proof. Set $h=\frac{\pi}{n}$ in Corollary 14. □

2. Inequalities for the Fourier coefficients

2.1. First, we establish a simple lemma on positive series.

Lemma 1. *Let $a_k\ge 0$ for $k\in\mathbb{N}$, $m,r\in\mathbb{R}$, $p\in(0,1)$, $r-mp<0$, $q_n=\sum_{k=1}^n k^m a_k$. Then*

$$\sum_{k=1}^\infty k^r a_k^p\le(mp-r)\sum_{n=1}^\infty n^{r-p(m+1)}q_n^p.$$

Proof. Set

$$t_n=\sum_{k=1}^n k^{mp}a_k^p,\quad n\ge 1;\quad t_0=0,\ \alpha=r-mp.$$

Then

$$\sum_{k=1}^l k^r a_k^p=\sum_{k=1}^l k^\alpha(t_k-t_{k-1})=l^\alpha t_l+\sum_{k=1}^{l-1}\{k^\alpha-(k+1)^\alpha\}t_k. \tag{27}$$

Since the series

$$\sum_{k=1}^\infty\{k^\alpha-(k+1)^\alpha\}t_k$$

is convergent, we have $\lim_{l\to\infty} l^\alpha t_l=0$. Indeed, let $\varepsilon>0$ be fixed. Then, by the Cauchy convergence test, there is a number n_ε such that for all $n\ge n_\varepsilon$,

$$\varepsilon>\sum_{k=n}^{2n-1}\{k^\alpha-(k+1)^\alpha\}t_k\ge t_n n^\alpha(1-2^\alpha).$$

Therefore, from (27) it follows that

$$\sum_{k=1}^\infty k^r a_k^p=\sum_{k=1}^\infty\{k^\alpha-(k+1)^\alpha\}t_k. \tag{28}$$

Applying Hölder's inequality for sums, we conclude that $t_k \le k^{1-p} q_k^p$. Further, we have

$$\{k^\alpha - (k+1)^\alpha\} k^{1-p} = -\alpha k^{1-p} \int\limits_k^{k+1} t^{\alpha-1}\, dt \le -\alpha \int\limits_k^{k+1} t^{\alpha-p}\, dt \le -\alpha k^{\alpha-p}.$$

The proof is completed by combining the inequalities obtained with (28). □

Remark 3. In connection with assertions similar to Lemma 1, see [7, pp. 306–308; 8].

2.2. For $f \in L_1$, set

$$S_k(f,x) = \frac{a_0(f)}{2} + \sum_{l=1}^{k} (a_l(f)\cos lx + b_l(f)\sin lx).$$

Theorem 4. *Let $f \in L_1$, $p \in (0,1)$, $m, n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $r - 2mp < 0$. Then*

$$\sum_{k=n}^{\infty} k^r \rho_k^{2p}(f) \le (2mp - r)\pi^{-p} \sum_{k=n}^{\infty} k^{r-p(2m+1)} \|S_k^{(m)}(f)\|_2^{2p}. \tag{29}$$

Proof. Setting $a_k = \rho_k^2(f)$ in Lemma 1, changing m for $2m$, and taking into account the relation

$$\|S_k^{(m)}(f)\|_2^2 = \pi \sum_{l=1}^{k} l^{2m} \rho_l^2(f),$$

we come to (29) with $n = 1$. It remains to apply the inequality obtained to the function $f - S_{n-1}(f)$. □

2.3. We will need the following two known assertions.

Theorem B (see [4, p. 230])**.** *Let $n, r \in \mathbb{N}$, $T \in H_n$. Then*

$$\|T^{(r)}\|_2 \le \left(\frac{n}{2}\right)^r \left\|\delta_{\frac{\pi}{n}}^r(T)\right\|_2.$$

By $W_p^{(r)}$ denote the set of 2π-periodic continuous functions whose derivatives of order $r-1$ are absolutely continuous on every interval, whereas the derivatives of order r belong to L_p.

Theorem C (see [9, p. 136])**.** *Let $r \in \mathbb{N}$, $1 \le p \le q \le \infty$, $f \in W_p^{(1)}$. Then*

$$\|\delta_h^{r+1}(f)\|_q \le |h|^{1-\frac{1}{p}+\frac{1}{q}} \|\delta_h^r(f')\|_p.$$

Theorem 5. *Let $1 \le q \le 2$, $p \in (0,1)$, $m, n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $r - 2mp < 0$, $f \in L_q$. Then*

$$\sum_{k=n}^{\infty} k^r \rho_k^{2p}(f) \le \frac{(2mp-r)\pi^{2p\left(1-\frac{1}{q}\right)}}{2^{(m+1)2p}} \sum_{k=n}^{\infty} k^{r-2p\left(1-\frac{1}{q}\right)} \|\delta_{\frac{\pi}{k}}^m(f)\|_q^{2p}. \tag{30}$$

Proof. Without loss of generality, we may assume that $a_0(f) = 0$. By $f^{(-1)}$ denote the primitive of f such that $\int\limits_{-\pi}^{\pi} f^{(-1)} = 0$. Using Theorems B and C, we derive

$$\begin{aligned} \|S_k^{(m)}(f)\|_2 &= \|S_k^{(m+1)}(f^{(-1)})\|_2 \le \left(\frac{k}{2}\right)^{m+1} \|\delta_{\frac{\pi}{k}}^{m+1}(S_k(f^{(-1)}))\|_2 \\ &\le \left(\frac{k}{2}\right)^{m+1} \|\delta_{\frac{\pi}{k}}^{m+1}(f^{(-1)})\|_2 \le \left(\frac{k}{2}\right)^{m+1} \left(\frac{\pi}{k}\right)^{\frac{3}{2}-\frac{1}{q}} \|\delta_{\frac{\pi}{k}}^m(f)\|_q. \end{aligned} \tag{31}$$

It remains to combine (29) with (31). □

Remark 4. For $q = 2$, inequality (30) can be changed for the stronger relation

$$\sum_{k=n}^{\infty} k^r \rho_k^{2p}(f) \leq (2mp - r)4^{-mp}\pi^{-p} \sum_{k=n}^{\infty} k^{r-p} \|\delta_{\frac{\pi}{k}}^m(f)\|_2^{2p},$$

which is obtained by combining (29) with Theorem B.

In conclusion, we indicate that Sec. 1 was written jointly by both authors, whereas Sec. 2 was written by V. V. Zhuk.

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