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APPROXIMATION OF PERIODIC FUNCTIONS BY MODIFIED STEKLOV AVERAGES IN L_2

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In the space L_2 of periodic functions, sharp (in the sense of constants) lower estimates for the deviation of the modified Steklov functions of the first and second orders in terms of the modulus of continuity are established. Similar results are also obtained for even continuous periodic functions with nonnegative Fourier coefficients in the space C. Bibliography: 3 titles.

Let L_2 be the space of 2π-periodic complex-valued functions f square-integrable on the period, and let

$$S_{h,1}(f,x) = \frac{1}{h}\int\limits_{-h/2}^{h/2} f(x+t)\,dt$$

and

$$S_{h,2}(f,x) = \frac{1}{h}\int\limits_{-h}^{h} f(x+t)\left(1-\left|\frac{t}{h}\right|\right)\,dt$$

be the Steklov functions of the first and second orders, respectively.

Set

$$U_{h,r}(f,x) = \frac{1}{3}(4S_{h,r}(f,x) - S_{2h,r}(f,x)),$$
$$U_{h,r,l}(f) = (E-(E-U_{h,r})^l)(f).$$

The main issue considered in the present paper is the question on the least constant $C(r,l,a)$ in the inequality

$$\omega_{4l}(f,ah)_2 \le C(r,l,a)\|f-U_{h,r,l}(f)\|_2$$

for $r=1$ and $r=2$, where $\omega_k(f,h)_2$ is the modulus of continuity of order k of a function f in the space L_2. The counterparts of the results obtained for the case of approximation of even continuous periodic functions with nonnegative Fourier coefficients in the space C are established in Sec. 3.2.

1. Introduction

1.1. In what follows, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, and $\mathbb{N}$ are the sets of complex, real, nonnegative real, integer, and positive integer numbers; L_p (with $1 \le p < \infty$) is the space of 2π-periodic complex-valued functions f p-integrable on the interval $Q=[-\pi,\pi]$ with the norm

$$\|f\|_p = \left(\int\limits_Q |f(x)|^p\right)^{1/p};$$

$L_\infty = C$ is the space of continuous 2π-periodic functions $f\colon \mathbb{R}\to\mathbb{C}$ with the norm

$$\|f\|_\infty = \|f\| = \max_{x\in\mathbb{R}}|f(x)|;$$

E is the identity operator in L_1.

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Functions given on subsets of $\mathbb{R}$ that have a removable singularity at a certain point and are not defined at this point are defined at it by continuity. In other cases, the symbol $0/0$ is understood as 0.

By $\delta_t^r(f,x)$ the central difference of order r of a function f with step t at a point x is denoted, i.e.,

$$\delta_t^r(f,x) = \sum_{m=0}^{r} (-1)^m C_r^m f\left(x + \tfrac{rt}{2} - mt\right).$$

For $f \in L_p$, and $r \in \mathbb{N}$, we set

$$\omega_r(f,h)_p = \sup_{|t|\le h} \|\delta_t^r(f)\|_p.$$

The quantity $\omega_r(f)_p$ is called the modulus of continuity of order r of the function f in the space L_p.

For $f \in L_1$, $h > 0$, $r - 1 \in \mathbb{N}$, and $x \in \mathbb{R}$, we set

$$S_{h,1}(f,x) = \frac{1}{h}\int_{-h/2}^{h/2} f(x+t)dt,$$

$$S_{h,r}(f,x) = S_{h,1}(S_{h,r-1}(f),x).$$

The function $S_{h,r}(f)$ is called the Steklov function of order r with step h for the function f. For $r \in \mathbb{N}$, we set

$$\psi_r(t) = \begin{cases} r \sum\limits_{0\le k<|t|+r/2} \frac{(-1)^k\left(|t|+\frac{r}{2}-k\right)^{r-1}}{k!(r-k)!} & \text{if } |t| \le \frac{r}{2}, \\ 0 & \text{if } |t| > \frac{r}{2}. \end{cases}$$

If $f \in L_1$ and $x \in \mathbb{R}$, then (see [1, p. 100])

$$S_{h,r}(f,x) = \int_{\mathbb{R}} f(x+th)\psi_r(t)dt.$$

Let $f \in L_1$, r, $m \in \mathbb{N}$. Then

$$S_{h,r,m}(f,x) = \frac{2}{C_{2m}^m}\sum_{k=1}^{m}(-1)^{k+1}C_{2m}^{m+k}S_{kh,r}(f,x) = \frac{2(-1)^{m+1}}{C_{2m}^m}\int_{\mathbb{R}_+}\delta_{th}^{2m}(f,x)\psi_r(t)dt + f(x).$$

If $1 \le p \le \infty$, $f \in L_p$, and $h > 0$, then

$$\begin{aligned} \|f - S_{h,r,m}(f)\|_p &= \left\| \frac{2}{C_{2m}^m}\int_{\mathbb{R}_+}\delta_{th}^{2m}(f)\psi_r(t)dt \right\|_p \le \frac{2}{C_{2m}^m}\int_{\mathbb{R}_+}\|\delta_{th}^{2m}(f)\|_p\psi_r(t)dt \\ &\le \frac{2}{C_{2m}^m}\int_{\mathbb{R}_+}\omega_{2m}(f,th)_p\psi_r(t)dt \qquad (1) \\ &\le \frac{2}{C_{2m}^m}\omega_{2m}(f,\tfrac{rh}{2})_p\int_{\mathbb{R}_+}\psi_r(t)dt = \frac{1}{C_{2m}^m}\omega_{2m}(f,\tfrac{rh}{2})_p. \end{aligned}$$

Let $l \in \mathbb{N}$, $f \in L_1$. We set

$$U_{h,r}(f,x) = S_{h,r,2}(f,x) = \frac{1}{3}(4S_{h,r}(f,x) - S_{2h,r}(f,x)),$$
$$U_{h,r,l}(f) = (E - (E - U_{h,r})^l)(f).$$

The paper mainly deals with the question on the least constant $C(r,l,a)$ in the inequality

$$\omega_{4l}(f,ah)_2 \le C(r,l,a)\|f - U_{h,r,l}(f)\|_2$$

for $r = 1$ and $r = 2$. In particular, it is proved (see Theorems 1 and 2) that

$$\sup_{h>0}\sup_{f\in L_2} \frac{\omega_{4l}(f,ah)_2}{\|f - U_{h,1,l}(f)\|_2} = 2^{4l}\big(30a^4\big)^l \quad \big(a \ge \tfrac{1}{2},\ l \in \mathbb{N}\big) \tag{2}$$

and

$$\sup_{h>0}\sup_{f\in L_2} \frac{\omega_{4l}(f,ah)_2}{\|f - U_{h,2,l}(f)\|_2} = 2^{4l}\big(\tfrac{45}{8}a^4\big)^l \quad \big(a \ge \tfrac{3}{4},\ l \in \mathbb{N}\big). \tag{3}$$

In connection with relations (2) and (3), it is appropriate to mention that for $l \in \mathbb{N}$, $r \in \mathbb{N}$, $h > 0$, $1 \le p \le \infty$, and $f \in L_p$, the following inequality holds:

$$\|f - U_{h,r,l}(f)\|_p \le C(r,l)\omega_{4l}(f,h)_p. \tag{4}$$

Relation (4) can be readily proved based on Theorem 1 in the monograph [2, p. 201], but we do not dwell on it.

1.2. Here, we present a brief overview of the results obtained in the paper. Introduce the following notation:

$$\alpha_r(x) = 1 - \frac{4}{3}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^r + \frac{1}{3}\left(\frac{\sin x}{x}\right)^r;$$

for $a > 0$,

$$D(a) = \sup_{x\in\mathbb{R}} \frac{\sin^4 ax}{\alpha_1(2x)}, \qquad G(a) = \sup_{x\in\mathbb{R}} \frac{\sin^4 ax}{\alpha_2(2x)}.$$

The paper consists of three sections. In Sec. 2, it is proved that $D(a) = 30a^4$ for $a \ge \frac{1}{2}$ and $G(a) = \frac{45}{8}a^4$ for $a \ge \frac{3}{4}$; also the values of $D(a)$ and $G(a)$ for some other values of a, obtained by a computing technique, are provided. In Sec. 3, relations (2) and (3) are established (some auxiliary results used in proving (2) and (3) are stated in Sec. 2), and the counterparts of relations (2) and (3) for the case of approximation of even continuous periodic functions with nonnegative Fourier coefficients in the space C are presented.

2. Auxiliary results

2.1. Lemma 1. *Let $a \ge \frac{1}{2}$. Then*

$$D(a) = 30a^4.$$

Proof. Set $b = \frac{1}{2}$,

$$f_a(x) = \frac{\sin^4 ax}{\alpha_1(2x)}.$$

As is readily seen

$$f_a(0) = \lim_{x\to 0} f_a(x) = 30a^4.$$

Thus, it is sufficient to prove that for all $x > 0$ and $a \ge b$,

$$f_a(x) \le 30a^4. \tag{5}$$

Assume that (5) is established for $a = b$. Then, taking into account that for $0 \le t \le \frac{3\pi}{4}$ and $\alpha \ge 1$,

$$|\sin \alpha t| \le \alpha \sin t, \tag{6}$$

for $0 < x < 3\pi/2$ we derive

$$f_a(x) = \frac{\sin^4 \frac{a}{b} bx}{\alpha_1(2x)} \le \left(\frac{a}{b}\right)^4 \frac{\sin^4 bx}{\alpha_1(2x)} \le \left(\frac{a}{b}\right)^4 30b^4 = 30a^4.$$

In the case where $x \ge 3\pi/2$ and $a \ge b$, we have

$$f_a(x) = \frac{\sin^4 ax}{1 - \frac{4}{3}\frac{\sin x}{x} + \frac{1}{3}\frac{\sin 2x}{2x}} \le \frac{1}{1 - \frac{4}{3x} - \frac{1}{6x}} = \frac{1}{1 - \frac{3}{2x}} \le \frac{\pi}{\pi - 1} < \frac{15}{8} \le 30a^4.$$

Prove the inequality $f_b(x) \le \frac{15}{8}$ for $x \in \mathbb{R}$. To this end, suffice it to ascertain that for $x > 0$,

$$\sin^4 \frac{x}{2} \le \frac{15}{8}\left(1 - \frac{4}{3}\frac{\sin x}{x} + \frac{1}{3}\frac{\sin 2x}{2x}\right). \tag{7}$$

Inequality (7) holds trivially for $x \ge \frac{45}{14}$. Therefore, it is sufficient to consider the case where $0 < x < \frac{45}{14}$. Using the known relations

$$\begin{aligned} \frac{\sin t}{t} &\ge \sum_{k=0}^{n} (-1)^k \frac{t^{2k}}{(2k+1)!} \quad \left(t \in \mathbb{R},\ \tfrac{n+1}{2} \in \mathbb{N}\right), \\ \frac{\sin t}{t} &\le \sum_{k=0}^{n} (-1)^k \frac{t^{2k}}{(2k+1)!} \quad \left(t \in \mathbb{R},\ \tfrac{n+2}{2} \in \mathbb{N}\right), \\ \sin^4 t &\le \frac{1}{2}\sum_{k=2}^{n} (-1)^{k+1} \frac{(2t)^{2k}}{(2k)!} + \frac{1}{8}\sum_{k=2}^{m} (-1)^k \frac{(4t)^{2k}}{(2k)!} \quad \left(t \in \mathbb{R}, \tfrac{n-1}{2}, \tfrac{m}{2} \in \mathbb{N}\right), \end{aligned} \tag{8}$$

we strengthen inequality (7) as follows:

$$\begin{aligned} &\frac{1}{2}\sum_{k=2}^{5} (-1)^{k+1} \frac{x^{2k}}{(2k)!} + \frac{1}{8}\sum_{k=2}^{6} (-1)^k \frac{(2x)^{2k}}{(2k)!} \\ &\quad - \frac{15}{8}\left(1 - \frac{4}{3}\sum_{k=0}^{6} (-1)^k \frac{x^{2k}}{(2k+1)!} + \frac{1}{3}\sum_{k=0}^{5} (-1)^k \frac{(2x)^{2k}}{(2k+1)!}\right) \le 0. \end{aligned} \tag{9}$$

In view of (9), we must prove that for $0 < x < \frac{45}{14}$,

$$\begin{aligned} &\frac{1}{2}\sum_{k=3}^{5} (-1)^{k+1} \frac{x^{2k}}{(2k)!} + \frac{1}{8}\sum_{k=3}^{6} (-1)^k \frac{(2x)^{2k}}{(2k)!} \\ &\quad - \frac{15}{8}\left(- \frac{4}{3}\sum_{k=3}^{6} (-1)^k \frac{x^{2k}}{(2k+1)!} + \frac{1}{3}\sum_{k=3}^{5} (-1)^k \frac{(2x)^{2k}}{(2k+1)!}\right) = x^6 p(x) \le 0. \end{aligned} \tag{10}$$

Straightforward computations show that $p\left(\frac{45}{14}\right) < 0$ and $p'(x) \ge 0$ for $x > 0$, implying that (10) holds for $0 < x < \frac{45}{14}$. □

Remark 1. *If* $0 < a \le \frac{1}{2}$, *then* $D(a) \le D\left(\frac{1}{2}\right) = \frac{15}{8}$.

Proof. The function $\sin t$ is increasing on $(0, \frac{\pi}{2}]$. Therefore, for $a \le \frac{1}{2}$ and $x \in (0, \pi]$, we have

$$f_a(x) = \frac{\sin^4 ax}{\alpha_1(2x)} \le \frac{\sin^4 \frac{1}{2}x}{\alpha_1(2x)} = f_{\frac{1}{2}}(x) \le \frac{15}{8}.$$

If $x > \frac{45}{14}$, then

$$f_a(x) \le \frac{1}{1 - \frac{4}{3}\frac{\sin x}{x} + \frac{1}{3}\frac{\sin 2x}{2x}} \le \frac{1}{1 - \frac{4}{3x} - \frac{1}{6x}} = \frac{1}{1 - \frac{3}{2x}} \le \frac{1}{1 - \frac{3\cdot 14}{2\cdot 45}} = \frac{15}{8}.$$

For $x \in \left(\pi, \frac{45}{14}\right)$, the function $\sin x$ is negative and increasing, whereas $\sin 2x$ is positive and increasing, whence

$$f_a(x) \le \frac{1}{1 - \frac{4}{3}\frac{\sin x}{x} + \frac{1}{3}\frac{\sin 2x}{2x}} \le \frac{1}{1 - \frac{4}{3}\frac{\sin \pi}{x} + \frac{1}{3}\frac{\sin 2\pi}{2x}} = 1 < \frac{15}{8}. \qquad \square$$

Lemma 2. *Let $a \ge \frac{3}{4}$. Then*

$$G(a) = \frac{45}{8}a^4.$$

Proof. Set $b = \frac{3}{4}$ and

$$g_a(x) = \frac{\sin^4 ax}{\alpha_2(2x)}.$$

As is readily seen,

$$g_a(0) = \lim_{x\to 0} g_a(x) = \frac{45}{8}a^4.$$

Consequently, it is sufficient to prove that for all $x > 0$ and $a \ge b$,

$$g_a(x) \le \frac{45}{8}a^4. \tag{11}$$

Assume that (11) is established for $a = b$. Then, applying inequality (6), we find that

$$g_a(x) = \frac{\sin^4 \frac{a}{b}bx}{\alpha_2(2x)} \le \left(\frac{a}{b}\right)^4 \frac{\sin^4 bx}{\alpha_2(2x)} \le \left(\frac{a}{b}\right)^4 \frac{45}{8}b^4 = \frac{45}{8}a^4$$

whenever $0 < bx < \frac{3\pi}{4}$, i.e., $0 < x < \pi$. If $x \ge \pi$, then

$$g_a(x) \le \frac{1}{\alpha_2(2x)} \le \frac{1}{1 - \frac{4}{3x^2}} \le \frac{3\pi^2}{3\pi^2 - 4} < \frac{45}{8}a^4.$$

Now we prove that $g_b(x) \le \frac{45}{8}\left(\frac{3}{4}\right)^4 = \frac{3645}{2048}$ for $x \in \mathbb{R}$. Write the latter inequality as

$$\sin^4 \frac{3x}{4} \le \frac{3645}{2048}\left(1 - \frac{4}{3}\left(\frac{\sin x}{x}\right)^2 + \frac{1}{3}\left(\frac{\sin 2x}{2x}\right)^2\right). \tag{12}$$

Inequality (12) holds trivially for $x \ge \frac{7}{4}$. Therefore, suffice it to consider the case $0 < x < \frac{7}{4}$. Using the known relations

$$\sin^2 t \ge \sum_{k=1}^{n}(-1)^{k+1}\frac{2^{2k-1}t^{2k}}{(2k)!} \qquad \left(t \in \mathbb{R},\ \tfrac{n}{2} \in \mathbb{N}\right),$$

$$\sin^2 t \le \sum_{k=1}^{n}(-1)^{k+1}\frac{2^{2k-1}t^{2k}}{(2k)!} \qquad \left(t \in \mathbb{R},\ \tfrac{n+1}{2} \in \mathbb{N}\right)$$

and inequality (8), we strengthen (12) as follows:

$$\begin{aligned}\frac{3645}{2048}\left(1 - \frac{4}{3}\sum_{k=1}^{7}(-1)^{k+1}\frac{2^{2k-1}x^{2k-2}}{(2k)!} + \frac{1}{3}\sum_{k=1}^{8}(-1)^{k+1}\frac{2^{2k-1}(2x)^{2k-2}}{(2k)!}\right)&\\ -\frac{1}{2}\sum_{k=2}^{7}(-1)^{k+1}\frac{\left(\frac{3}{2}x\right)^{2k}}{(2k)!} - \frac{1}{8}\sum_{k=2}^{6}(-1)^k\frac{(3x)^{2k}}{(2k)!} \ge 0.&\end{aligned} \tag{13}$$

In order to prove (13), it is sufficient to show that for $0 < y < \frac{49}{16}$,

$$q(y) = \frac{3645}{2048}\left(\frac{4}{3}\sum_{k=4}^{7}(-1)^k\frac{2^{2k-1}y^{k-4}}{(2k)!} - \frac{1}{3}\sum_{k=4}^{8}(-1)^k\frac{2^{4k-3}y^{k-4}}{(2k)!}\right)$$
$$-\frac{1}{2}\sum_{k=3}^{7}(-1)^{k+1}\frac{(\frac{3}{2})^{2k}y^{k-3}}{(2k)!} - \frac{1}{8}\sum_{k=3}^{6}(-1)^k\frac{3^{2k}y^{k-3}}{(2k)!} \geq 0.$$

We strengthen the latter inequality as

$$v(y) = q(y) - \frac{351y}{5\cdot 2^{16}} \geq 0.$$

Straightforward computations demonstrate that $v\left(\frac{49}{16}\right) > 0$ and $v'(y) < 0$ for $y > 0$. It follows that (12) holds for $0 < x < \frac{7}{4}$. □

Remark 2. *If* $0 < a \leq \frac{3}{4}$, *then* $G(a) \leq G\left(\frac{3}{4}\right) = \frac{3645}{2048}$.

Proof. For $a \leq \frac{3}{4}$, the function $\sin^4 ax$ is an increasing function of x on $\left(0, \frac{2\pi}{3}\right]$. Therefore, on the interval indicated, $g_a(x) \leq g_{\frac{3}{4}}(x) \leq G\left(\frac{3}{4}\right)$. If $x > \frac{2\pi}{3} > 2$, then

$$g_a(x) \leq \frac{1}{1-\frac{4}{3x^2}} \leq \frac{3}{2} < \frac{3645}{2048}.$$ □

2.2. Below, we provide values of $D(a)$ and $G(a)$ obtained by a computing technique. Set

$$f_a(x) = \frac{\sin^4 ax}{\alpha_1(2x)}$$

and consider this function on the interval $[0, +\infty)$. By $x_0 = x_0(a)$ denote the point at which $D(a)$ is attained. Enumerate the maxima of the function $f_a(x)$ in the increasing order of x. By $n_0 = n_0(a)$ denote the number of the maximum corresponding to x_0.

Set

$$g_a(x) = \frac{\sin^4 ax}{\alpha_2(2x)}$$

and consider this function on the interval $[0, +\infty)$. By $x_0 = x_0(a)$ denote the point at which $G(a)$ is attained. Enumerate the maxima of the function $g_a(x)$ in the increasing order of x. Let $n_0 = n_0(a)$ denote the number of the maximum corresponding to x_0.

2.3. If $d_k \in \mathbb{C}$, then, by definition,

$$\sum_{k=-\infty}^{\infty} d_k = \sum_{k\in\mathbb{Z}} d_k = d_0 + \sum_{k=1}^{\infty}(d_{-k} + d_k).$$

Let $f \in L_1$. Then

$$c_k(f) = \frac{1}{2\pi}\int_Q f(t)e^{-ikt}dt$$

are the Fourier coefficients, and

$$\sigma(f, x) = \sum_{k=-\infty}^{\infty} c_k(f)e^{ikt}$$

is the Fourier series.

Table 1. Values of the function $D(a)$.

a	$f_a(0)$	x_0	n_0	$D(a)$
1	max	0	1	30
0.9	max	0	1	19.683
0.8	max	0	1	12.288
0.7	max	0	1	7.203
0.6	max	0	1	3.888
0.5	max	0	1	1.875
0.45	max	0	1	1.230188
0.43142	max	0	1	1.039256
0.43141	max	32.792169	5	1.039267
0.425	max	33.258949	5	1.042971
0.4125	min	26.674585	4	1.052067
0.4	min	27.435558	4	1.043096
0.35	min	13.589278	2	1.074429
0.3	min	26.261611	3	1.041487
0.25	min	6.852485	1	1.046843
0.2	min	7.920941	1	1.205494
0.15	min	52.276633	3	1.02588
0.1	min	14.648387	1	1.073388
0.05	min	32.856915	2	1.029758

Lemma 3. *Let $a > 0$, $h > 0$, $r \in \mathbb{N}$, $l \in \mathbb{N}$, $f \in L_2$, and let*

$$D_r(a,h) = \sup_{k\in\mathbb{Z}} \frac{\sin^4 \frac{akh}{2}}{\alpha_r(kh)}.$$

Then

$$\|\delta_{ah}^{4l}(f)\|_2 \le 2^{4l} D_r^l(a,h) \|f - U_{h,r,l}(f)\|_2.$$

Proof. Using Parseval's identity

$$\|g\|_2^2 = \int_Q |g|^2 = 2\pi \sum_{k\in\mathbb{Z}} |c_k(g)|^2$$

for $g \in L_2$ and the relations

$$\delta_{ah}^{4l}(f,x) = 2^{4l} \sum_{k\in\mathbb{Z}} c_k(f) e^{ikx} \sin^{4l} \frac{akh}{2},$$

$$f(x) - U_{h,r,l}(f,x) = \sum_{k\in\mathbb{Z}} c_k(f) e^{ikx} \left(1 - \frac{4}{3}\left(\frac{\sin\frac{kh}{2}}{\frac{kh}{2}}\right)^r + \frac{1}{3}\left(\frac{\sin kh}{kh}\right)^r\right)^l,$$

we derive

$$\|\delta_{ah}^{4l}(f)\|_2^2 = (2\pi) 2^{8l} \sum_{k\in\mathbb{Z}} |c_k(f)|^2 \sin^{8l} \frac{akh}{2},$$

$$\|f - U_{h,r,l}(f)\|_2^2 = 2\pi \sum_{k\in\mathbb{Z}} |c_k(f)|^2 \alpha_r^{2l}(kh).$$

Table 2. Values of the function $G(a)$.

a	$g_a(0)$	x_0	n_0	$G(a)$
1	max	0	1	5.625
0.9	max	0	1	3.690563
0.8	max	0	1	2.304
0.75	max	0	1	1.779785
0.732	max	0	1	1.614979
0.7315	min	0.149508	1	1.610578
0.725	min	0.59855	1	1.555982
0.7125	min	0.9879	1	1.463396
0.7	min	1.249736	1	1.38486
0.65	min	1.920253	1	1.172278
0.6	min	2.375294	1	1.063513
0.55	min	2.761446	1	1.013748
0.5	min	3.141593	1	1
0.45	min	3.562653	1	1.011964
0.4	min	4.029026	1	1.043056
0.35	min	4.500236	1	1.066309
0.3	min	5.068232	1	1.040981
0.25	min	6.283185	1	1
0.2	min	7.827639	1	1.022169
0.15	min	10.569266	1	1.009132
0.1	min	15.707963	1	1
0.05	min	31.415927	1	1

Therefore,

$$\begin{aligned}\|\delta_{ah}^{4l}(f)\|_2^2 &= (2\pi)2^{8l}\sum_{k\in\mathbb{Z}}|c_k(f)|^2\frac{\sin^{8l}\frac{akh}{2}}{\alpha_r^{2l}(kh)}\alpha_r^{2l}(kh)\\ &\le \sup_{k\in\mathbb{Z}}\frac{\sin^{8l}\frac{akh}{2}}{\alpha_r^{2l}(kh)}(2\pi)2^{8l}\sum_{k\in\mathbb{Z}}|c_k(f)|^2\alpha_r^{2l}(kh) = 2^{8l}D_r^{2l}(a,h)\|f-U_{h,r,l}(f)\|_2^2.\end{aligned}\tag{14}$$

The proof is completed by taking the square roots of both sides of relation (14). □

Remark 3. *For the function $f(x)=\cos x$ we have*

$$\|\delta_{ah}^{4l}(f)\|_2 = 2^{4l}\left(\frac{\sin^4\frac{ah}{2}}{\alpha_r(h)}\right)^l\|f-U_{h,r,l}(f)\|_2,$$

whence

$$\sup_{h>0}\frac{\|\delta_{ah}^{4l}(f)\|_2}{\|f-U_{h,r,l}(f)\|_2} = 2^{4l}\sup_{x\in\mathbb{R}}\left(\frac{\sin^4 ax}{\alpha_r(2x)}\right)^l.$$

Corollary 1. *Let $a>0$, r, $l\in\mathbb{N}$. Then*

$$\sup_{h>0}\sup_{f\in L_2}\frac{\|\delta_{ah}^{4l}(f)\|_2}{\|f-U_{h,r,l}(f)\|_2} = 2^{4l}\sup_{x\in\mathbb{R}}\left(\frac{\sin^4 ax}{\alpha_r(2x)}\right)^l.$$

In order to prove Corollary 1, it is sufficient to combine Lemma 3 with Remark 3.

3. Main results

3.1. Theorem 1. *Let $a \geq \frac{1}{2}$, $l \in \mathbb{N}$. Then*

$$\sup_{h>0}\sup_{f\in L_2}\frac{\omega_{4l}(f,ah)_2}{\|f-U_{h,1,l}(f)\|_2}=2^{4l}\left(30a^4\right)^l.$$

Proof. In view of Remark 1 and Lemma 1 in Sec. 2, for $a \geq \frac{1}{2}$ we have

$$\sup_{0<t\leq 1} D(at)=30a^4.$$

Using this fact and applying Lemma 3, we find that

$$\begin{aligned}\omega_{4l}(f,ah)_2=\sup_{0\leq t\leq 1}\|\delta_{ath}^{4l}(f)\|_2 &\leq 2^{4l}\left(\sup_{0<t\leq 1} D(at)\right)^l\|f-U_{h,1,l}(f)\|_2\\ &\leq 2^{4l}\left(30a^4\right)^l\|f-U_{h,1,l}(f)\|_2.\end{aligned}$$

Therefore,

$$\sup_{h>0}\sup_{f\in L_2}\frac{\omega_{4l}(f,ah)_2}{\|f-U_{h,1,l}(f)\|_2}\leq 2^{4l}\left(30a^4\right)^l.$$

The opposite inequality is obvious by virtue of Corollary 1 in Sec.2. □

Theorem 2. *Let $a \geq \frac{3}{4}$, $l \in \mathbb{N}$. Then*

$$\sup_{h>0}\sup_{f\in L_2}\frac{\omega_{4l}(f,ah)_2}{\|f-U_{h,2,l}(f)\|_2}=2^{4l}\left(\frac{45}{8}a^4\right)^l.$$

The proof of Theorem 2 is similar to that of Theorem 1 and is based on Remark 2, Lemmas 2 and 3, and Corollary 1 in Sec. 2.

Corollary 2. *Let $a \geq \frac{1}{2}$, $h > 0$, $f \in L_2$. Then*

$$\frac{1}{480a^4}\omega_4(f,ah)_2\leq\|f-S_{h,1,2}(f)\|_2\leq\frac{1}{6}\omega_4\left(f,\tfrac{h}{2}\right)_2.$$

In order to prove Corollary 2, it is sufficient to combine Theorem 1 (with $l = 1$) and inequality (1) with $r = 1$ and $m = 2$.

In particular, Corollary 2 implies that for any $f \in L_2$ and $h > 0$,

$$\frac{1}{30}\omega_4\left(f,\tfrac{h}{2}\right)_2\leq\|f-S_{h,1,2}(f)\|_2\leq\frac{1}{6}\omega_4\left(f,\tfrac{h}{2}\right)_2.$$

Corollary 3. *Let $a \geq \frac{3}{4}$, $h > 0$, $f \in L_2$. Then*

$$\frac{1}{90a^4}\omega_4(f,ah)_2\leq\|f-S_{h,2,2}(f)\|_2\leq\frac{1}{6}\omega_4(f,h)_2.$$

In order to prove Corollary 3, suffice it to combine Theorem 2 (with $l = 1$) and inequality (1) with $r = 2$ and $m = 2$.

In particular, Corollary 3 implies that for any $f \in L_2$ and $h > 0$,

$$\frac{1}{90}\omega_4(f,h)_2\leq\|f-S_{h,2,2}(f)\|_2\leq\frac{1}{6}\omega_4(f,h)_2.$$

3.2. By A denote the set of even real functions f from C with the Fourier coefficients

$$a_k(f) = \frac{1}{\pi} \int\limits_Q f(x) \cos kx dx \geq 0, \quad k = 0, 1, 2 \dots .$$

As is known [3, p. 277], if $f \in A$, then its Fourier series uniformly converges on $\mathbb{R}$, and the following relation holds:

$$f(x) = \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} a_k(f) \cos kx. \tag{15}$$

Based on (15), we find that if $f \in A$, then

$$\delta_t^{4l}(f, x) = 2^{4l} \sum_{k=1}^{\infty} a_k(f) \sin^{4l} \frac{kt}{2} \cos kx,$$

$$f(x) - U_{h,r,l}(f, x) = \sum_{k=1}^{\infty} a_k(f) \left(1 - \frac{4}{3} \left(\frac{\sin \frac{kh}{2}}{\frac{kh}{2}} \right)^r + \frac{1}{3} \left(\frac{\sin kh}{kh} \right)^r \right)^l \cos kx,$$

$$\|\delta_t^{4l}(f)\| = 2^{4l} \sum_{k=1}^{\infty} a_k(f) \sin^{4l} \frac{kt}{2},$$

$$\|f - U_{h,r,l}(f)\| = \sum_{k=1}^{\infty} a_k(f) \alpha_r^l(kh).$$

Using the above relations and arguing as in proving Theorems 1 and 2, we readily obtain the following assertions.

Theorem 1′. *Let* $a \geq \frac{1}{2}$, $l \in \mathbb{N}$. *Then*

$$\sup_{h>0} \sup_{f \in A} \frac{\omega_{4l}(f, ah)_\infty}{\|f - U_{h,1,l}(f)\|} = 2^{4l} \left(30a^4\right)^l.$$

Theorem 2′. *Let* $a \geq \frac{3}{4}$, $l \in \mathbb{N}$. *Then*

$$\sup_{h>0} \sup_{f \in A} \frac{\omega_{4l}(f, ah)_\infty}{\|f - U_{h,2,l}(f)\|} = 2^{4l} \left(\frac{45}{8} a^4\right)^l.$$

Translated by L. Yu. Kolotilina.

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