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BOUNDS FOR THE INVERSES OF GENERALIZED NEKRASOV MATRICES

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The paper considers upper bounds for the infinity norm of the inverse for matrices in two subclasses of the class of (nonsingular) H-matrices, both of which contain the class of Nekrasov matrices. The first one has been introduced recently and consists of the so-called S-Nekrasov matrices. For S-Nekrasov matrices, the known bounds are improved. The second subclass consists of the so-called QN- *(quasi-Nekrasov) matrices, which are defined in the present paper. For* QN*-matrices, an upper bound on the infinity norm of the inverses is established. It is shown that in application to Nekrasov matrices the new bounds are generally better than the known ones. Bibliography: 15 titles.*

1. Introduction and preliminaries

The paper considers two classes of (nonsingular) H-matrices, both containing the Nekrasov matrices, and upper bounds on the infinity norm of the inverse for matrices in these classes. The first class SN is the known class of S-Nekrasov matrices. The second one is a new class of matrices, which are referred to as the QN- (quasi-Nekrasov) matrices. We show that QN-matrices are nonsingular and, moreover, they are H-matrices. On the other hand, the class of QN-matrices contains the class N of Nekrasov matrices. Thus, matrices from both classes SN and QN can be regarded as generalized Nekrasov matrices.

Recall some definitions and facts, which will be used in what follows.

A matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is called an H-matrix whenever its comparison matrix $\mathcal{M}(A) = (m_{ij})$, defined by the relation

$$m_{ij} = \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & i \neq j, \end{cases}$$

is a nonsingular M-matrix. In accordance with this definition, all H-matrices are nonsingular.

A matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \geq 2$, is called a Nekrasov matrix if

$$|a_{ii}| > h_i(A), \quad i = 1, \ldots, n, \tag{1.1}$$

where the quantities $h_i(A)$ are defined by the following recursive relations:

$$h_1(A) = r_1(A) = \sum_{j\neq 1} |a_{1j}|; \quad h_i(A) = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|} h_j(A) + \sum_{j=i+1}^{n} |a_{ij}|, \quad i = 2, \ldots, n. \tag{1.2}$$

The nonsingularity of Nekrasov matrices was established by Gudkov in [6]. The facts that the class N contains the class SDD of strictly diagonally dominant matrices and is itself contained in the class of nonsingular H-matrices were established by Robert [13].

Note that as it follows from the definition, all the diagonal entries of Nekrasov matrices are nonzero.

In matrix terms, the vector $h(A) = (h_i(A))$ can be written as

$$h(A) = |D|(|D| - |L|)^{-1}|U|e = |D|[I_n - (|D| - |L|)^{-1}\mathcal{M}(A)]e, \tag{1.3}$$

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where $e = [1, \ldots, 1]^T \in \mathbb{R}^n$ is the unit vector, I_n is the identity matrix of order n, and $A = D + L + U$ is the standard splitting of a matrix $A \in \mathbb{C}^{n\times n}$ into its diagonal (D), strictly lower triangular (L), and strictly upper triangular (U) parts, respectively. Thus, condition (1.1) amounts to the relation (see [13])

$$(|D| - |L|)^{-1}|U|e = [I_n - (|D| - |L|)^{-1}\mathcal{M}(A)]e < e, \tag{1.4}$$

and (1.1) actually is the condition of strict diagonal dominance of the Z-matrix

$$(|D| - |L|)^{-1}\mathcal{M}(A) = I_n - (|D| - |L|)^{-1}|U|,$$

obtained from the comparison matrix $\mathcal{M}(A)$ by premultiplying it with the lower triangular matrix $(|D| - |L|)^{-1}$.

In [9], the following upper bound on the norm of the inverse to a Nekrasov matrix was established.

Theorem 1.1. *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ be a Nekrasov matrix of order $n \geq 2$. Then*

$$\|A^{-1}\|_\infty \leq \max_{i\in\langle n\rangle} \frac{z_i(A)}{|a_{ii}| - h_i(A)}, \tag{1.5}$$

where we denote $\langle n\rangle = \{1, \ldots, n\}$.

Here and in what follows, the vector $z(A) = (z_i(A))$ is defined by the relation

$$z(A) = |D|(|D| - |L|)^{-1}e. \tag{1.6}$$

As was shown in [9], the bound (1.5) generally improves the earlier bounds proposed by Cvetković et al. in [1] and, for an SDD matrix $A = (a_{ij})$, the bound (1.5) is at least as good as the classical Varah bound [14]

$$\|A^{-1}\|_\infty \leq \frac{1}{\min_{i\in\langle n\rangle}\{|a_{ii}| - r_i(A)\}}, \tag{1.7}$$

i.e.,

$$\max_{i\in\langle n\rangle} \frac{z_i(A)}{|a_{ii}| - h_i(A)} \leq \frac{1}{\min_{i\in\langle n\rangle}\{|a_{ii}| - r_i(A)\}}, \tag{1.8}$$

where

$$r_i(A) = \sum_{j\neq i} |a_{ij}|, \quad i = 1, \ldots, n,$$

and equality in (1.8) occurs if and only if

$$\min_{i\in\langle n\rangle}\{\mathcal{M}(A)e\}_i = \min_{i\in\langle n\rangle} \frac{|a_{ii}| - h_i(A)}{z_i(A)}.$$

Given a nonempty subset $S \subseteq \langle n\rangle$, the notion of SDD matrices can be generalized as follows (see [4, 15]). Define the partial sums

$$r_i^S(A) = \sum_{j\in S\setminus\{i\}} |a_{ij}|, \quad i \in S. \tag{1.9}$$

In these terms, a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \geq 2$, is said to be S-SDD (S-strictly diagonally dominant) if the following two conditions are fulfilled:

$$|a_{ii}| > r_i^S(A) \quad \text{for all} \quad i \in S \tag{1.10}$$

and

$$\left[|a_{ii}| - r_i^S(A)\right] \left[|a_{jj}| - r_j^{\bar{S}}(A)\right] > r_i^{\bar{S}}(A)\, r_j^S(A) \quad \text{for all} \quad i \in S \quad \text{and} \quad j \in \bar{S}. \tag{1.11}$$

The S-SDD matrices (under a different name) were first introduced in [5], where it was proved that they are H-matrices. Essentially the same matrix class was also considered in [7] as a special case of block matrices satisfying pseudoblock diagonal dominance conditions of the Ostrowski–Brauer type. Under the name of PBDD(n_1, n_2) essentially the same matrix class also appeared in [8].

Obviously, the SDD matrices form a proper subclass of the class of S-SDD matrices, unless $S = \langle n \rangle$.

We will need the following upper bound on the inverses of S-SDD matrices, which was originally established in [11] and proved in a different way in [8].

Theorem 1.2. *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \geq 2$, be an S-SDD matrix for a nonempty proper subset S of the index set $\langle n \rangle$. Then*

$$\|A^{-1}\|_\infty \leq \max_{\substack{i\in S,\\ j\in \bar{S}}} \max\left\{\rho_{ij}^S(A),\ \rho_{ji}^{\bar{S}}(A)\right\}, \tag{1.12}$$

where

$$\rho_{ij}^S(A) = \frac{|a_{ii}| - r_i^S(A) + r_j^S(A)}{(|a_{ii}| - r_i^S(A))(|a_{jj}| - r_j^{\bar{S}}(A)) - r_i^{\bar{S}}(A) r_j^S(A)}, \quad i \in S, \quad j \in \bar{S}. \tag{1.13}$$

The paper is organized as follows. Section 2 considers the S-Nekrasov matrices. For such matrices, a new upper bound on the infinity norm of the inverse is established; it is proved that it generally improves the previous bounds. Also it is shown that for Nekrasov matrices the new bound is better than the bound of Theorem 1.1.

The QN-matrices are introduced in Sec 3. It is shown that the class of QN-matrices is a subclass of the H-matrices and contains the Nekrasov matrices. For the norm of the inverse to a QN-matrix, an upper bound is established. For Nekrasov matrices, the bound obtained is shown to improve the bound of Theorem 1.1.

2. Upper bounds for the inverse of an SN-matrix

The class SN of S-Nekrasov matrices, where S is a nonempty proper subset of the index set, was defined in [3] in terms of the quantities

$$h_1^S(A) = r_1^S(A); \quad h_i^S(A) = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|} h_j^S(A) + \sum_{\substack{j\geq i+1\\ j\in S}} |a_{ij}|, \quad i = 2, \dots, n. \tag{2.1}$$

A matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \geq 2$, is said to be an S-Nekrasov (shortly, SN-) matrix if

$$|a_{ii}| > h_i^S(A) \quad \text{for all} \quad i \in S \tag{2.2}$$

and

$$\left[|a_{ii}| - h_i^S(A)\right] \left[|a_{jj}| - h_j^{\bar{S}}(A)\right] > h_i^{\bar{S}}(A)\, h_j^S(A) \quad \text{for all} \quad i \in S \quad \text{and} \quad j \in \bar{S}. \tag{2.3}$$

Denote

$$e^S = (e_i^S), \quad e_i^S = \begin{cases} 1, & i \in S, \\ 0, & i \in \bar{S}. \end{cases}$$

Then, as is not difficult to realize, relations (2.1) can be written in matrix-vector form as

$$h^S(A) = |L||D|^{-1} h^S(A) + |U| e^S,$$

implying that

$$h^S(A) = |D|(|D| - |L|)^{-1} |U| e^S. \tag{2.4}$$

As is readily seen from the definitions, the Nekrasov matrices form a subclass of the S-Nekrasov matrices. On the other hand, the class SN contains the class S-SDD (see [3, 2]).

In [2], for the inverses of S-Nekrasov matrices the following two upper bounds were established.

Theorem 2.1. *Let S be a nonempty proper subset of the set $\langle n\rangle$, $n \ge 2$, and let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ be an SN-matrix. Then*

$$\|A^{-1}\|_\infty \le \max_{i\in\langle n\rangle} z_i(A) \cdot \max_{\substack{i\in S,\\ j\in\bar S}} \max\left\{\chi_{ij}^S(A),\ \chi_{ji}^{\bar S}(A)\right\}, \tag{2.5}$$

and

$$\|A^{-1}\|_\infty \le \max_{i\in\langle n\rangle} \frac{z_i(A)}{|a_{ii}|} \cdot \max_{\substack{i\in S,\\ j\in\bar S}} \max\left\{\widetilde{\chi}_{ij}^S(A),\ \widetilde{\chi}_{ji}^{\bar S}(A)\right\}, \tag{2.6}$$

where the vector $z(A)$ is defined in (1.6) *and*

$$\chi_{ij}^S(A) = \frac{|a_{ii}| - h_i^S(A) + h_j^S(A)}{\left[|a_{ii}| - h_i^S(A)\right]\left[|a_{jj}| - h_j^{\bar S}(A)\right] - h_i^{\bar S}(A)h_j^S(A)}, \quad i\in S, \quad j\in\bar S; \tag{2.7}$$

$$\widetilde{\chi}_{ij}^S(A) = \frac{|a_{ii}||a_{jj}| - |a_{jj}|h_i^S(A) + |a_{ii}|h_j^S(A)}{\left[|a_{ii}| - h_i^S(A)\right]\left[|a_{jj}| - h_j^{\bar S}(A)\right] - h_i^{\bar S}(A)h_j^S(A)}, \quad i\in S, \quad j\in\bar S. \tag{2.8}$$

As indicated in [2], in the particular case of Nekrasov matrices, the bounds (2.5) and (2.6) improve the corresponding bounds presented in [1].

The bounds of Theorem 2.1 are improved in the following theorem, whose proof is based on the same idea as that of Theorem 1.1.

Theorem 2.2. *Let S be a nonempty proper subset of the set $\langle n\rangle$, $n \ge 2$, and let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ be an SN-matrix. Then*

$$\|A^{-1}\|_\infty \le \max_{\substack{i\in S,\\ j\in\bar S}} \max\left\{\xi_{ij}^S(A),\ \xi_{ji}^{\bar S}(A)\right\}, \tag{2.9}$$

where

$$\xi_{ij}^S(A) = \frac{z_j(A)\left[|a_{ii}| - h_i^S(A)\right] + z_i(A)h_j^S(A)}{\left[|a_{ii}| - h_i^S(A)\right]\left[|a_{jj}| - h_j^{\bar S}(A)\right] - h_i^{\bar S}(A)h_j^S(A)}, \quad i\in S, \quad j\in\bar S. \tag{2.10}$$

Proof. Since $A \in S$N, then the matrix

$$C = (c_{ij}) = |D|(|D| - |L|)^{-1}\mathcal{M}(A) = |D| - |D|(|D| - |L|)^{-1}|U| \tag{2.11}$$

is an S-SDD matrix by Theorem 3.2 in [10]. Define the diagonal matrix $\Delta = \operatorname{diag}(\delta_1, \dots, \delta_n)$ from the condition

$$\Delta e = |D|(|D| - |L|)^{-1}e = z(A).$$

Then Δ has positive diagonal entries and

$$\Delta^{-1}|D|(|D| - |L|)^{-1}e = e,$$

so that

$$\|\Delta^{-1}|D|(|D| - |L|)^{-1}\|_\infty = 1. \tag{2.12}$$

Using (2.11) and (2.12), we derive

$$\begin{aligned}\|\mathcal{M}(A)^{-1}\|_\infty &= \|(\Delta^{-1}C)^{-1}\left[\Delta^{-1}|D|(|D|-|L|)^{-1}\right]\|_\infty \\ &\le \|(\Delta^{-1}C)^{-1}\|_\infty \cdot \|\Delta^{-1}|D|(|D|-|L|)^{-1}\|_\infty \\ &= \|(\Delta^{-1}C)^{-1}\|_\infty.\end{aligned} \tag{2.13}$$

From relation (2.4),

$$h^S(A) = |D|(|D|-|L|)^{-1}|U|e^S,$$

and (2.11) it follows that

$$h_i^S(A) = |a_{ii}| - \{Ce^S\}_i, \quad i = 1, \dots, n,$$

or, since C has positive diagonal and nonpositive off-diagonal entries,

$$|c_{ii}| - r_i^S(C) = |a_{ii}| - h_i^S(A), \quad i \in S, \tag{2.14}$$

and

$$r_j^S(C) = h_j^S(A), \quad j \in \bar{S}. \tag{2.15}$$

By applying Theorem 1.2 to the S-SDD matrix $\Delta^{-1}C$ and taking into account relations (2.13) and (2.14)–(2.15), we obtain

$$\|\mathcal{M}(A)^{-1}\|_\infty \le \max_{\substack{i\in S,\\ j\in\bar{S}}} \max\left\{\rho_{ij}^S(\Delta^{-1}C),\ \rho_{ji}^{\bar{S}}(\Delta^{-1}C)\right\} = \max_{\substack{i\in S,\\ j\in\bar{S}}} \max\left\{\xi_{ij}^S(A),\ \xi_{ji}^{\bar{S}}(A)\right\}.$$

Now, in order to complete the proof, it only remains to recall that, by the Ostrowski theorem [12], for the H-matrix A we have

$$\|A^{-1}\|_\infty \le \|\mathcal{M}(A)^{-1}\|_\infty. \qquad \square$$

Below, we demonstrate that the new bound (2.9) of Theorem 2.2 improves both bounds of Theorem 2.1 and also that for Nekrasov matrices, the bound (2.9) generally improves the known bound (1.5).

Theorem 2.3. *Let S be a nonempty proper subset of the set $\langle n\rangle$, $n \ge 2$, and let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ be an SN-matrix. Then*

$$\max_{\substack{i\in S,\\ j\in\bar{S}}} \max\left\{\xi_{ij}^S(A),\ \xi_{ji}^{\bar{S}}(A)\right\} \le \max_{i\in\langle n\rangle} z_i(A) \cdot \max_{\substack{i\in S,\\ j\in\bar{S}}} \max\left\{\chi_{ij}^S(A),\ \chi_{ji}^{\bar{S}}(A)\right\} \tag{2.16}$$

and

$$\max_{\substack{i\in S,\\ j\in\bar{S}}} \max\left\{\xi_{ij}^S(A),\ \xi_{ji}^{\bar{S}}(A)\right\} \le \max_{i\in\langle n\rangle} \frac{z_i(A)}{|a_{ii}|} \cdot \max_{\substack{i\in S,\\ j\in\bar{S}}} \max\left\{\widetilde{\chi}_{ij}^S(A),\ \widetilde{\chi}_{ji}^{\bar{S}}(A)\right\}, \tag{2.17}$$

where the quantities $\xi_{ij}^S(A)$, $\chi_{ij}^S(A)$, and $\widetilde{\chi}_{ij}^S(A)$ are defined in (2.10), (2.7), *and* (2.8), *respectively.*

Proof. In order to prove (2.16), it is sufficient to ascertain that

$$\xi_{ij}^S(A) \le \max_{i\in\langle n\rangle} z_i(A) \cdot \chi_{ij}^S(A), \quad i \in S, \quad j \in \bar{S}. \tag{2.18}$$

Indeed, since the denominators of $\xi_{ij}^S(A)$ and $\chi_{ij}^S(A)$ coincide, (2.18) stems from the trivial inequality

$$z_j(A)\left[|a_{ii}| - h_i^S(A)\right] + z_i(A)h_j^S(A) \le \max_{i\in\langle n\rangle} z_i(A) \cdot \left[|a_{ii}| - h_i^S(A) + h_j^S(A)\right].$$

In order to establish (2.17), we denote

$$\alpha = \max_{i\in\langle n\rangle} \frac{z_i(A)}{|a_{ii}|}.$$

Then we have

$$z_i(A) \le \alpha|a_{ii}| \quad \text{and} \quad z_j(A) \le \alpha|a_{jj}|.$$

By using the latter inequalities, we obtain

$$z_j(A)\left[|a_{ii}| - h_i^S(A)\right] + z_i(A)h_j^S(A) \le \alpha\left[|a_{ii}||a_{jj}| - |a_{jj}|h_i^S(A) + |a_{ii}|h_j^S(A)\right],$$

which shows that the numerator of $\xi_{ij}^S(A)$ never exceeds that of $\max_{i\in\langle n\rangle}\left\{\frac{z_i(A)}{|a_{ii}|}\right\}\widetilde{\chi}_{ij}^S(A)$. It only remains to observe that the denominators of the two fractions are the same. □

Theorem 2.4. *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ be a Nekrasov matrix. Then*

$$\max_{\substack{i\in S,\\ j\in \bar{S}}} \max\left\{\xi_{ij}^S(A),\ \xi_{ji}^{\bar{S}}(A)\right\} \le \max_{i\in\langle n\rangle} \frac{z_i(A)}{|a_{ii}| - h_i(A)}, \tag{2.19}$$

where $\xi_{ij}^S(A)$ are defined in (2.10).

Proof. Observe that as it follows from their proofs, the bounds (1.5) and (2.9) are the Varah bound (1.7) and the bound (1.12), respectively, on the inverse of the same SDD matrix $C = |D|(|D| - |L|)^{-1}\mathcal{M}(A)$. Consequently, inequality (2.19) stems from the known fact (see [8]) that for an SDD matrix, the bound (1.12) is generally sharper than (1.7). □

3. QN-matrices

A matrix $A = D+L+U \in \mathbb{C}^{n\times n}$, $n \ge 2$, with nonzero diagonal entries is called a QN-matrix if the matrix

$$G = M^{-1}\mathcal{M}(A) = I_n - M^{-1}|L||D|^{-1}|U| \tag{3.1}$$

is strictly diagonally dominant. Here and in what follows, we denote

$$M = (|D| - |L|)|D|^{-1}(|D| - |U|) = \mathcal{M}(A) + |L||D|^{-1}|U|. \tag{3.2}$$

Obviously, the matrix M is monotone, i.e., it is invertible and M^{-1} is a nonnegative matrix.

Since, by (3.1), G is a Z-matrix (i.e., its off-diagonal entries are nonpositive), from the property of strict diagonal dominance of G it follows that it is a nonsingular M-matrix, and the strict diagonal dominance amounts to the condition

$$Ge = M^{-1}\mathcal{M}(A)e = (I_n - M^{-1}|L||D|^{-1}|U|)e > 0. \tag{3.3}$$

Thus, $A \in$ QN if and only if

$$e > M^{-1}|L||D|^{-1}|U|e. \tag{3.4}$$

First we prove that QN-matrices are H-matrices.

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$, $n \ge 2$, be a* QN*-matrix. Then A is an H-matrix.*

Proof. By (3.1), we have

$$\mathcal{M}(A) = MG,$$

so that $\mathcal{M}(A)$ is a product of a monotone matrix times an M-matrix. Therefore, $\mathcal{M}(A)$ is a monotone matrix. This means that $\mathcal{M}(A)$ is an M-matrix, whereas A is an H-matrix. □

Next we show that the class QN contains the class N.

Theorem 3.2. *Let $A \in \mathbb{C}^{n\times n}$, $n \ge 2$, be a Nekrasov matrix. Then A is a QN-matrix.*

Proof. Indeed, since $A \in \mathrm{N}$, the Z-matrix

$$C = (|D| - |L|)^{-1}\mathcal{M}(A) = I_n - (|D| - |L|)^{-1}|U|$$

is strictly diagonally dominant, i.e., the vector Ce is positive. But then the vector

$$Ge = (|D| - |U|)^{-1}|D|Ce$$

is positive a fortiori. This means that the Z-matrix $G = M^{-1}\mathcal{M}(A)$ is strictly diagonally dominant, whence A is a QN-matrix. □

Now we provide an upper bound on the infinity norm of the inverse to a QN-matrix.

Theorem 3.3. *Let $A \in \mathbb{C}^{n\times n}$, $n \geq 2$, be a QN-matrix. Then*

$$\|A^{-1}\|_\infty \leq \max_{i\in\langle n\rangle} \frac{\{M^{-1}e\}_i}{\{M^{-1}\mathcal{M}(A)e\}_i}. \tag{3.5}$$

Proof. The matrix A being a QN-matrix, the matrix G defined in (3.1) is strictly diagonally dominant, and we have

$$\mathcal{M}(A) = MG. \tag{3.6}$$

Now define the diagonal matrix $\Delta = \operatorname{diag}(\delta_1, \dots, \delta_n)$ via the relation

$$M^{-1}e = \Delta e. \tag{3.7}$$

Observe that since M is monotone, Δ has positive diagonal entries. By (3.7), we have

$$(M\Delta)^{-1}e = e.$$

For the monotone matrix $M\Delta$, the latter relation means that

$$\|(M\Delta)^{-1}\|_\infty = 1. \tag{3.8}$$

Using (3.6) and (3.8), we derive

$$\|\mathcal{M}(A)^{-1}\|_\infty = \|(\Delta^{-1}G)^{-1}(M\Delta)^{-1}\|_\infty \leq \|(\Delta^{-1}G)^{-1}\|_\infty \; \|(M\Delta)^{-1}\|_\infty = \|(\Delta^{-1}G)^{-1}\|_\infty. \tag{3.9}$$

Since, by Theorem 3.1, A is an H-matrix, by the Ostrowski theorem [12], we have

$$\|A^{-1}\|_\infty \leq \|\mathcal{M}(A)^{-1}\|_\infty. \tag{3.10}$$

Now the proof is completed by combining (3.9) with (3.10) and applying the classical Varah bound (1.7) to the SDD M-matrix $\Delta^{-1}G$,

$$\|(\Delta^{-1}G)^{-1}\|_\infty \leq \frac{1}{\min\limits_{i\in\langle n\rangle}\{\Delta^{-1}Ge\}_i} = \max_{i\in\langle n\rangle} \frac{\delta_i}{\{Ge\}_i} = \max_{i\in\langle n\rangle} \frac{\{M^{-1}e\}_i}{\{M^{-1}\mathcal{M}(A)e\}_i}.$$ □

As is readily seen, in view of (1.6) and (1.3), the bound (3.5) can also be written in the form

$$\|A^{-1}\|_\infty \leq \max_{i\in\langle n\rangle} \frac{\{(|D| - |U|)^{-1}z(A)\}_i}{\{(|D| - |U|)^{-1}(|D|e - h(A))\}_i},$$

exhibiting the interrelation between (3.5) and (1.5).

The theorem below claims that for a Nekrasov matrix, the bound (3.5) of Theorem 3.3 is in general tighter that the bound (1.5) of Theorem 1.1.

Theorem 3.4. *Let $A \in \mathbb{C}^{n\times n}$, $n \geq 2$, be an N-matrix. Then*

$$\max_{i\in\langle n\rangle} \frac{\{M^{-1}e\}_i}{\{M^{-1}\mathcal{M}(A)e\}_i} \leq \max_{i\in\langle n\rangle} \frac{z_i(A)}{|a_{ii}| - h_i(A)}. \tag{3.11}$$

Proof. Denote

$$\widetilde{M} = (|D| - |L|)|D|^{-1}. \tag{3.12}$$

Then we have (see (1.6))

$$\widetilde{M}^{-1}e = |D|(|D| - |L|)^{-1}e = z(A) \tag{3.13}$$

and

$$\widetilde{M}^{-1}\mathcal{M}(A)e = |D|\left[I_n - (|D| - |L|)^{-1}|U|\right]e.$$

In view of (1.3), the latter relation yields

$$\{\widetilde{M}^{-1}\mathcal{M}(A)e\}_i = |a_{ii}| - h_i(A), \quad i = 1, \ldots, n. \tag{3.14}$$

From (3.13) and (3.14) we obtain

$$\max_{i\in\langle n\rangle} \frac{\{\widetilde{M}^{-1}e\}_i}{\{\widetilde{M}^{-1}\mathcal{M}(A)e\}_i} = \max_{i\in\langle n\rangle} \frac{z_i(A)}{|a_{ii}| - h_i(A)}. \tag{3.15}$$

Denote

$$\alpha = \max_{i\in\langle n\rangle} \frac{\{\widetilde{M}^{-1}e\}_i}{\{\widetilde{M}^{-1}\mathcal{M}(A)e\}_i}. \tag{3.16}$$

Then we obviously have

$$\{\widetilde{M}^{-1}e\}_i \le \alpha\{\widetilde{M}^{-1}\mathcal{M}(A)e\}_i, \quad i = 1, \ldots, n,$$

or, in vector notation,

$$\widetilde{M}^{-1}e \le \alpha\widetilde{M}^{-1}\mathcal{M}(A)e.$$

Premultiplying the relation obtained by the nonnegative matrix $(|D| - |U|)^{-1}$ and using (3.2), we arrive at the inequality

$$M^{-1}e \le \alpha M^{-1}\mathcal{M}(A)e,$$

which means that

$$\{M^{-1}e\}_i \le \alpha\{M^{-1}\mathcal{M}(A)e\}_i, \quad i = 1, \ldots, n. \tag{3.17}$$

Now, in view of (3.15), (3.16), and (3.17), we have

$$\max_{i\in\langle n\rangle} \frac{z_i(A)}{|a_{ii}| - h_i(A)} = \max_{i\in\langle n\rangle} \frac{\{\widetilde{M}^{-1}e\}_i}{\{\widetilde{M}^{-1}\mathcal{M}(A)e\}_i} = \alpha \ge \max_{i\in\langle n\rangle} \frac{\{M^{-1}e\}_i}{\{M^{-1}\mathcal{M}(A)e\}_i}.$$

This completes the proof of the theorem. □

In conclusion, it is worth mentioning that if $A = D + L + U$ is a QN-matrix and

$$B = (D + L)D^{-1}(D + U) = A + LD^{-1}U,$$

then the (preconditioned) matrix

$$B^{-1}A = I_n - B^{-1}LD^{-1}U$$

is an SDD matrix, along with the matrix $G = M^{-1}\mathcal{M}(A)$.

Indeed, by Ostrowski's theorem, we have

$$|(D + L)^{-1}| \le (|D| - |L|)^{-1} \quad \text{and} \quad |(D + U)^{-1}| \le (|D| - |U|)^{-1},$$

whence $|B^{-1}| \le M^{-1}$. This implies that

$$\mathcal{M}(B^{-1}A)e \ge \left[I_n - |B^{-1}||L||D|^{-1}|U|\right]e \ge \left[I_n - M^{-1}|L||D|^{-1}|U|\right]e = Ge > 0,$$

where (3.3) has been used.

Thus, not only the comparison matrix $\mathcal{M}(A)$ is transformed into an SDD matrix when premultiplied by M^{-1}, but the same holds for the QN-matrix A premultiplied by B^{-1}.

In particular, the splitting $A = B - LD^{-1}U$ is a convergent monotone splitting of a QN-matrix A, and $\rho(B^{-1}LD^{-1}U) < 1$, where ρ is the spectral radius.

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