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CALABI–BERNSTEIN-TYPE PROBLEMS FOR SOME NONLINEAR EQUATIONS ARISING IN LORENTZIAN GEOMETRY

M. Caballero, A. Romero, and R. M. Rubio UDC 514.7

CONTENTS

1. Introduction

The classical Calabi–Bernstein theorem asserts that the only entire (i.e., defined on all of $\mathbb{R}^2$) solutions to the maximal surface equation in Lorentz–Minkowski space-time $\mathbb{L}^3$ are the affine functions

$$u(x, y) = ax + by + c \quad \text{with} \quad a^2 + b^2 < 1.$$

This relevant uniqueness result was first proved by Calabi [9] and later extended for the maximal hypersurface equation in $\mathbb{L}^{n+1}$ by Cheng and Yau [10]. It can be also stated in terms of the local complex representation of the surface [12, 15]. Moreover, a direct simple proof of that result, inspired from [11], which uses only the Liouville theorem on harmonic functions on $\mathbb{R}^2$, was given in [20]. There, even local estimates of the Gauss curvature are presented, which implies the Calabi–Bernstein theorem [2, 13] (see also [1] for more details and related results).

The maximal surface equation in $\mathbb{L}^3$ may be widely generalized as follows. Let f be a positive smooth function defined on an open interval I of $\mathbb{R}$. Consider the class of smooth real valued functions u defined on a domain Ω of $\mathbb{R}^2$ such that $u(\Omega) \subset I$ and $|Du| < f(u)$, where $|Du|$ is the length of the usual gradient

$$|Du| = \left(\frac{\partial u}{\partial x}, \ \frac{\partial u}{\partial y}\right)$$

of u. If such a function u is extremal under interior variation of the functional

$$u \mapsto \iint f(u)\sqrt{f(u)^2 - |Du|^2}\, dx \wedge dy,$$

then it satisfies the following elliptic partial differential equation:

$$\left(f(u)^2 - \left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial^2 u}{\partial x^2} + \left(f(u)^2 - \left(\frac{\partial u}{\partial x}\right)^2\right)\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} - f(u)f'(u)(f(u)^2 - |Du|^2) = 0, \quad \text{(A.1)}$$

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$$|Du| < f(u). \tag{A.2}$$

Note that the constraint (A.2) is just the ellipticity condition for Eq. (A.1). This variational problem naturally arises from Lorentzian Geometry. In fact, put $M := I \times \mathbb{R}^2$ endowed with the Lorentzian metric

$$\langle \cdot, \cdot \rangle = -\pi_I^*(dt^2) + f(\pi_I)^2 \pi_{\mathbb{R}^2}^*(g), \tag{1}$$

where π_I and $\pi_{\mathbb{R}^2}$ denote the projections onto I and $\mathbb{R}^2$, respectively, and g is the usual Riemannian metric of $\mathbb{R}^2$. According to the terminology of [19, p. 204], $(M, \langle \cdot, \cdot \rangle)$ is a warped product with base $(I, -dt^2)$, fiber $(\mathbb{R}^2, g)$, and warping function f. We will refer to $(M, \langle \cdot, \cdot \rangle)$ as a Robertson–Walker (RW) space-time.

For each $u \in C^\infty(\Omega)$ such that $u(\Omega) \subset I$, the induced metric on Ω, via its graph $\{(u(x,y), x, y) : (x,y) \in \Omega\}$ in M is given by

$$g_u = -du^2 + f(u)^2 g, \tag{2}$$

which is positive definite if and only if u satisfies (A.2). When $g(u)$ is Riemannian, then

$$f(u)\sqrt{f(u)^2 - |Du|^2}\, dx \wedge dy$$

is its area element, and the previous functional is the area functional $\mathcal{A}$. A function u satisfying (A.2) is a critical point of $\mathcal{A}$ if and only if its (space-like) graph has zero mean curvature. We will refer to Eq. (A.1) as the maximal surface equation in the RW space-time M. On the other hand, in a much more general setting, the constant mean curvature space-like hypersurface equation has been extensively studied not only for its mathematical interest (see [10]) but also because it is important in general relativity (see [18]). Specially, several uniqueness results have been obtained for graphs of solutions of this PDE in some relevant n-dimensional space-times (see [3–5, 17, 21]). We note that, in the first three references, the unknown is defined on a compact manifold, whereas in the last one the manifold is not necessarily compact although the existence of a local maximum of an externally notable function on the graph is assumed.

Coming back to our main interest here, observe that for any warping function f, the constant function $u = t_0$ is a solution to Eq. (A.1) if and only if $f'(t_0) = 0$. Then the following question is natural: *When are the functions $u = t_0$ with $f'(t_0) = 0$, the only entire solutions to Eq.* (A.1)?

On the other hand, we can also ask: *When does Eq.* (A.1) *have no entire solution*?

We will deal with these questions assuming that f is not locally constant, in this case M is called a proper RW space-time, and that M satisfies a natural curvature condition which is expressed in terms of the derivatives of f as $(\log f)'' \le 0$ (see the following section). Thus, we have the following theorem (see [16]).

Theorem 1.1. *If f is not locally constant, the inequalities* $\inf(f) > 0$ *and* $(\log f)'' \le 0$ *hold, and there exists $t_0 \in I$ such that $f'(t_0) = 0$, then the unique entire solution to Eq.* (A.1) *is $u = t_0$.*

Theorem 1.2. *If f is such that* $\inf(f) > 0$, $\sup(f) < \infty$, $(\log f)'' \le 0$, *and f' has no zeros, then there exists no entire solution to Eq.* (A.1).

The key facts to prove these results are:

(i) On any maximal surface of a proper RW space-time such that $(\log f)'' \le 0$, there exists a positive superharmonic function which is constant if and only if the surface is an open portion of a space-like slice $t = t_0$ with $f'(t_0) = 0$.
(ii) Given a space-like graph S such that $\sup(f(t)|_s) < \infty$, $t := \pi_I \circ x$, its metric is conformally related to a metric g^* which is complete when the graph is entire and $\inf(f(t)|_s) > 0$.
(iii) On any maximal graph S such that $\sup(f(t)|_s) < \infty$, g^* has nonnegative Gauss curvature.

Intuitively, a Lorentzian metric of the family given in (1) may be thought of as a perturbation of the flat metric $g_0 = -dt^2 + dx^2 + dy^2$ of Lorentz–Minkowski space-time $\mathbb{L}^3$, close to g_0 if the warping function f is close to the constant function 1. As was shown in [16], for a natural topology in the class of RW space-times with fiber $\mathbb{R}^2$, there exist proper RW space-times close to $\mathbb{L}^3$, where the maximal surface equation Eq. (A.1) has no solution, and there exist proper RW space-times close to $\mathbb{L}^3$ where Eq. (A.1) has only one solution.

2. The Null Convergence Condition

On any RW space-time M, there exists a vector field

$$\xi := f(\pi_I)\partial_t, \tag{3}$$

which is time-like and satisfies

$$\bar{\nabla}_X = f'(\pi_I)X \tag{4}$$

for any X tangent to M, where $\bar{\nabla}$ is the Levi-Civita connection of the Lorentzian metric (1) of M [19, Corollary 7.35]. Thus, ξ is conformal with

$$\mathcal{L}_\xi\langle\cdot,\cdot\rangle = 2f'(\pi_I)\langle\cdot,\cdot\rangle$$

and its metrically equivalent 1-form is closed.

Since M is 3-dimensional, its curvature tensor is completely determined by its Ricci tensor, and this obviously depends on f; in fact, M is flat if and only if f is constant (see [19, Corollary 7.43]). Here, we are interested in the case where M is not flat but its curvature satisfies a natural geometric assumption arising from Relativity theory. In fact, this assumption on a space-time is a necessary condition in order that the space-time obeys Einstein's equation. So, we recall that a Lorentzian manifold obeys the null convergence condition (NCC), when its Ricci tensor $\overline{\mathrm{Ric}}$ satisfies

$$\overline{\mathrm{Ric}} \geq 0$$

for any null tangent vector Z, i.e., $Z \neq 0$ satisfies $\langle Z, Z\rangle = 0$. Taking into account that the fiber of the RW space-time M is flat, and making use again of [19, Corollary 7.43], we obtain

$$\overline{\mathrm{Ric}}(Z,Z) = -(\log f)''\langle Z, \partial_t\rangle^2$$

for any null tangent vector Z. Therefore, a RW space-time M obeys NCC if and only if its warping function satisfies

$$(\log f)'' \leq 0. \tag{5}$$

3. Positive Superharmonic Functions

Let M be a RW space-time and let $x : S \to M$ be a (connected) immersed space-like surface in M. The unitary time-like vector field

$$\partial_t := \frac{\partial}{\partial t} \in \mathfrak{X}(M)$$

determines a time-orientation on M. It allows us to construct $N \in \mathfrak{X}^\perp(S)$ as the only, globally defined, unitary time-like normal vector field on S in the same time-orientation of $-\partial_t$. Thus, from the incorrect Cauchy–Schwartz inequality [19, Proposition 5.30], we have

$$\langle N, \partial_t\rangle \geq 1$$

and $\langle N, \partial_t\rangle = 1$ holds at a point p if and only if $N(p) = -\partial_t(p)$. A space-like slice is a space-like surface x such that $\pi_I \circ x$ is a constant. A space-like surface is a space-like slice if and only if it is orthogonal to ∂_t or, equivalently, orthogonal to ξ.

Denote by

$$\partial_t^T := \partial_t + \langle N, \partial_t\rangle N$$

the tangential component of ∂_t on S. It is not difficult to see that

$$\nabla t = -\partial_t^T, \tag{6}$$

where ∇t is the gradient of $t := \pi_I \circ x$. Now, from the Gauss formula, taking into account $\xi^T = f(t)\partial_t^T$ and (6), the Laplacian of t satisfies

$$\Delta t = -\frac{f'(t)}{f(t)}\{2 + |\nabla t|\} + \operatorname{trace}(A), \tag{7}$$

where $f(t) := f \circ t$, $f'(t) := f' \circ t$ and A is the shape operator associated to N. The function $H := -\frac{1}{2}\operatorname{trace}(A)$ is called the mean curvature of S relative to N. A space-like surface S with $H = 0$ is called maximal. In fact, $H = 0$ if and only if S is (locally) a critical point of the area functional. Note that, with our choice of N, the shape operator of $t = t_0$ is

$$A = \frac{f'(t_0)}{f(t_0)} I$$

and

$$H = \frac{f'(t_0)}{f(t_0)}.$$

For any maximal surface S, we obtain from (7) that t is harmonic if and only if $f'(t) = 0$ for any $t \in I$. Assume now that f is not locally constant (in particular, this holds when f is analytic and nonconstant). In this case, we have that $f(t)$ is constant if and only if t is constant, [16, Lemma 2.1] and, therefore, t is harmonic if and only if $t = t_0$ with $f'(t_0) = 0$. This contrasts with the case of maximal surfaces in $\mathbb{L}^3$ (and, of course, of minimal surfaces in Euclidean space $\mathbb{R}^3$), where the coordinates of the immersion are harmonic functions. This fact is crucial to introduce the (local) conformal Weierstrass representation of the surface, which allows one to express in terms of conformal data the local geometry of the surface. Using (7), we obtain

$$\Delta f(t) = -2\frac{f'(t)^2}{f(t)} + f(t)(\log f)''(t)|\nabla t|^2. \tag{8}$$

Thus, if it is assumed that M satisfies NCC, then $\Delta f(t) \leq 0$, that is, $f(t)$ is a positive superharmonic function on S.

4. The Gauss Curvature

The Gauss curvature K of a maximal surface S in M, taking into account the Gauss equation and the expression for the Ricci tensor of M [19, Corollary 7.43], satisfies

$$K = \frac{f'(t)^2}{f(t)^2} - (\log f)''(t)|\partial_t^T|^2 + \frac{1}{2}\operatorname{trace}(A^2), \tag{9}$$

where

$$\frac{f'(t)^2}{f(t)^2} - (\log f)''(t)|\partial_t^T|^2$$

is the sectional curvature in M of the tangent plane $dx_p(T_pS)$ at any point $p \in S$.

Now we consider the function $\langle N, \xi\rangle$ on S, where ξ is given by (3). We have

$$\nabla\langle N, \xi\rangle = -A\xi^T,$$

where

$$\xi^T := \xi + \langle N, \xi\rangle N$$

is the tangential component on S. Therefore,

$$|\nabla\langle N, \xi\rangle|^2 = \frac{1}{2}\operatorname{trace}(A^2)\{\langle N, \xi\rangle^2 - f(t)^2\}. \tag{10}$$

A direct computation, using the Codazzi equation, gives

$$\nabla\langle N,\xi\rangle = \overline{\mathrm{Ric}}(N,\xi^T) + \mathrm{trace}(A^2)\langle N,\xi\rangle.$$

Making use again of [19, Cor. 7.43], we have

$$\overline{\mathrm{Ric}} = -(\log f)''(t)|\partial_t^T|^2\langle N,\xi\rangle.$$

Thus,

$$\Delta\langle N,\xi\rangle = \{K - \frac{f'(t)^2}{f(t)^2} + \frac{1}{2}\,\mathrm{trace}(A^2)\}\langle N,\xi\rangle. \tag{11}$$

Note that the Gauss curvature of a maximal surface in the RW space-time M, which obeys NCC, satisfies $K \geq \dfrac{f'(t)^2}{f(t)^2}$ as a direct consequence of (9). Moreover, the equality holds if and only if S is totally geodesic [16, Proposition 3.1].

5. Proof of Theorems 1.1 and 1.2

On a space-like graph defined by u, the unitary time-like normal vector field in the same time-orientation of $-\partial_t$ is given by

$$N = \frac{-f(u)}{\sqrt{f(u)^2 - |Du|^2}}\left(1,\ \frac{1}{f(u)^2}\frac{\partial u}{\partial x},\ \frac{1}{f(u)^2}\frac{\partial u}{\partial y}\right).$$

Assume that $\epsilon := \inf(f) > 0$ and the graph is entire. Consider the Riemannian metric

$$g' := \langle N,\xi\rangle^2 g_u \tag{12}$$

on $\mathbb{R}^2$. It is easy to see that $L' \geq \epsilon^2 L$, where L' (respectively, L) denotes the length with respect to g' (respectively, the usual metric g of $\mathbb{R}^2$) of a given smooth curve on $\mathbb{R}^2$. This implies that any divergent curve on $\mathbb{R}^2$ has infinite g'-length. Therefore, g' is complete.

Now note that t_0 must be the only critical point of f and that f attains at t_0 its global maximum. We set $\lambda := \sup(f)$ and consider the Riemannian metric

$$g^* = (\langle N,\xi\rangle + \lambda)^2 g_u \tag{13}$$

on $\mathbb{R}^2$. Completeness of g' easily gives that g^* is also complete. The advantage of g^* over g' is that we can control its Gauss curvature. In fact, we have that the Gauss curvature K^* of g^* is nonnegative. In order to show that, note that

$$K - (\langle N,\xi\rangle + \lambda)^2 K^* = \Delta\log(\langle N,\xi\rangle + \lambda),$$

where K denotes the Gauss curvature g_u, and using (10) and (11), we have

$$\Delta\log(\langle N,\xi\rangle + \lambda) \leq K,$$

which gives $K^* \geq 0$.

Taking into account the conformal invariance of superharmonic functions, from (8), we have that $f(t)$ is a positive superharmonic function of $(\mathbb{R}^2, g^*)$. From a classical result by Ahlfors and Blanc–Fiala–Huber (see, e.g., [14]), a complete two-dimensional Riemannian manifold with nonnegative Gauss curvature is parabolic. Therefore, $(\mathbb{R}^2, g^*)$ is parabolic and $f(t)$ must be constant. Thus, $u(x,y)$ is equal to a constant t_0 for all $(x,y) \in \mathbb{R}^2$ with $f'(t_0) = 0$, completing the proof of Theorem 1.1. Finally, Theorem 1.2 follows when we arrive at the existence of a zero of the derivative of the warping function, contradicting to the assumption.

Remark 5.1. It is also possible to prove Theorems 1.1 and 1.2 from a local approach. In fact, there is a local inequality on any maximal surface which implies these results (see [22]).

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M. Caballero
Universidad de Córdoba, Córdoba, Spain
E-mail: magdalena.caballero@uco.es
A. Romero
Universidad de Granada, Granada, Spain
E-mail: aromero@ugr.es
R. M. Rubio
Universidad de Córdoba, Córdoba, Spain
E-mail: rmrubio@uco.es