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ASYMPTOTIC BEHAVIOR OF THE SPECTRUM OF PSEUDODIFFERENTIAL OPERATORS OF VARIABLE ORDER

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We consider compact selfadjoint pseudodifferential operators under the assumption that the decay order of symbols with respect to ξ depends on a point x. We show that the asymptotic Weyl formula is valid for such operators. Bibliography: 12 *titles.*

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1 Introduction

Let $A = A^* > 0$ be a compact operator in a Hilbert space H, and let $\lambda_k(A)$ be a nonincreasing sequence of eigenvalues of A enumerated with taken into account multiplicity. We introduce the eigenvalue distribution function $N(t, A) \overset{\text{def}}{=} \#\{k \mid \lambda_k(A) > t^{-1}\}$. The asymptotic behavior of the distribution function as $t \to \infty$ was studied in many works with focus on integral operators with kernels having singularity on the diagonal. Such operators are traditionally used in the theory of partial differential equations. Recently, it was discovered that the spectral theory of integral operators can be also applied to the theory of random processes. Namely, to study small deviations of Gaussian random processes, it is required to study asymptotic distribution of eigenvalues of operators whose kernels have variable singularity order on the diagonal.

It is convenient to study integral operators with kernels possessing singularity on the diagonal by considering such operators as elements of a suitable algebra of pseudodifferential operators. Suppose that Ω is a bounded domain in $\mathbb{R}^d$ and the pseudodifferential operator $A = A^* > 0$ defined by

$$Au(x) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} \int\limits_{\Omega} e^{i(x-y)\xi} a(x,\xi) u(y)\, dy\, d\xi, \quad u \in C_0^\infty(\Omega),$$

is compact in $L_2(\Omega)$. Let the symbol $a(x,\xi)$ be asymptotically homogeneous at infinity, i.e.,

$$a(x,\xi) = a_0(x,\xi) + o(|\xi|^{-m}), \quad |\xi| \to +\infty,$$
$$a_0(x,\rho\xi) = \rho^{-m} a_0(x,\xi), \quad \rho > 0.$$

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Since $A = A^* > 0$, we have $a_0(x,\xi) \geqslant 0$. For such symbols satisfying certain smoothness conditions the asymptotic Weyl formula holds (cf. [1, 2] and the review [3])

$$N(t,A) \sim (2\pi)^{-d}\mathrm{vol}_{2d}\{x,\xi \in \Omega\times\mathbb{R}^d |\, \mathrm{Re}\ a(x,\xi) > t^{-1}\}, \quad t \to +\infty. \tag{1.1}$$

Hereinafter, vol_d is the d-dimensional volume of a set. The asymptotic homogeneity of the symbol leads to the explicit formula

$$N(t,A) \sim (2\pi)^{-d}\mathrm{vol}_{2d}\{x,\xi \in \Omega\times\mathbb{R}^d |\, a_0(x,\xi) > 1\}\cdot t^{d/m}, \quad t \to +\infty. \tag{1.2}$$

Let the homogeneity order of the symbol depend on x,

$$a(x,\rho\xi) = \rho^{-(m+\varphi(x))}a(x,\xi)(1+o(1)), \quad \rho \to +\infty, \tag{1.3}$$

and let $\varphi(x) \in C^\infty(\overline{\Omega})$, $\varphi(x) \geqslant 0$, $x \in \Omega$. We set $\Gamma = \{x \in \overline{\Omega} |\, \varphi(x) = 0\}$. If the d-dimensional Lebesgue measure of Γ is positive, then the asymptotic formula (1.2) remains valid. If the measure of Γ is equal to zero, then from the classical results we can only conclude that $N(t,A)$ increases slower than $t^{m/d}$, but faster than $t^{m/d-\varepsilon}$ for any $\varepsilon > 0$. By perturbation theory [4, 3], variations of the symbol outside a neighborhood of Γ does not affect the asymptotics of the spectrum.

The traditional proof of the Weyl formula (1.1) is divided into following steps. First, we verify the formula for model operators on a torus with symbols independent of x (in this case, the spectrum is explicitly computed). Second, we estimate the distribution function in suitable classes of symbols. Finally, we "close" the asymptotics. None of these steps remains valid in the case of variable singularity order.

The goal of this paper is to prove the Weyl formula (1.1) for the class of hypoelliptic pseudodifferential operators including operators with symbols (1.3). For this purpose we use the method of approximate spectral projection.

2 Method of Approximate Spectral Projection

The method of approximate spectral projection for obtaining asymptotics of eigenvalue distribution was proposed in [5, 6] and developed by many authors (cf. the reviews [7, 8]). We note that many results were obtained in [9, 7]. In these works, the spectrum of unbounded (pseudo)differential operators was studied. The asymptotic behavior of the spectrum of operators with variable order (1.3) was not studied earlier. In this paper, we modify the method of approximate spectral projection for spectral asymptotics of compact selfadjoint operators.

Lemma 2.1. *Suppose that A is a selfadjoint nonnegative operator in a Hilbert space H, $V(t)$, $t > 1$, is a nondecreasing positive function such that $V(t) \to +\infty$ as $t \to +\infty$, $E(t)$, $0 \leqslant E(t) \leqslant I$, is a family of selfadjoint nuclear operators in H with traces satisfying the following conditions for some $c, \nu > 0$:*

(1) $\mathrm{tr}\, E(t) = V(t) + O(W(t, ct^{1-\nu}))$ *as* $t \to +\infty$,

(2) $\mathrm{tr}\,(E(t)(I - E(t))) = O(W(t, ct^{1-\nu}))$ *as* $t \to +\infty$,

where $W(t,\tau) = V(t+\tau) - V(t-\tau)$. Then the following assertions hold.

(A) *If for all $u \in H$ and $t > 1$*

$$((tA - I)E(t)u, E(t)u) \geqslant -ct^{-\nu}\|u\|^2, \tag{2.1}$$

then for all sufficiently large t

$$N(t,A) \geqslant V(t) - CW(t, ct^{1-\nu}). \tag{2.2}$$

(B) *If for all* $u \in H$ *and* $t > 1$

$$((tA - I)(I - E(t))u, (I - E(t))u) \leqslant ct^{-\nu}\|u\|^2, \tag{2.3}$$

then for all sufficiently large t

$$N(t,A) \leqslant V(t) + CW(t, ct^{1-\nu}). \tag{2.4}$$

Proof. We follow the proof of a similar assertion for the spectrum of an unbounded operator (cf. [8, Section 15] and [6, Section 28]). Let $L(t) \subset H$ be the linear span of eigenvectors of the operator $E(t)$ corresponding to the eigenvalues λ_k larger than 1/2. Since $0 \leqslant \lambda_k \leqslant 1$, we have

$$\operatorname{tr} E(t) - \dim L(t) = \sum_{2\lambda_k > 1} (\lambda_k - 1) + \sum_{2\lambda_k \leqslant 1} \lambda_k \leqslant 2 \sum_{2\lambda_k \leqslant 1} (1 - \lambda_k)\lambda_k \leqslant 2\operatorname{tr}\big(E(t)(I - E(t))\big),$$

$$\operatorname{tr} E(t) - \dim L(t) \geqslant - \sum_{2\lambda_k > 1} (1 - \lambda_k) \geqslant -2 \sum_{2\lambda_k > 1} (1 - \lambda_k)\lambda_k \geqslant -2\operatorname{tr}\big(E(t)(I - E(t))\big).$$

By Condition (2),

$$|\operatorname{tr} E(t) - \dim L(t)| = O(W(t, ct^{1-\nu})), \quad t \to +\infty. \tag{2.5}$$

The operator $E(t)$ is a one-to-one mapping from $L(t)$ to itself. For $u \in L(t)$ we set $v = E(t)u$. By $\|E(t)u\| \geqslant 1/2\|u\|$, from (2.1) it follows that $((tA - I)v, v) \geqslant -4ct^{-\nu}\|v\|^2$, which leads to $((tA - (1 - c_1 t^{-\nu})I)v, v) > 0$ for $c_1 > 4c$ and for all $v \in L(t)$, $v \neq 0$. By the variational principle, $N(t(1 - c_1 t^{-\nu})^{-1}, A) \geqslant \dim L(t)$ and the equality (2.5) implies the lower estimate (2.2).

Further, $L(t)$ is a reducing subspace for $E(t)$. For $u \in L(t)^\perp$ we have $v = u - E(t)u \in E(t)^\perp$, $\|u\| \leqslant 2\|v\|$, and $I - E(t)$ is a one-to-one mapping from the space $L(t)^\perp$ to itself. As in the proof of the lower estimate, from (2.3) it follows that $((tA - (1 + c_1 t^{-\nu})I)v, v) < 0$ for all $v \in L(t)^\perp$, $v \neq 0$. By the variational principle, $N(t(1 + c_1 t^{-\nu})^{-1}, A) \leqslant \dim L(t)$. Then the equality (2.5) implies the upper estimate (2.4). □

Remark 2.1. If $V(t)$ regularly varies at infinity with order $r > 0$, i.e., $tV'(t)/V(t) \to r$ as $t \to +\infty$, then $W(t, ct^{1-\nu}) \leqslant Ct^{-\nu}V(t)$ and the distribution function is estimated by $V(t)(1 \pm Ct^{-\nu})$.

Remark 2.2. The operator A in Lemma 2.1 can depend on the parameter t. If $A = A(t)$ satisfies (2.1) or (2.3), then $N(t, A(t))$ satisfies (2.2) or (2.4) respectively.

3 Calculus of Pseudodifferential Operators

It is convenient to verify the inequalities in the assumptions of Lemma 2.1 if the operator A belongs to a suitable algebra of pseudodifferential operators, where composition theorems hold and it is possible to prove a Gårding type inequality for operators with strongly elliptic symbols.

We describe some classes of pseudodifferential operators which will be used below. The operators and symbols depend on the parameter $t \geqslant 1$. Let $p(t, x, \xi)$ be a positive infinitely smooth weight function with respect to $x, \xi \in \mathbb{R}^d$. By definition, the class $S_{\rho,\delta,\varkappa}(p)$ consists of infinitely smooth functions $a(t, x, \xi)$ such that for any multiindices α and β

$$|\partial_x^\alpha \partial_\xi^\beta a(t, x, \xi)| \leqslant c_{\alpha,\beta} t^{-\varkappa|\beta|} p(t, x, \xi)(1 + |\xi|)^{\delta|\alpha| - \rho|\beta|}, \quad x, \xi \in \mathbb{R}^d, \tag{3.1}$$

$0 \leqslant \delta < \rho \leqslant 1, \varkappa \in \mathbb{R}$. In the case $p(t, x, \xi) \equiv (1 + |\xi|^2)^{m/2}$ and $\varkappa = 0$, we have the known Hörmander class $S^m_{\rho,\delta}$.

For fixed t the classes $S_{\rho,\delta,\varkappa}(p)$ are included to the general classes of Weyl–Hörmander calculus [10]. These classes are described by a metric (a quadratic form) g in $\mathbb{R}^{2d}$ and a weight function p. For our classes the metric

$$g_{x,\xi}(y, \eta) = (1 + |\xi|^2)^{-\delta/2} |y|^2 + (1 + |\xi|^2)^{\rho/2} |\eta|^2$$

is the same as in the standard classes $S^m_{\rho,\delta}$ and, consequently, satisfies all the required conditions (cf. [10, Section 18]). The following estimates are a rather simple sufficient condition for applying the calculus of pseudodifferential operators (cf. [9, Lemmas 1.1 and 1.2]):

$$|\partial_{x_j} p(x, \xi)| \leqslant C p(x, \xi)(1 + |\xi|)^\delta, \quad |\partial_{\xi_j} p(x, \xi)| \leqslant C p(x, \xi)(1 + |\xi|)^{-\rho}, \quad 1 \leqslant j \leqslant d, \tag{3.2}$$

$$p(x, \xi) \leqslant C p(y, \eta)((1 + |\xi|)^N + (1 + |\eta|)^N). \tag{3.3}$$

An important example of a weight function p is provided by the function $p(x, \xi) = (1 + |\xi|^2)^{\varphi(x)}$, where $\varphi \in C^\infty(\mathbb{R}^d)$, $\varphi(x) = \text{const}$ with sufficiently large $|x|$. For this weight function the inequality (3.2) is satisfied with $\rho = 1$ and any $\delta > 0$. The inequality (3.3) is obvious.

We introduce the pseudodifferential operator $A(t, x, D_x)$ with Weyl symbol $a(t, x, \xi)$ by

$$(A(t, x, D_x)u)x = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} e^{i(x-y)\xi} a(t, \frac{x+y}{2}, \xi) u(y) dy d\xi,$$

where u belongs to the Schwarz class. We denote by $\Psi_{\rho,\delta,\varkappa}(p)$ the class of such operators with symbols in $S_{\rho,\delta,\varkappa}(p)$. For classes of symbols with the weight function $(1 + |\xi|^2)^{m/2}$ we preserve the standard notation $S^m_{\rho,\delta,\varkappa}$ and denote by $\Psi^m_{\rho,\delta,\varkappa}$. the classes of corresponding operators.

If $p = p(t, x, \xi)$ satisfies (3.2) and (3.3) uniformly with respect to t, then the following assertion about composition and boundedness in L_2 holds (cf. [10, Section 18] and [9, Section 1]).

Proposition 3.1. *Suppose that a_k are Weyl symbols of the operators $A_k \in \Psi_{\rho,\delta,\varkappa}(p_k)$, $k = 1, 2$. Then $A_1 A_2 \in \Psi_{\rho,\delta,\varkappa}(p_1 p_2)$ and for any n the Weyl symbol of the operator $A_1 A_2$ is expanded into the sum*

$$\sum_{|\alpha|+|\beta|<n} \frac{(-1)^{|\beta|}}{\alpha! \beta!} (-2i)^{|\alpha+\beta|} \partial_x^\alpha \partial_\xi^\beta a_1 \partial_\xi^\alpha \partial_x^\beta a_2 + r_n,$$

where $r_n \in S_{\rho,\delta,\varkappa}(p_1 p_2 (1 + |\xi|^2)^{-n(\rho-\delta)/2})$.

Proposition 3.2. *If the weight function p is bounded in $\mathbb{R}^{2d}$ uniformly with respect to $t > 1$, then the operator $A \in \Psi_{\rho,\delta,\varkappa}(p)$, $\varkappa \geqslant 0$, is bounded in $L_2(\mathbb{R}^d)$ uniformly with respect to t.*

We recall that an operator with a real Weyl symbol is formally selfadjoint in L_2.

The following assertion contains the G$\mathring{a}$rding inequality (cf. [7, Section 1, Lemma 5′]).

Proposition 3.3. *Suppose that for a real positive symbol $a \in S_{\rho,\delta,\varkappa}(p)$, $\varkappa > 0$, some $N \in \mathbb{R}$ and $t > 1$, $x, \xi \in \mathbb{R}^d$*

$$a(t,x,\xi) \geqslant c_0 t^{-N}(1+|\xi|)^{-N}, \quad c_0 > 0, \tag{3.4}$$

$$|\partial_x^\alpha \partial_\xi^\beta a(t,x,\xi)| \leqslant C_{\alpha,\beta} a(t,x,\xi) t^{-\varkappa|\beta|}(1+|\xi|)^{\delta|\alpha|-\rho|\beta|} \quad \forall \alpha, \beta, \tag{3.5}$$

i.e., $a(t,x,\xi)$ is an elliptic symbol in $S_{\rho,\delta,\varkappa}(a)$. Then there is $t_0 > 1$ such that for $t > t_0$

$$(A(t,x,D_x)u,u)_{L_2(\mathbb{R}^d)} \geqslant 0, \quad u \in C_0^\infty(\mathbb{R}^d).$$

4 Asymptotics of Spectrum

Suppose that Ω is a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary S, φ is an infinitely smooth nonnegative function in $\mathbb{R}^d$, Γ is the set of zeros of φ, and $\Gamma \subset \overline{\Omega}$. We assume that $\varphi(x) \equiv const > 0$ for sufficiently large $|x|$. We consider a pseudodifferential operator $A(x, D_x)$ with Weyl symbol $a(x,\xi) \in S_{\rho,\delta,0}((1+|\xi|^2)^{-(m+\varphi(x))/2})$, $0 \leqslant \delta < \rho \leqslant 1$, $m > 0$. We assume that the real symbol $a(x,\xi)$ is formally *hypoelliptic*, i.e.,

(1) for some $c_0, m_0 > 0$

$$a(x,\xi) \geqslant c_0|\xi|^{-m_0} \quad \forall\, x \in \Omega,\ |\xi| > R, \tag{4.1}$$

(2)

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leqslant c_{\alpha,\beta}(1+|\xi|)^{\delta|\alpha|-\rho|\beta|} a(x,\xi), \quad |\xi| > R, \tag{4.2}$$

(3) (the nondegeneracy condition) for any $0 < \nu < (\rho-\delta)/2m_0$

$$\mathrm{vol}_d\{\xi \in \mathbb{R}^d |\, a(x,\xi) > t^{-1}\} = v(x) t^{\frac{d}{m+\varphi(x)}}(1+O(t^{-\nu})), \quad t \to +\infty, \tag{4.3}$$

uniformly with respect to $x \in \overline{\Omega}$, where v is a continuous positive function in $\overline{\Omega}$.

Condition (3) holds if $a(x,\xi) = a_0(x,\xi) + O(|\xi|^{-m-(\rho-\delta)})$ as $|\xi| \to +\infty$, $a_0(x,\xi)$ is homogeneous in ξ with degree $-(m+\varphi(x))$ and $v(x) = \mathrm{vol}_d\{\xi \in \mathbb{R}^d |\, a_0(x,\xi) > 1\} > 0$ in $\overline{\Omega}$.

Let $A_\Omega := rAe$, where r and e are the operators of extension by zero and restriction on Ω.

Theorem 4.1. *For the spectrum of the operator A_Ω with symbol satisfying conditions* (1)–(3) *the following asymptotic Weyl formula holds*:

$$N(t, A_\Omega) = (2\pi)^{-d}\mathrm{vol}_{2d}\{x, \xi \in \Omega \times \mathbb{R}^d |\, a(x,\xi) > t^{-1}\}(1+O(t^{-\nu})), \quad t \to +\infty, \tag{4.4}$$

which, in view of (4.3), *leads to the explicit Laplace integral*

$$N(t, A_\Omega) = (2\pi)^{-d} \int\limits_\Omega t^{\frac{d}{m+\varphi(x)}} v(x) dx (1+O(t^{-\nu})), \quad t \to +\infty. \tag{4.5}$$

Conditions (1)–(3) are restrictions on the behavior of the symbol $a(x,\xi)$ only in $\Omega \times \mathbb{R}^d$. However, the symbol a can vary outside Ω (extend from $\Omega \times \mathbb{R}^d$ to $\mathbb{R}^d \times \mathbb{R}^d$) in such a way that these conditions are uniformly satisfied with respect to $\mathbb{R}^d \times \mathbb{R}^d$. If the symbol varies outside $\Omega \times \mathbb{R}^d$, then the operator A_Ω is replaced with the operator $A_\Omega + Q$, where Q is an operator with infinitely smooth kernel in $\overline{\Omega \times \Omega}$ such that the asymptotics of $N(t, A_\Omega)$ remains unchanged. Therefore, we will assume that Conditions (1)–(3) are uniformly satisfied with respect to $x, \xi \in \mathbb{R}^d \times \mathbb{R}^d$.

Asymptotics of the integral in (4.5) is defined by a small neighborhood of Γ, where φ attains the minimum. As a rule, this asymptotics is computed with accuracy up to $O((\ln t)^{-\infty})$, so that a particular value of ν is not essential.

To prove the theorem, we construct approximate spectral projections of the operator A and verify the assumptions of Lemma 2.1. This will be done in Lemmas 5.1–5.4 below.

The assumption that S is Lipschitz can be weakened in many cases, the asymptotics (4.4), (4.5) is preserved (possibly with a worse estimate for the remainder) for a larger class of domains. Slightly modifying the proof of Theorem 4.1, we obtain the following assertion.

Theorem 4.2. *Let A_Ω be an operator such that its symbol satisfies Conditions* (1)–(3)*. Then the following assertions hold.*

1. *If Ω is an arbitrary bounded domain in $\mathbb{R}^d$, $\Gamma \cap S = \varnothing$, and $\varphi(x) \geqslant \sigma$ for some $\sigma > 0$ uniformly with respect to $x \in S$, then formulas* (4.4) *and* (4.5) *are preserved with the reminder of order $O(t^{-\widetilde{\nu}})$ for any $\widetilde{\nu} < \min\Big\{\dfrac{\rho-\delta}{2m_0}, \dfrac{m\sigma}{d(d+\sigma)}\Big\}$.*

2. *If S is not Lipschitz, but $\mathrm{vol}_d\{x \in \mathbb{R}^d |\ dist(x,S) \leqslant s\} \leqslant Cs^\tau$ for some $\tau < 1$ for $s < s_0$, then formulas* (4.4) *and* (4.5) *are preserved with the remainder of order $O(t^{-\widetilde{\nu}})$ for any $\widetilde{\nu} < \dfrac{\rho-\delta}{2m_0}\tau$.*

5 Construction of Approximate Spectral Projection

We consider parameters $\nu, \varkappa > 0$ such that $(\nu+\varkappa) < (\rho-\delta)/2m_0$. We set $\rho' = \rho - (\nu+\varkappa)m_0$ and $\delta' = \delta + (\nu+\varkappa)m_0$. We note that $0 \leqslant \delta < \delta' < \rho' < \rho \leqslant 1$. We introduce the functions $\chi_\pm(s) \in C^\infty(\mathbb{R})$, $0 \leqslant \chi_\pm(s) \leqslant 1$, by

$$\chi_\pm(s) = \begin{cases} 1, & s > \mp t^{-\nu}, \\ 0, & s < \mp 2^{\pm 1} t^{-\nu}. \end{cases}$$

The derivatives of $\chi_\pm$ are different from zero only for $\mp 2^{\pm1} t^{-\nu} \leqslant s \leqslant \mp t^{-\nu}$, and the following estimate holds: $|\chi_\pm^{(k)}(s)| \leqslant c_k t^{\nu k}$. We set

$$\chi_\pm(t,x,\xi) := \chi_\pm(ta(x,\xi) - 1). \tag{5.1}$$

We introduce three cut-off functions $\zeta_-(x,t)$, $\zeta_0(x,t)$, and $\zeta_+(x,t)$. For all $t > 1$ we set $\zeta_- \in C_0^\infty(\Omega)$, $\zeta_-(x,t) = 1$ if $\mathrm{dist}\,(x,S) > t^{-\nu}$. Further, ζ_0, $\zeta_+ \in C_0^\infty(R^d)$ for all $t > 1$ and

$$\zeta_0(x,t) = \begin{cases} 1, & x \in \Omega, \\ 0, & \mathrm{dist}\,(x,S) > t^{-\nu}. \end{cases}$$

The cut-off function ζ_+ covers the cut-off function ζ_0:

$$\zeta_+(x,t) = \begin{cases} 1, & \zeta_0(x,t) \neq 0, \\ 0, & \mathrm{dist}\,(x,S) > 2t^{-\nu}. \end{cases}$$

For all these cut-off functions we have $0 \leqslant \zeta(x,t) \leqslant 1$, $|\partial_x^\alpha \zeta(x,t)| \leqslant c_\alpha t^{\nu|\alpha|}$, $t > 1$. Denote by $\zeta = \zeta(t)$ the operator of multiplication by $\zeta(x,t)$. We note that

$$t^\nu \leqslant C(1+|\xi|)^{\max\{(\rho-\rho'),(\delta'-\delta)\}} \tag{5.2}$$

on the support of the derivatives of $\chi_\pm(t,x,\xi)$.

Lemma 5.1. 1. *The symbols $\chi_\pm(t,x,\xi)$ belong to the class $S^0_{\rho',\delta',\varkappa}$. The corresponding operators in $L_2(\mathbb{R}^d)$ satisfy the estimate*

$$-\varepsilon I \leqslant \chi_\pm(t,x,D_x) \leqslant (1+\varepsilon)I \tag{5.3}$$

for any $\varepsilon > 0$ and $t \geqslant t_0(\varepsilon)$.

2. *Let ζ be the operator of multiplication by one of the cut-off functions ζ_-, ζ_0, ζ_+. Then $[\zeta,\chi_\pm] := \zeta\chi_\pm(t,x,D_x) - \chi_\pm(t,x,D_x)\zeta$ is a pseudodifferential operator of class $\Psi^{-\delta}_{\rho',\delta',\varkappa}$.*

3. *There exists t_0 such that for all $t > t_0$ and $u \in C_0^\infty(\mathbb{R}^d)$*

$$((tA(x,D_x)-I)\chi_-(t,x,D_x)u, \chi_-(t,x,D_x)u)_{L_2(\mathbb{R}^d)} \geqslant t^{-\nu}\|u\|^2_{L_2(\mathbb{R}^d)}, \tag{5.4}$$

$$((tA(x,D_x)-I)(I-\chi_+(t,x,D_x))u, (I-\chi_+(t,x,D_x))u)_{L_2(\mathbb{R}^d)} \leqslant t^{-\nu}\|u\|^2_{L_2(\mathbb{R}^d)}. \tag{5.5}$$

Proof. 1. Let us estimate the derivatives:

$$\begin{aligned} &|\partial_x^\alpha\partial_\xi^\beta \chi_\pm(t,x,\xi)| = |\partial_x^\alpha\partial_\xi^\beta \chi_\pm(ta(x,\xi)-1)| \\ &= \Bigg| \sum_{\substack{\alpha^1+\ldots+\alpha^k=\alpha,\ \beta^1+\ldots+\beta^l=\beta \\ \alpha^i,\beta^j>0}} c_{\alpha^1,\ldots,\alpha^k,\beta^1,\ldots,\beta^l}\, t^{k+l}\chi_\pm^{(k+l)}(ta(x,\xi)-1) \prod_{1\leqslant i\leqslant k,\ 1\leqslant j\leqslant l} \partial_x^{\alpha^i}\partial_\xi^{\beta^j} a(x,\xi)\Bigg| \\ &\leqslant c_{\alpha,\beta} t^{\nu(|\alpha|+|\beta|)}(1+|\xi|)^{\delta|\alpha|-\rho|\beta|} \leqslant c'_{\alpha,\beta} t^{-\varkappa(|\alpha|+|\beta|)}(1+|\xi|)^{\delta'|\alpha|-\rho'|\beta|}, \end{aligned} \tag{5.6}$$

where the estimate (5.2) was used.

The inequalities (5.3) are the Gårding inequalities for the pseudodifferential operators $\varepsilon I + \chi_\pm(t,x,D_x)$ and $(1+\varepsilon)I - \chi_\pm(t,x,D_x)$ with Weyl symbols in $S^0_{\rho',\delta',\varkappa}$. The symbols are strongly elliptic in $S^0_{\rho',\delta',\varkappa}$ and satisfy the assumptions of Proposition 3.3.

2. The commutator is a pseudodifferential operator with amplitude $\chi_\pm(t,(x+y)/2,\xi)(\zeta(x,t)-\zeta(y,t))$:

$$\big([\zeta,\chi_\pm]u\big)(x) = (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(x-y)\xi}\chi_\pm(t,\frac{x+y}{2},\xi)(\zeta(x,t)-\zeta(y,t))u(y)dyd\xi. \tag{5.7}$$

We have

$$\zeta(x,t)-\zeta(y,t) = \sum_{j=1}^d (x_j-y_j)\Phi_j(x,y,t),$$

where

$$\Phi_j(x,y,t) = \int_0^1 (\zeta)_{x_j}(y+s(x-y),t)ds, \quad |\partial_{x,y}^\alpha \Phi_j(x,y,t)| \leqslant c_\alpha t^{\nu(|\alpha|+1)}.$$

Integrating by parts in (5.7), we see that the commutator can be prescribed by the amplitude

$$\sum_{j=1}^{d} \Phi_j(x,y,t)\partial_{\xi_j}\chi_\pm(t,(x+y)/2,\xi).$$

Estimating the derivatives of the amplitude, as in the proof of Assertion 1, we find that this amplitude belongs to the class $S^{-\delta}_{\rho',\delta',\varkappa}$.

3. We consider only the estimate (5.4) since the proof of (5.5) is similar. This estimate is the Gårding inequality for the operator $P_-(t) := \chi_-(t,x,D_x)(tA-I)\chi_-(t,x,D_x)+t^{-\nu}$. We set

$$p_-(t,x,\xi) := (ta(x,\xi)-1)\chi_-(t,x,\xi)^2+t^{-\nu}.$$

We show that p_- is the principal symbol of the operator $P_-(t)$ and the assumptions of Proposition 3.3 are satisfied. By construction, $p_-(t,x,\xi) \geqslant t^{-\nu}$. From (4.2) and (5.6) it follows that

$$\left|(ta(x,\xi)-1)\partial_x^\sigma\partial_\xi^\tau((\chi_-(t,x,\xi))^2)\right| \leqslant C_{\sigma,\tau}t^{-\varkappa(|\sigma|+|\tau|)}(1+|\xi|)^{-\rho'|\tau|+\delta'|\sigma|}p_-(t,x,\xi) \tag{5.8}$$

and for $|\alpha|+|\beta|>0$

$$\begin{aligned}&\left|\partial_x^\alpha\partial_\xi^\beta(ta(x,\xi)-1)\partial_x^\sigma\partial_\xi^\tau((\chi_-(t,x,\xi))^2)\right| \\ &\qquad\leqslant C_{\alpha,\beta,\sigma,\tau}t^{-\varkappa(|\sigma|+|\tau|)}(1+|\xi|)^{-\rho|\beta|+\delta|\alpha|-\rho'|\tau|+\delta'|\sigma|}ta(x,\xi).\end{aligned} \tag{5.9}$$

On the support of the derivatives of $\chi_-(t,x,\xi)$, we have $\frac{1}{2}t^{-\nu} \leqslant ta(x,\xi)-1 \leqslant t^{-\nu}$. By the choice of parameters ρ, ρ', δ, δ', from the inequalities (5.8) and (5.9) it follows that

$$\begin{aligned}&\left|\partial_x^\alpha\partial_\xi^\beta(ta(x,\xi)-1)\partial_x^\sigma\partial_\xi^\tau((\chi_-(t,x,\xi))^2)\right| \\ &\qquad\leqslant C_{\alpha,\beta,\sigma,\tau}t^{-\varkappa(|\sigma|+|\tau|+|\alpha|+|\beta|)}(1+|\xi|)^{-\rho'(|\beta|+|\tau|)+\delta'(|\alpha|+|\sigma|)}p_-(t,x,\xi).\end{aligned} \tag{5.10}$$

By these estimates and (3.2), $p_-(t,x,\xi)$ is a weight function and $p_-(t,x,\xi) \in S_{\rho',\delta',\varkappa}(p_-(t,x,\xi))$. By Proposition 3.1, the complete symbol of composition $\chi_-(t,x,D_x)(tA-I)\chi_-(t,x,D_x)$ is the asymptotic sum of the series of derivatives estimated in (5.10). By these estimates, the complete symbol differs from p_- by a summand of class $S_{\rho',\delta',\varkappa}((1+|\xi|^2)^{-(\rho'-\delta')/2}p_-(t,x,\xi))$. The estimate (5.4) holds in view of Proposition 3.3. □

We introduce two families of approximate spectral projections. Let $\mathscr{E}_\pm(t) := 3(\chi_\pm(t,x,D_x))^2 - 2(\chi_\pm(t,x,D_x))^3$ be approximate spectral projections in $L_2(\mathbb{R}^d)$. We define $E_\pm(t)$ by

$$\begin{aligned}E_-(t) &:= \zeta_-(t)\mathscr{E}_-(t)\zeta_-(t), \quad E_- : L_2(\Omega) \mapsto C_0^\infty(\Omega),\\ E_+(t) &:= \zeta_+(t)\mathscr{E}_+(t)\zeta_+(t), \quad E_+ : L_2(\mathbb{R}^d) \mapsto C_0^\infty(\mathbb{R}^d).\end{aligned}$$

Since $0 \leqslant 3s^2-2s^3 \leqslant 1$ for $-1 \leqslant s \leqslant 3/2$, from (5.3) it follows that $0 \leqslant \mathscr{E}_\pm$, $E_+ \leqslant I$ in $L_2(R^d)$ and $0 \leqslant E_- \leqslant I$ in $L_2(\Omega)$ for all t and large t_0,. We set

$$V(t) := (2\pi)^{-d}\int\limits_\Omega v(x)t^{\frac{d}{m+\varphi(x)}}\,dx.$$

It is easy to show that $tV'(t)/V(t) \to d/m$ as $t \to +\infty$ so that the function V is regular. By Remark 2.1, we replace $W(t,ct^{1-\nu})$ with $Ct^{-\nu}V(t)$ in estimates for traces and the spectrum distribution function.

Lemma 5.2. *The traces of operators $E_\pm(t)$ in $L_2(\mathbb{R}^d)$ admit the asymptotics*

$$\operatorname{tr} E_\pm(t) = V(t)(1 + O(t^{-\nu})), \quad t \to +\infty,$$
$$\operatorname{tr}\big(E_\pm(t)(I - E_\pm(t))\big) = O(t^{-\nu}V(t)), \quad t \to +\infty.$$

Proof. We consider only the operator E_- since E_+ is treated in a similar way. For every t the operator $E_-(t)$ is an integral operator with smooth compactly supported kernel in $\Omega \times \Omega$:

$$(2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{i(x-y)\xi} \zeta_-(x,t) e_-(t, (x+y)/2, \xi) \zeta_-(y,t) d\xi,$$

where $e_-(t,x,\xi) \in S^0_{\rho',\delta',\varkappa}$ is the Weyl symbol of $\mathscr{E}_-(t)$. The trace of $E_-(t)$ is the integral

$$\operatorname{tr} E_-(t) = (2\pi)^{-d} \int\limits_{\Omega} \int\limits_{\mathbb{R}^d} (\zeta_-(x,t))^2 e_-(t,x,\xi) dx d\xi. \tag{5.11}$$

Up to symbols of class $S^{-\infty}_{\rho',\delta',\varkappa}$ contributing to the integral (5.11) with order $O(t^{-\infty})$, we have

$$e_-(t,x,\xi) = \begin{cases} 1, & ta(x,\xi) \geqslant 1 + t^{-\nu}, \\ 0, & ta(x,\xi) \leqslant 1 + t^{-\nu}/2. \end{cases} \tag{5.12}$$

Estimating the integral and taking into account (4.3), we get

$$\begin{aligned} \int\limits_{R^d} e_-(t,x,\xi) d\xi &= \int\limits_{ta(x,\xi)>1} d\xi + O\Bigg(\int\limits_{1<ta(x,\xi)<1+t^{-\nu}} d\xi \Bigg) + O(t^{-\infty}) \\ &= t^{\frac{d}{m+\varphi(x)}} v(x) + O\Big(t^{\frac{d}{m+\varphi(x)}} (1 - (1+t^{-\nu})^{-\frac{d}{m+\varphi(x)}} \Big) + O(t^{\frac{d}{m}-\nu}) \\ &= t^{\frac{d}{m+\varphi(x)}} v(x) + O(t^{\frac{d}{m}-\nu}). \end{aligned}$$

Since S is Lipschitz, the measure of the boundary strip of width $t^{-\nu}$ has order $t^{-\nu}$. Therefore, integrating over Ω, we find

$$\operatorname{tr} E_-(t) = V(t) + O(t^{\frac{d}{m}-\nu}) + O\Bigg(\int\limits_{0<\zeta_-(x,t)<1} t^{\frac{d}{m+\varphi(x)}} dx \Bigg) = V(t) + O(t^{\frac{d}{m}-\nu}).$$

We consider the operator $E_-(I - E_-)$. We write

$$(E_-)^2 = \zeta_- \mathscr{E}_- (\zeta_-)^2 \mathscr{E}_- \zeta_- = (\zeta_-)^2 (\mathscr{E}_-)^2 (\zeta_-)^2 + \zeta_- Q_1 \zeta_-.$$

The operator Q_1 includes the commutators of ζ_- and $\mathscr{E}_-$:

$$Q_1 := \zeta_- \mathscr{E}_- [(\zeta_-), \mathscr{E}_-] - [(\zeta_-), \mathscr{E}_-] \mathscr{E}_- \zeta_- - ([(\zeta_-), \mathscr{E}_-])^2.$$

Thus, $E_-(I - E_-) = \zeta_- Q_2 \zeta_- - \zeta_- Q_1 \zeta_-$, where $Q_2 := \mathscr{E}_- - \zeta_-(\mathscr{E}_-)^2 \zeta_-$. From (5.12) and Proposition 3.1 it follows that, up to symbols of class $S^{-\infty}_{\rho',\delta',\varkappa}$, the support of the symbol of Q_2 belongs to the set

$$\{t, x, \xi |\ 0 < \zeta_-(x,t) < 1 \text{ or } t^{-\nu}/2 \leqslant ta(x,\xi) - 1 \leqslant t^{-\nu}\}. \tag{5.13}$$

The trace of $\zeta_- Q_2 \zeta_-$ has order $O(t^{-\nu}V(t))$.

It remains to show that $\operatorname{tr}(\zeta_- Q_1 \zeta_-)$ is estimated in the same way. Since $[\zeta_-, \chi_-] \in \Psi^{-\delta}_{\rho',\delta',\varkappa}$ and $\chi_- \in \Psi^0_{\rho',\delta',\varkappa}$ (Assertions 1 and 2 of Lemma 5.1), the commutator $[\zeta_-, \chi_-^k]$ belongs to the class $\Psi^{-\delta}_{\rho',\delta',\varkappa}$, $k \in \mathbb{N}$, and, consequently, $[\zeta_-, \mathscr{E}_-] \in \Psi^{-\delta}_{\rho',\delta',\varkappa}$. Up to symbols of order $-\infty$, the support of the symbol of this commutator, together with the support of the symbol of Q_1, belongs to the set (5.13) and, consequently, $\operatorname{tr}(\zeta_- Q_1 \zeta_-) = O(t^{-\nu}V(t))$. □

Lemma 5.3. *There exist C and t_0 such that for all $t > t_0$*

$$((tA - I)E_-(t)u, E_-(t)u) \geqslant -Ct^{-\nu}\|u\|^2, \quad u \in L_2(\Omega). \tag{5.14}$$

Proof. Taking the commutator of $\mathscr{E}_-(t)$ and ζ_-, we find

$$\begin{aligned} &((tA - I)E_-(t)u, E_-(t)u) = ((tA - I)\mathscr{E}_-(t)\zeta_-^2 u, \mathscr{E}_-(t)\zeta_-^2 u) \\ &+ 2\operatorname{Re}((tA - I)\zeta_-\mathscr{E}_-(t)\zeta_- u, [\zeta_-, \mathscr{E}_-(t)]\zeta_- u) + ((tA - I)[\zeta_-, \mathscr{E}_-(t)]\zeta_- u, [\zeta_-, \mathscr{E}_-(t)]\zeta_- u). \end{aligned} \tag{5.15}$$

We represent $\mathscr{E}_-(t) = \chi(t, x, D_x)Q(t)$, where $Q(t) := 3\chi(t, x, D_x) - 2\chi(t, x, D_x)^2$ is a bounded operator in L_2. The inequality (5.4) yields the lower estimate for the first term:

$$((tA - I)\mathscr{E}_-(t)\zeta_-^2 u, \mathscr{E}_-(t)\zeta_-^2 u) \geqslant -t^{-\nu}\|Q(t)\zeta_-^2 u\|^2 \geqslant -Ct^{-\nu}\|u\|^2.$$

Let us prove that the moduli of the last two terms in (5.15) are estimated by $Ct^{-\nu}\|u\|^2_{L_2}$. Since the L_2-norms of the operators $\mathscr{E}_-(t) \in \Psi^0_{\rho',\delta',\varkappa}$ (Assertion 1 of Lemma 5.1) and ζ_- are bounded uniformly with respect to $t > 1$, it suffices to verify that the L_2-norm of the operator $(tA - I)[\zeta_-, \mathscr{E}_-(t)]$ does not exceed $Ct^{-\nu}$. For this purpose we prove that $t^\nu(tA - I)[\zeta_-, \mathscr{E}_-(t)] \in \Psi^0_{\rho',\delta',\varkappa}$. Let $k_-(t, x, \xi)$ be the Weyl symbol of the commutator $[\zeta_-, \mathscr{E}_-(t)]$. In the proof of Lemma 5.2, we found that $k_-(t, x, \xi) \in S^{-\delta}_{\rho',\delta',\varkappa}$. The Weyl symbol of the composition $(tA - I)[\zeta_-, \mathscr{E}_-(t)]$ is represented by the asymptotic sum (cf. Proposition 3.1)

$$\sum_{\alpha,\beta} c_{\alpha,\beta}\partial_x^\alpha \partial_\xi^\beta (ta(x, \xi) - 1)\partial_\xi^\alpha \partial_x^\beta k_-(t, x, \xi). \tag{5.16}$$

The modulus of the term with $\alpha = \beta = 0$ is estimated by $Ct^{-\nu}(1 + |\xi|)^{-\delta}$ since, up to symbols of order $-\infty$, the support of k_- belongs to the set (5.13)

We consider the term with $|\alpha| + |\beta| > 0$ in (5.16). By Assertion 2 of Lemma 5.1 and the inequalities (4.2),

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta (ta(x, \xi) - 1)\partial_\xi^\alpha \partial_x^\beta k_-(t, x, \xi)| &\leqslant C_{\alpha,\beta} t^{-\varkappa(|\alpha|+|\beta|)}(1 + |\xi|)^{-\delta-(\rho'-\delta)|\beta|-(\rho-\delta')|\alpha|} \\ &\leqslant C'_{\alpha,\beta} t^{-\nu-\varkappa(|\alpha|+|\beta|)}(1 + |\xi|)^{-\delta-(\rho'-\delta')(|\alpha|+|\beta|)}. \end{aligned}$$

The last inequality is valid because of the estimate (5.2) and the choice of ρ, δ, ρ', δ'.

The derivatives of the series (5.16) with respect to x, ξ are estimated in a similar way. Thus, $t^\nu(tA - I)[\zeta_-, \mathscr{E}_-(t)] \in \Psi^0_{\rho',\delta',\varkappa}$ and we obtain an estimate for the last two terms in (5.15). □

We set $A_0(t) := \zeta_0(t)A(x, D_x)\zeta_0(t)$, where $\zeta_0(t)$ is the operator of multiplication by $\zeta_0(x, t)$ in $L_2(\mathbb{R}^d)$. For the sake of brevity we write ζ_0 instead of $\zeta_0(t)$.

Lemma 5.4. *There exist C and t_0 such that for all $t > t_0$*

$$((tA_0(t) - I)(I - E_+(t))u, (I - E_+(t))u) \leqslant Ct^{-\nu}\|u\|^2, \quad u \in L_2(\mathbb{R}^d). \tag{5.17}$$

Proof. As in the proof of Lemma 5.3, we consider the commutator of ζ_0 and $\mathscr{E}_+(t)$. Taking into account that the norm of ζ_0 is equal to 1 and $\zeta_0(x,t)\zeta_+(x,t) = \zeta_0(x,t)$, we have

$$\begin{aligned}
&((tA_0(t) - I)(I - E_+(t))u, (I - E_+(t))u) \leqslant ((tA - I)\zeta_0(I - \zeta_+\mathscr{E}_+(t)\zeta_+)u, \zeta_0(I - \zeta_+\mathscr{E}_+(t)\zeta_+)u) \\
&= ((tA - I)(I - \mathscr{E}_+(t))\zeta_0 u, (I - \mathscr{E}_+(t))\zeta_0 u) - 2\mathrm{Re}\,((tA - I)[\zeta_0, \mathscr{E}_+(t)]\zeta_+ u, (I - \mathscr{E}_+(t))\zeta_0 u) \\
&+ ((tA - I)[\zeta_0, \mathscr{E}_+(t)]\zeta_+ u, [\zeta_0, \mathscr{E}_+(t)]\zeta_+ u).
\end{aligned} \tag{5.18}$$

Further arguments repeat the proof of Lemma 5.3. We write $I - \mathscr{E}_+(t) = (I - \chi_+(t, x, D_x))\widetilde{Q}(t)$, where $\widetilde{Q}(t) = I + \chi_+ - 2\chi_+^2$ is a bounded operator in L_2. The first term on the right-hand side of (5.18) is estimated with the help of the inequality (5.5):

$$((tA - I)(I - \mathscr{E}_+(t))\zeta_+ u, (I - \mathscr{E}_+(t))\zeta_+ u) \leqslant t^{-\nu}\|\widetilde{Q}(t)\zeta_+ u\|^2 \leqslant Ct^{-\nu}\|u\|^2.$$

The moduli of the remaining two terms are estimated by $Ct^{-\nu}\|u\|^2$ since the $L_2(\mathbb{R}^d)$-norm of $(tA - I)[\zeta_+, \mathscr{E}_+(t)]$ does not exceed $Ct^{-\nu}$ (cf. the proof of Lemma 5.3). □

Proof of Theorem 4.1. The required assertion follows from Lemmas 5.1–5.4 which verify the assumptions of Lemma 2.1. From Lemma 5.3 and Assertion (A) of Lemma 2.1 we obtain the lower estimate for the spectrum distribution function of the operator A_Ω : $N(t, A_\Omega) \geqslant V(t)(1 - Ct^{-\nu})$ with $t \geqslant t_0$. Lemma 5.3, Assertion (B) of Lemma 2.1, together with Remark 2.2, and the obvious inequality $A_\Omega \leqslant A_0(t)$ in $L_2(\mathbb{R}^d)$ yield the upper estimate $N(t, A_\Omega) \leqslant N(t, A_0(t)) \leqslant V(t)(1 + Ct^{-\nu})$. □

6 Examples

We describe the simplest example of a situation where the assumptions of Theorem 4.1 are satisfied. Let A be a pseudodifferential operator with Weyl symbol $a(x, \xi)$ in the Hörmander class $S^{-m}_{1,\delta}$ ($m > 0$) with any $\delta > 0$. The symbol a is asymptotically homogeneous at infinity, i.e., $a(x, \xi) = a_0(x, \xi) + O(|\xi|^{-(m+\theta)})$ as $|\xi| \to +\infty$ for any $\theta < 1$, and the function $a_0(x, \xi) \geqslant 0$ is homogeneous with respect to ξ with degree $-(m + \varphi(x))$. Let the symbol $a(x, \xi)$ be formally hypoelliptic, i.e., (4.1) and (4.2) hold with parameters $\rho = 1$ and any $\delta > 0$, whereas the function $v(x) = \mathrm{vol}_d\{\xi \in \mathbb{R}^d | \, a_0(x, \xi) > 1\}$ is positive in $\overline{\Omega}$. Then the assumptions of Theorem 4.1 are satisfied and the spectrum distribution function of the corresponding operator A_Ω admits the asymptotic formula (4.5) for any $\nu < 1/2m_0$.

The order of spectrum asymptotics essentially depends on the behavior of φ near Γ. We mention some simplest examples of applying the Laplace method (cf., for example, [11]) to computations of asymptotics of the integral in formula (4.5).

6.1. Let Γ be a set of positive measure. Since $\varphi(x) \equiv 0$ on Γ, we have $\partial^\alpha \varphi(x) \equiv 0$ on Γ, and the spectrum asymptotics is of purely power character. However, the remainder converges to zero slower than any negative power of logarithm:

$$N(t, A_\Omega) = (2\pi)^{-d} \int\limits_\Omega t^{\frac{d}{m+\varphi(x)}} v(x) dx (1 + O(t^{-\nu})) = (2\pi)^{-d}\, t^{\frac{d}{m}} \int\limits_\Gamma v(x)\, dx (1 + O(r(t)),$$

where $r(t)(\ln t)^\epsilon \to +\infty$ as $t \to +\infty$ for any $\epsilon > 0$.

6.2. Suppose that the measure of Γ is equal to zero and Γ is a smooth surface of codimension k, $1 \leqslant k \leqslant d$ (Γ is a point if $k = d$). We denote by $n_1(x), \cdots n_k(x)$ the orthonormal basis of normals to Γ at a point $x \in \Gamma$. In the situation of a general position, the $k \times k$-matrix of second order derivatives $\varphi''_{nn}(x) := (\varphi_{n_i n_j}(x))$ is nonsingular. Then we have the asymptotic series in powers of $(\ln t)^{-1}$:

$$N(t, A_\Omega) = (2\pi)^{-d} \int\limits_\Omega t^{\frac{d}{m+\varphi(x)}} v(x) dx (1 + O(t^{-\nu})) \sim t^{\frac{d}{m}} (\ln t)^{-\frac{k}{2}} \sum_{j=0}^{\infty} p_j (\ln t)^{-j}.$$

The leading term of the asymptotics is expressed as

$$p_0 = (2\pi)^{-d+\frac{k}{2}} \int\limits_\Gamma \left(\det \varphi''_{nn}(x)\right)^{-\frac{1}{2}} v(x)\, d\Gamma(x),$$

for $k < d$ and

$$p_0 = (2\pi)^{-\frac{k}{2}} \left(\det \varphi''_{nn}(x^*)\right)^{-\frac{1}{2}} v(x^*)$$

for $k = d$, i.e., $\Gamma = \{x^*\} \in \Omega$. In the case $\Gamma = \{x^*\} \in \Omega$, in the general case, the asymptotics of the integral (4.5) is determined by the asymptotics of the volume of $\{x \in \Omega | \, \varphi(x) < s\}$ as $s \to +0$ (cf. [11]). In the degenerate case $\det \varphi''_{nn}(x^*) = 0$, it is possible to obtain the asymptotics of this volume in terms of the Newton polyhedron for the function φ (cf. [12]).

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