DOI 10.1007/s10958-015-2368-7
Journal of Mathematical Sciences, Vol. 207, No. 2, May, 2015

ATTRACTORS OF m-HESSIAN EVOLUTIONS

N. M. Ivochkina*
St. Petersburg State University of Architecture and Civil Engineering
4, 2-nd Krasnoarmeiskaia Str. St. Petersburg 190005, Russia
mail@NI1570.spb.edu

N. V. Filimonenkova
St. Petersburg State University of Architecture and Civil Engineering
2-nd Krasnoarmeiskaia St. 4, St. Petersburg 190005, Russia
NF33@yandex.ru

UDC 517.95

We study the asymptotic behavior of C^2-evolutions $u = u(x,t)$ under a given action of the m-Hessian evolution operators and boundary conditions. We obtain sufficient (close to necessary) conditions for the convergence of solutions to the first initial-boundary value problem for the m-Hessian evolution equations to stationary functions as $t \to \infty$. Bibliography: 18 titles.

Dedicated to Professor N. N. Uraltseva

1 Introduction

The development of the theory of stationary Hessian equations [1]–[4] naturally gives rise to the study of Hessian evolution equations. Apparently, the first examples appeared in [5], where fully nonlinear equations, in particular, the parabolic Monge–Ampère equation (cf. also [6])

$$-u_t \det u_{xx} = f > 0, \quad (x,t) \in Q_T = \Omega \times (0;T), \quad \Omega \subset R^n, \tag{1.1}$$

were studied within the framework of the Krylov theory of Bellman equations. Some sufficient conditions for the existence of admissible solutions to the first initial-boundary value problem for the equations

$$-u_t + \mathrm{tr}_m^{\frac{1}{m}} u_{xx} = f > 0, \quad (x,t) \in Q_T = \Omega \times (0;T), \quad m = 1, \ldots, n, \tag{1.2}$$

were obtained in [7]. Equations (1.2) look like the most natural fully nonlinear counterpart of the heat equation, $m = 1$. On the other hand, the first initial-boundary value problem for the logarithmic parabolic Hessian equations

$$-u_t + \log\, \mathrm{tr}_{m,l} u_{xx} = f, \quad \mathrm{tr}_{m,l} u_{xx} := \frac{\mathrm{tr}_m u_{xx}}{\mathrm{tr}_l u_{xx}}, \quad 0 \leqslant l < m \leqslant n, \tag{1.3}$$

* To whom the correspondence should be addressed.

Translated from *Problems in Mathematical Analysis* **78**, January 2015, pp. 103-110.

was involved in the study of Hessian integral norms introduced in [8], where it was proved that the first initial-boundary value problem for Equations (1.3) is solvable in the case $l = 0$. For general l and m the solvability was established in [9], where also the asymptotic behavior of admissible solutions was studied. Poincaré type inequalities for the Hessian integral norms were proved in [9]. Further generalizations of Equations (1.3) can be found in [10].

In this paper, we study the asymptotic behavior of the classical solution to the first initial–boundary value problem for the m-Hessian evolution equation

$$E_m[u] = f, \quad u|_{\partial' Q_T} = \varphi, \quad 1 \leqslant m \leqslant n, \tag{1.4}$$

where $\partial' Q_T = (\Omega \times \{t = 0\}) \cup (\partial\Omega \times [0;T])$, Ω is a bounded domain in R^n,

$$E_m[u] := -u_t T_{m-1}[u] + T_m[u], \quad (x,t) \in \overline{Q}_T, \tag{1.5}$$

$T_m[u] = T_m(u_{xx}) := \mathrm{tr}_m u_{xx}$, and u_{xx} is the Hesse matrix of u in the spatial variables. If $T_0(u_{xx}) \equiv 1$, then $T_1[u] = \Delta u$ and (1.5) is the heat operator, i.e., in this case, the problem (1.4) is the classical first initial–boundary value problem for the heat equation.

The m-Hessian evolution operators (1.5), including the case $m = n+1$, were introduced in [11]. Sufficient conditions for the solvability of the problem (1.4) in a weak (approximate) sense were obtained by using a parabolic version of the Aleksandrov–Bakel'man maximum principle [12, 13, 6]. In [11], the existence of an m-admissible solution was a priori assumed. Sufficient solvability conditions were obtained in [14]. Similar approaches were applied in [6] to the parabolic Monge–Ampère equation (1.1).

To illustrate our results, we consider the two-dimensional case, i.e., $\Omega \subset R^2$. If $m = 1$, then (1.5) is the heat operator and $E_2[u] = -u_t \Delta u + \det u_{xx}$. Assume that $u \in C^{2,1}(\overline{\Omega} \times [0;\infty))$ and $E_2[u] > 0$. Let $\mathbf{u}$ be a strictly convex function of class C^2 in $\overline{\Omega}$. Then there exists $\nu = \nu[\mathbf{u}]$ such that $\det \mathbf{u}_{xx} \geqslant \nu > 0$.

Theorem 1.1. *Assume that there is a point $x_0 \in \Omega$ such that $\Delta u(x_0, 0) > 0$ and*

$$\lim_{t\to\infty} |u(x,t) - \mathbf{u}(x)| = 0, \quad x \in \partial\Omega,$$

$$\lim_{t\to\infty} |E_2[u] - \det \mathbf{u}_{xx}| = 0, \quad x \in \Omega.$$

Then $\lim_{t\to\infty} |u(x,t) - \mathbf{u}(x)| = 0$ for all $x \in \overline{\Omega}$.

We may say that, under the assumptions of Theorem 1.1, $\mathbf{u} = \mathbf{u}(x)$ attracts $\{u = u(x,t)\}$ if $\mathbf{f} = \det \mathbf{u}_{xx}$, $x \in \overline{\Omega}$, and $\Phi = \mathbf{u}|_{\partial\Omega}$ attract $\{f = E_2[u]\}$, $(x,t) \in \overline{\Omega} \times [0;\infty)$, and $\{\varphi = u(x,t)\}$, $(x,t) \in \partial\Omega \times [0;\infty)$, respectively. We note that the function $u(x,0)$ in Theorem 1.1 is not necessarily convex.

Remark 1.2. In the case $E_2[u] > 0$, $(x,t) \in \overline{Q}_T$, there are no points $x \in \Omega$ such that $\Delta u(x,0) = 0$. Indeed, in the opposite case, the eigenvalues of u_{xx} either have different signs or vanish. Hence $E_2[u](x,0) = \det u_{xx}(x,0) \leqslant 0$, which is impossible.

Suppose that $\Delta u(x,0) < 0$, $x \in \overline{\Omega}$, i.e., there is no point x_0 as in Theorem 1.1. If the Dirichlet problem $\det \mathbf{v}_{xx} = \det \mathbf{u}_{xx}$, $\mathbf{v}|_{\partial\Omega} = -\Phi$ possesses a convex solution $\mathbf{v}$, then $-\mathbf{v}$ attracts $u(x,t)$ in view of Theorem 1.1.

Corollary 1.3. *Let $\partial\Omega \in C^{4+\alpha}$, $\alpha > 0$, be strictly convex, and let $u \in C^2(\overline{\Omega} \times [0;\infty))$. Assume that $E_2[u] > 0$ and there is $\mathbf{f} \in C^{2+\alpha}(\overline{\Omega})$, $\mathbf{f} > 0$, such that $\lim_{t\to\infty} |E_2[u] - \mathbf{f}| = 0$. If $\lim_{t\to\infty} |u(x,t)| = 0$, $x \in \partial\Omega$, then the convex solution to the Dirichlet problem*

$$\det \mathbf{u}_{xx} = \mathbf{f}, \quad \mathbf{u}|_{\partial\Omega} = 0 \tag{1.6}$$

attracts u or $-u$.

Corollary 1.3 means that the problem (1.6) has exactly two solutions: a convex solution $\mathbf{u}$ and $-\mathbf{u}$. By Remark 1.2, $u(x,t)$ converges to $\mathbf{u}$ if $\Delta u(x,0) > 0$ and to $-\mathbf{u}$ in the opposite case.

The aforesaid can be extended to the case of an arbitrary dimension, but for this purpose new geometric and algebraic notions should be introduced.

The main result of the paper is formulated in Section 3 and is proved in Section 5. The proof is based on a comparison theorem which requires construction of barriers. For such barriers we take solutions of auxiliary first order linear ordinary differential equations (cf. Section 4).

2 Preliminaries

We denote by $\mathrm{Sym}\,(N)$ the space of symmetric $N \times N$ matrices and by $T_p(S)$ the p-trace of a matrix $S \in \mathrm{Sym}\,(N)$, i.e., the sum of all principal p-minors of S, $1 \leqslant p \leqslant N$. We set $T_0(S) := 1$.

Definition 2.1 (cf. [1])**.** A matrix $S \in \mathrm{Sym}\,(N)$ is *m-positive* if $S \in K_m$, where

$$K_m = \{S : T_p(S) > 0,\ p = 1, \ldots, m\}. \tag{2.1}$$

The cones K_m play an important role in the theory of m-Hessian partial differential equations and admit different definitions. One of such definitions can be extracted from [15].

Definition 2.2. The cone K_m is the component of positiveness of the function $T_m(S)$ in $\mathrm{Sym}\,(N)$ containing $S = Id$.

Introducing the scalar product $(S^1, S^2) := tr(S^1 S^2)$, we can regard $\mathrm{Sym}\,(N)$ as a metric space equipped with the norm $||S||^2 = (S,S)$. In that sense, the cone (2.1) is an open set and $T_m(S) = 0$ for $S \in \partial K_m$. According to Definition 2.2, the set of nonnegative definite $N \times N$ matrices belongs to $\overline{K}_m$ for all $1 \leqslant m \leqslant N$.

Remark 2.3. By [15], the matrix Id in Definition 2.2 can be replaced with an arbitrary matrix $S_0 \in K_m$. By the Gårding theory, the function $T_m^{\frac{1}{m}}$ is concave in $\overline{K}_m$ [16] and, consequently, T_m is nonnegative monotone in $\overline{K}_m$, i.e., $T_m(S^1 + S^2) \geqslant T_m(S^i)$, $i = 1, 2$, for $S^1, S^2 \in \overline{K}_m$.

We consider the following subspace of $\mathrm{Sym}\,(N)$:

$$\mathbf{S}^{ev} = \{S^{ev} = (s_{kl})_0^n,\ s_{00} = s,\ s_{0i} = s_{i0} = 0,\ S = (s_{ij})_1^n \in \mathrm{Sym}\,(n)\} \tag{2.2}$$

and denote

$$E_m(s,S) := T_m(S^{ev}) = sT_{m-1}(S) + T_m(S), \quad 1 \leqslant m \leqslant n, \tag{2.3}$$

$$K_m^{ev} = \{s, S : E_p(s,S) > 0,\ p = 1, \ldots, m\}. \tag{2.4}$$

Denote by $S^{\langle i_1,\ldots,i_k\rangle} \in \mathrm{Sym}(N-k)$ the matrix obtained from $S \in \mathrm{Sym}(N)$ by crossing out rows and columns numbered by $i_1, \ldots, i_k$.

Then the following assertions hold.

(1) Let $1 < i < N$, and let $S^{\langle i\rangle}$ be $(m-1)$-positive. Then S is m-positive if and only if $T_m(S) > 0$.

(2) S is m-positive if and only if there exists a collection of numbers $(i_1, \dots, i_{m-1}) \subset \{i\}_1^N$ such that $T_m(S) > 0$, $T_{m-k}(S^{\langle i_1,\dots,i_k\rangle}) > 0$.

In the case $m = N$, $(i_1, \dots, i_{N-2}) = (N, \dots, 2)$ assertion (2) is the classic Sylvester criterion.

Using (1), (2) it is possible to specify the description of K_m^{ev} as follows:

$$K_m^{ev} = \{s, S : E_m(s, S) > 0, \quad S \in K_{m-1}\}. \tag{2.5}$$

Suppose that $\Omega \subset R^n$ is a bounded domain, $Q_T = \Omega \times (0;T)$, $\partial'' Q_T = \partial\Omega \times [0;T]$, $\partial' Q_T = (\Omega \times \{0\}) \cup \partial'' Q_T$, $u \in C^{2,1}(\overline{Q}_T)$. We introduce functional analogs of (2.2), (2.3), (2.4) by letting $S^{ev}[u]$ with $s[u] = -u_t$, $S[u] = u_{xx}$:

$$E_m[u] := T_m(S^{ev}[u]) = -u_t T_{m-1}(u_{xx}) + T_m(u_{xx}), \quad 1 \leqslant m \leqslant n, \tag{2.6}$$

$$\mathbf{K}_m^{ev}(\overline{Q}_T) = \{u \in C^{2,1}(\overline{Q}_T) : S^{ev}[u] \in K_m^{ev}, (x,t) \in (\overline{Q}_T)\}, \tag{2.7}$$

where u_{xx} is the Hesse matrix of u.

Unlike (2.5), the cone (2.7) is closed on bounded sets in $C^{2,1}(\overline{Q}_T)$. Indeed, let $u \in C^{2,1}(\overline{Q}_T)$. Then the set $\{S^{ev}[u], u \in C^{2,1}(\overline{Q}_T)\}$ is compact in $\mathrm{Sym}\,(n+1)$, which implies that for $u \in \mathbf{K}_m^{ev}(\overline{Q}_T)$ there exists $\nu = \nu[u] > 0$ such that $E_m[u] \geqslant \nu, (x,t) \in \overline{Q}_T$.

Definition 2.4. The operator (2.6) is called the *m-Hessian evolution operator* and $u \in \mathbf{K}_m^{ev}(\overline{Q}_T)$ is referred to as an *m-admissible evolution* in $\overline{Q}_T$.

3 The Main Result

We consider the problem (1.4), (1.5) for the operator $E_m[u]$ defined by (2.6):

$$E_m[u] = -u_t T_{m-1}(u_{xx}) + T_m(u_{xx}) = f, \quad u|_{\partial' Q_T} = \varphi, \quad 1 \leqslant m \leqslant n, \tag{3.1}$$

and the Dirichlet problem

$$T_m[\mathbf{u}] = \mathbf{f}, \quad x \in \Omega, \quad \mathbf{u}|_{\partial\Omega} = \Phi. \tag{3.2}$$

We formulate the main result of this paper.

Theorem 3.1. *Suppose that*

(i) *$f \geqslant \nu_m > 0$ and there is a point $x_0 \in \Omega$ where the matrix $\varphi_{xx}(x_0, 0)$ is $(m-1)$-positive,*

(ii) *the problem* (3.1) *has a solution $u \in C^{2,1}(\overline{\Omega} \times [0;\infty))$,*

(iii) *the problem* (3.2) *has an m-admissible solution $\mathbf{u}$ in $\overline{\Omega}$,*

(iv) *$\lim_{t\to\infty} |f(x,t) - \mathbf{f}(x)| = 0$, $x \in \overline{\Omega}$, and $\lim_{t\to\infty} |\varphi(x,t) - \Phi(x)| = 0$, $x \in \partial\Omega$.*

Then $\lim_{t\to\infty} |u(x,t) - \mathbf{u}(x)| = 0$, $x \in \overline{\Omega}$.

The proof of Theorem 3.1 is contained in Sections 4 and 5.

Theorem 3.2. *Let Assumption* (i) *be satisfied. Assume that there is a point $x_1 \in \Omega$ such that a matrix $\varphi_{xx}(x_1, 0)$ is not $(m-1)$-positive. Then there are no solutions of class $C^{2,1}(\overline{Q}_T)$ to the problem* (3.1) *if $T > 0$ is small.*

Necessary conditions for the solvability of the problem (3.1) were obtained in Theorem 1.2 in [14]. We formulate a refined version of this theorem. Denote by $\mathbf{k}_{m-1}[\partial\Omega]$ the $m-1$-curvature of $\partial\Omega$ (cf. the definition in [18]).

Theorem 3.3. *Assume that* $\partial\Omega \in C^{4+\alpha}$, $\mathbf{k}_{m-1}[\partial\Omega] > 0$, $f > 0$, $f \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$, $\varphi \in C^{4+\alpha,2+\alpha}(\partial' Q_T)$, $\varphi(x,0) \in \mathbf{K}_{m-1}(\overline{\Omega})$, *and* f, φ *satisfy the compatibility condition*

$$-\varphi_t(x,0)T_{m-1}(\varphi_{xx}(x,0)) + T_m(\varphi_{xx}(x,0)) - f(x,0) = 0, \quad x \in \partial\Omega. \tag{3.3}$$

Then there exists a unique solution $u \in \mathbf{K}_m^{ev}(\overline{Q}_T)$ *to the problem* (3.1) *in* $C^{2,1}(\overline{Q}_T)$; *moreover,* $u \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$.

By Theorem 3.2, the condition $\varphi(x,0) \in \mathbf{K}_{m-1}(\overline{\Omega})$ is necessary for the solvability of the problem (3.1) in $C^{2,1}(\overline{Q}_T)$ provided that n is odd. It is also necessary in a certain sense if n is even since $-\varphi(x,0) \in \mathbf{K}_{m-1}(\overline{\Omega})$.

An analog of Theorem 3.3 for the problem (3.2) was proved in [3, Theorem 1.2]. We formulate this result in our terminology.

Theorem 3.4. *Suppose that* $\partial\Omega \in C^{4+\alpha}$, $\mathbf{f} \in C^{2+\alpha}(\overline{\Omega})$, $\Phi(x) \in C^{4+\alpha}(\partial\Omega)$. *If* $\mathbf{k}_{m-1}[\partial\Omega] > 0$, $\mathbf{f} > 0$, *then there exists a unique m-admissible solution* $\mathbf{u}$ *to the problem* (3.2).

As was shown in [3], the condition $\mathbf{k}_{m-1}[\partial\Omega] > 0$ is necessary for the solvability of the problem (3.2) if $\mathbf{\Phi} = \mathrm{const}$ (cf. also [16]). More precisely, if there is a point $M_0 \in \partial\Omega$ such that $\mathbf{k}_{m-1}[\partial\Omega](M_0) < 0$, then there are no solutions of class $C^2(\overline{\Omega})$ to Equation (3.2) with constant Dirichlet data. The following assertion extends this result to m-Hessian evolution equations.

Theorem 3.5. *Let* $\partial\Omega \in C^{4+\alpha}$, $f \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$, $f \geqslant \nu_m > 0$, $\varphi \in C^{4+\alpha,2+\alpha}(\partial' Q_T)$, $\varphi(x,0) \in \mathbf{K}_{m-1}(\overline{\Omega})$, $1 < m \leqslant n$. *Assume that there are* $x_0 \in \partial\Omega$, $t_0 \in (0;T)$, *and* $r > 0$ *such that* $\mathbf{k}_{m-1}[\partial\Omega](x_0) < 0$, $\varphi_t(x_0,t_0) \geqslant 0$, $\varphi(x,t_0) = C = \mathrm{const}$ *in* $B_r(x_0)$. *Then there are no* $C^{2,1}$*-solutions to the problem* (3.1), (3.3) *for* $t \geqslant t_0$.

Proof. Assume the contrary. Let there exist such a solution $u \in C^{2,1}(\overline{Q}_{t_0})$. Then u is unique, and belongs to $\mathbf{K}_m^{ev}(\overline{Q}_{t_0})$. Hence $u(x,t_0) \in \mathbf{K}_{m-1}(\overline{\Omega})$. However, from (3.1) it follows that $T_m(u_{xx})(x_0,t_0) > 0$ and, by continuity, $u(x,t_0) \in \mathbf{K}_m(\overline{\Omega}) \cap B_{r_1}(x_0)$ with some $0 < r_1 < r$. Moreover, $\partial\Omega \cap B_{r_1}(x_0)$ is a level surface of an m-admissible function since $\varphi(x,t_0) = C$ in $B_r(x_0)$ and, consequently, $\mathbf{k}_{m-1}[\partial\Omega](x_0) > 0$. We arrive at a contradiction. □

To prove Theorem 3.1, we need the following assertion.

Theorem 3.6. *Let* $v, w \in C(\overline{Q}_T)$ *be such that* $v \in C^{2,1}(Q_T)$ *and* $w \in \mathbf{K}_m^{ev}(Q_T)$. *If*

$$E_m[v] - E_m[w] \leqslant 0, \quad (x,t) \in Q_T, \quad (w-v)|_{\partial' Q_T} \leqslant 0, \tag{3.4}$$

then $w \leqslant v$ *in* $\overline{Q}_T$.

Proof. For $w^\varepsilon = w - \varepsilon t$, $\varepsilon > 0$ we have

$$T_{m-1}[w^\varepsilon] = T_{m-1}[w] > 0, \quad E_m[w^\varepsilon] = \varepsilon T_{m-1}[w] + E_m[w] > E_m[w], \quad (x,t) \in Q_T. \tag{3.5}$$

Taking into account that w is an m-admissible evolution and using (2.5), (2.6), we see that $w^\varepsilon \in \mathbf{K}_m^{ev}(Q_T)$. From (3.4) and (3.5) it follows that

$$E_m[v] - E_m[w^\varepsilon] < 0, \quad (x,t) \in Q_T, \quad (w^\varepsilon - v)|_{\partial' Q_T} \leqslant 0. \tag{3.6}$$

Assume that

$$\sup_{Q_T}(w^\varepsilon - v) = (w^\varepsilon - v)(x_0, t_0), \quad (x_0, t_0) \in \overline{Q}_T \backslash \partial' Q_T, \tag{3.7}$$

and denote $S^{ev}[\,\cdot\,](x_0,t_0) = S_0^{ev}[\,\cdot\,]$. From (3.7) it follows that $S_0^{ev}[v - w^\varepsilon] \geqslant \mathbf{0}$. Since K_m^{ev} is a convex set, we have $S_0^{ev}[v] = S_0^{ev}[w^\varepsilon] + S_0^{ev}[v - w^\varepsilon] \in K_m^{ev}$. Since E_m is monotone in K_m^{ev} (cf. Remark 2.3), we find $E_m[v] - E_m[w^\varepsilon] \geqslant 0$ which contradicts the first inequality in (3.6). Hence the assumption (3.7) is impossible. Therefore, (x_0, t_0) belongs to $\partial' Q_T$ and the second inequality in (3.6) is valid for all $(x,t) \in \overline{Q}_T$. This is equivalent to the inequality $w - v \leqslant \varepsilon T$. Letting ε to zero, we obtain the required assertion. □

4 Functions θ, σ, and V

We consider the Cauchy problem for the linear ordinary differential equation

$$\theta' + b(\theta + h) = 0, \quad t \geqslant 0, \quad \theta(0) = \theta_0, \quad b = \text{const} > 0. \tag{4.1}$$

We have

$$\theta = \exp(-bt)\left(\theta_0 - b\int_0^t \exp(b\tau)h(\tau)d\tau\right). \tag{4.2}$$

The following two assertions are obvious.

Lemma 4.1. *Let θ be a solution to Equation* (4.1) *with $h = h^+ > 0$, $(h^+)'(t) \leqslant 0$, $h_0^+ < 1/m$. If $\theta_0 + h_0^+ \leqslant 0$, then*

$$\theta(t) + h^+(t) \leqslant 0, \quad \theta'(t) \geqslant 0. \tag{4.3}$$

If, in addition, $\lim_{t\to\infty} h^+(t) = \overline{h}^+$, *then* $\lim_{t\to\infty} \theta(t) = -\overline{h}^+$ *and* $\lim_{t\to\infty} \theta'(t) = 0$.

Lemma 4.2. *Assume that $h = -h^- < 0$, $(h^-)'(t) \leqslant 0$, $\theta_0 - h_0^- \geqslant 0$. Then*

$$\theta(t) - h^-(t) \geqslant 0, \quad \theta'(t) \leqslant 0. \tag{4.4}$$

If, in addition, $\lim_{t\to\infty} h^-(t) = \underline{h}^-$, *then* $\lim_{t\to\infty} \theta(t) = \underline{h}^-$ *and* $\lim_{t\to\infty} \theta'(t) = 0$.

We introduce the function $\sigma = \sigma(t)$ by

$$(1+\sigma)^m = 1 + m\theta, \quad 1 \leqslant m \leqslant n. \tag{4.5}$$

Denote by θ^+ the function θ from Lemma 4.1 and by σ^+ the corresponding solution to Equation (4.5). Since $h_0^+ < 1/m$, the function σ^+ is well defined, and (4.5), (4.3) imply

$$m\theta^+(t) \leqslant \sigma^+(t) < \theta^+(t) < 0, \quad (\sigma^+(t))' > 0. \tag{4.6}$$

Denote by θ^- the function θ from Lemma 4.2 and by σ^- the corresponding solution to Equation (4.5). Then an analog of (4.6) in the case $\theta_0 \geqslant 1$ is written as

$$0 < \frac{(1 + m\theta_0^-)^{\frac{1}{m}} - 1}{\theta_0^-}\theta^-(t) < \sigma^-(t) < \theta^-(t), \quad (\sigma^-(t))' \leqslant 0. \tag{4.7}$$

To prove Theorem 3.1, we use Theorem 3.6 with barriers of the form

$$V = \sigma(\mathbf{u} - A) + \mathbf{u}, \quad (x,t) \in \overline{Q}_T, \tag{4.8}$$

where $\sigma = \sigma(t)$ and a positive constant $A > 0$ are to be chosen, whereas $\mathbf{u} = \mathbf{u}(x)$ is a given C^2-function. The following identity plays a crucial role in the further considerations:

$$E_m[V] = (A - \mathbf{u})T_{m-1}[\mathbf{u}]\theta' + mT_m[\mathbf{u}](\theta + 1). \tag{4.9}$$

We will assume that $\mathbf{u}$ is an m-admissible solution to the problem (3.2) in Ω and there are parameters ν_m, μ_k such that

$$0 < \nu_m \leqslant \mathbf{f}, \quad T_k[\mathbf{u}] \leqslant \mu_k, \quad k = m-1, m, \quad x \in \Omega. \tag{4.10}$$

5 Asymptotic Behavior of m-Hessian Evolutions

We obtain an upper bound for m-admissible evolutions in a bounded cylinder Q_T.

Lemma 5.1. *Suppose that $u \in \mathbf{K}_m^{ev}(Q_T) \cap C(\overline{Q}_T)$ and $E_m[u] \geqslant \nu > 0$. Assume that there exist nonincreasing functions $h_i^+ = h_i^+(t) > 0$, $i = 1, 2$, such that*

$$(u - \Phi)|_{\partial' Q_T} \leqslant h_1^+, \quad \frac{1}{m}\left(1 - \frac{E_m[u]}{\mathbf{f}}\right) \leqslant h_2^+, \quad (x,t) \in Q_T. \tag{5.1}$$

Then

$$u(x,t) - \mathbf{uu}(x) \leqslant m(2mh_1^+(0) + \operatorname{osc}\Omega\mathbf{u})(-\theta^+(t)), \quad (x,t) \in \overline{Q}_T, \tag{5.2}$$

where $\theta^+ = \theta$ is given by (4.2) *with*

$$\begin{aligned} b^+ &= \frac{m\nu_m}{(2mh_1^+(0) + \operatorname{osc}_\Omega \mathbf{u})\mu_{m-1}}, \\ -h = h^+ &= \max\left\{\frac{\max\{1 - \nu/\mu_m; 1/2\}}{mh_1^+(0)} h_1^+(t); h_2^+(t)\right\}, \end{aligned} \tag{5.3}$$

the constants ν_m, μ_k, $k = m-1, m$, satisfy (4.10), *and $\theta_0^+ = h_0$.*

Proof. Applying Theorem 3.6 with $w = u$, we reduce the proof to construction of an upper barrier $V = V^+$ (cf. (4.8)). We begin by constructing the function $\sigma = \sigma^+$ in (4.8). By the inequalities $0 < h_0^+ < 1/m$ and (4.3), we have $-1 < -m\theta^+ < 0$. Hence the function $\sigma = \sigma^+$ is uniquely defined by (4.5) and the relation (4.6) holds. By (4.3), $0 < -\sigma t \leqslant h_1^+(t)$.

We define the constant A in (4.8) by

$$A = A^+ = A_1^+ + \sup_\Omega \mathbf{u}, \quad A_1^+ = 2mh_1^+(0). \tag{5.4}$$

By (5.3), $u - V^+ \leqslant 0$ on the parabolic boundary of Q_T. Thus, the second inequality in (3.4) holds. To prove the first inequality in (3.4), we use (4.9) and represent $E_m[V^+] - E_m[u]$ as

$$E_m[V^+] - E_m[u] = (A - \mathbf{u})T_{m-1}[\mathbf{u}]\Big((\theta^+)' + \frac{m\mathbf{f}}{(A - \mathbf{u})T_{m-1}[\mathbf{u}]}\Big(\theta^+ + \frac{\mathbf{f} - E_m[u]}{m\mathbf{f}}\Big)\Big). \tag{5.5}$$

By (5.3), the relations (4.3), (4.1), and (5.5) imply

$$E_m[V^+] - E_m[u] \leqslant (A - \mathbf{u})T_{m-1}[\mathbf{u}]\left((\theta^+)' + b(\theta^+ + h^+)\right) = 0,$$

i.e., the first inequality in (3.4) holds. Thus, $w = u$, $v = V^+$ satisfy the assumptions of Theorem 3.6. Hence $u - V^+ \leqslant 0$ in $\overline{Q}_T$. The inequality (5.2) is verified by a direct computation. □

Theorem 3.6 provides lower bounds for $C^{2,1}$-evolutions under certain conditions that are weaker than (5.1).

Lemma 5.2. *Let* $u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$. *We assume that there exists a nonincreasing function* $h^- > 0$ *such that*

$$(u - \Phi)|_{\partial' Q_T} \geqslant -h^-, \quad \frac{1}{m}\left(1 - \frac{E_m[u]}{\mathbf{f}}\right) \geqslant -h^-, \quad (x,t) \in Q_T. \tag{5.6}$$

Then

$$u(x,t) - \mathbf{u}(x) \geqslant -(A_1^- + \operatorname{osc}_\Omega \mathbf{u})\theta^-(t), \quad A_1^- = \frac{h_0^-}{((1 + mh_0^-)^{\frac{1}{m}} - 1)}, \quad (x,t) \in \overline{Q}_T, \tag{5.7}$$

where $\theta^- = \theta$ *is given by* (4.2) *with*

$$b^- = \frac{m\nu_m}{(A_1^- + \operatorname{osc}_\Omega \mathbf{u})\mu_{m-1}}, \quad \theta_0^- = h_0^-, \tag{5.8}$$

ν_m, μ_k, $k = m-1, m$, *are the constants in* (4.10).

Proof. We apply Theorem 3.6 to $w = V^- = \sigma^-(\mathbf{u} - (A_1^- + \sup_\Omega \mathbf{u})) + \mathbf{u}$, $v = u$, where σ^- satisfies (4.5) with $\theta = \theta^-$, which requires the inclusion $V^- \in \mathbf{K}_m^{ev}(Q_T)$. To ensure this inclusion, we note that $V^-_{xx} = (\sigma^- + 1)\mathbf{u}_{xx} \in K_m \subset K_{m-1}$ for all $x \in \overline{\Omega}$, $t \in [0;T]$ in view of the inequality $\sigma^- > 0$ and the choice of $\mathbf{u}$ (cf. Remark 2.3). By (2.5) and (2.7), it suffices to verify the inequality $E_m[V^-] > 0$. Indeed, the relations (4.9), (4.1), (4.4), (5.8) and the second inequality in (5.6) imply

$$E_m[V^-] > (A_1^- + \sup_\Omega \mathbf{u} - \mathbf{u})T_{m-1}[\mathbf{u}]((\theta^-)' + b^-\theta^-) > b^-h^- > 0, \quad (x,t) \in Q_T. \tag{5.9}$$

In a similar way, we obtain the first inequality in (3.4) in Q_T, i.e., $E_m[V^-] - E_m[u] \geqslant 0$.

By (4.4), (4.7) and the choice of A_1^- (cf. (5.2)), we have

$$(u - V^-)|_{\partial' Q_T} \geqslant -h^- + A_1^- \frac{(1 + mh_0^-)^{\frac{1}{m}}}{h_0^-}\theta^- \geqslant 0.$$

By Theorem 3.6, $u - V^- \geqslant 0$ for all $(x,t) \in \overline{Q}_T$ and, consequently, we obtain (5.7). □

If the assumptions of Lemmas 5.1 and 5.2 are satisfied for all $T > 0$, then $\mathbf{u}$ attracts all solutions to the problem (3.1) with data subject to Assumptions (i)–(iv) in $C(Q)$, $Q = \Omega \times [0;\infty)$.

Lemmas 5.1 and 5.2 lead to the following more general assertion.

Theorem 5.3. *Let $u \in C^{2,1}(Q) \cap C(\overline{Q})$. Assume that there is a point $(x_0, t_0) \in Q$ such that $u_{xx}(x_0, t_0)$ is an m-positive matrix, $0 < \nu \leqslant E_m[u] \leqslant \mu$ in $Q_0 = \Omega \times [t_0; \infty)$, and* (5.1), (5.6) *are satisfied in Q_0. Then $|u|_Q$ is bounded independently on t. If, in addition,*

$$\lim_{t\to\infty} h(t) = 0, \quad h = \max\{h^+; h^-\}, \quad t > t_0, \tag{5.10}$$

then $\mathbf{u} = \mathbf{u}(x)$ attracts $u = u(x,t)$ in $C(\overline{Q})$.

It is obvious that Assumption (iv) of Theorem 3.1 guarantees the existence of a function h in (5.10). Moreover, Assumptions (i)–(iii) are sufficient for the validity of the remaining assumptions of Theorem 5.3 with $t_0 = 0$. Hence Theorem 3.1 is a special case of Theorem 5.3.

To conclude the paper, we consider the heat operator $E_1[u] = -u_t + \Delta u$. In this case, $\mathbf{u}$ is a solution to the Poisson equation

$$\Delta \mathbf{u} = \mathbf{f}, \quad \mathbf{u} \in C^2(\Omega) \cap C(\overline{\Omega}). \tag{5.11}$$

Corollary 5.4. *Suppose that $u \in C^{2,1}(Q) \cap C(\overline{\Omega} \times [0; \infty))$ and $|\mathbf{f}| \leqslant \mu_1$. We assume that*

$$\begin{aligned} &\lim_{t\to\infty} E_1[u](x,t) = \mathbf{f}(x), \quad x \in \Omega, \\ &\lim_{t\to\infty} u(x,t) = \mathbf{u}(x), \quad x \in \partial\Omega. \end{aligned} \tag{5.12}$$

Then

$$\lim_{t\to\infty} u(x,t) = \mathbf{u}(x), \quad x \in \overline{\Omega}. \tag{5.13}$$

Proof. The equality (5.13) does not follow from Theorem 5.3 with $m = 1$ since it is not assumed that $\mathbf{f} \geqslant \nu > 0$ in (5.11). Therefore, we represent $\mathbf{u}$ as

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2, \quad \mathbf{u}_1 = \frac{1}{2}(\mathbf{u} + Cx^2), \quad \mathbf{u}_2 = \frac{1}{2}(\mathbf{u} - Cx^2), \quad C = \frac{\mu + \nu}{2n}, \tag{5.14}$$

with some $\nu > 0$. It is obvious that $\mathbf{u}_1$ and $-\mathbf{u}_2$ are solutions to problems similar to (5.11) with $\mathbf{f}_i \geqslant \nu/2$, $i = 1, 2$. With (5.14) we associate the evolutions u_i, $i = 1, 2$, $u = u_1 + u_2$, satisfying some relations similar to (5.12). We see that all the assumptions of Theorem 5.3 with $m = 1$ hold for u_1, $-u_2$, which leads to the required relation (5.13). □

Acknowledgments

The work is supported by "Nauchnye Shkoly" (grant No. 1771.2014.1), Sankt Petersburg State University (grant No. 6.38.670.2013), and the Russian Foundation for Basic Research (grant No. 15-01-07650).

References

1. N. M. Ivochkina, "A description of the stability cones generated by differential operators of Monge–Ampère type" [in Russian], *Mat. Sb.* **122**, No. 2, 265–275 (1983); English transl.: *Math. USSR, Sb.* **50**, 259–268 (1985).

2. N. M. Ivochkina, "Solution of the Dirichlet problem for some equations of Monge–Ampère type" [in Russian], *Mat. Sb.* **128**, No. 3, 403–415 (1985); English transl.: *Math. USSR-Sb.* **56**, No. 2, 403–415 (1987).

3. L. Caffarelly, L. Nirenberg, and J. Spruck, "The Dirichlet problem for nonlinear second order elliptic equations III. Functions of the eigenvalues of the Hessian," *Acta Math.* **155**, 261–301 (1985).

4. N. V. Filimonenkova, "On the classical solvability of the Dirichlet problem for nondegenerate m-Hessian equations" [in Russian], *Probl. Mat. Anal.* **60**, 89–110 (2011); English transl.: *J. Math. Sci., New York* **178**, No. 6, 666–694 (2011).

5. N. V. Krylov, *Nonlinear Elliptic and Parabolic Equations of the Second Order* [in Russian], Nauka, Moscow (1985); English transl.: Reidel, Dordrecht (1987).

6. K. Tso, "On an Aleksandrov–Bakel'man type maximum principle for second-order parabolic equations," *Commun. Partial Differ. Equations* **10**, 543–553 (1985).

7. N. M. Ivochkina and O. A. Ladyzhenskaya, "On parabolic problems generated by some symmetric functions of the eigenvalues of Hessian," *Topol. Methods Nonlinear Anal.* **4**, No. 1, 19–29 (1994).

8. X.-J. Wang, "A class of fully nonlinear elliptic equations and related functionals," *Indiana Univ. Math. J.* **43**, No. 1, 25–54 (1994).

9. N. S. Trudinger and X.-J. Wang, "A Poincaré type inequality for Hessian integrals," *Calc. Var. Partial Differ. Equ.* **6**, No. 4, 315–328 (1998).

10. K.-S. Chou and X.-J. Wang, "A variational theory of the Hessian equations," *Commun. Pure Appl. Math.* **54**, No. 9, 1029–1064 (2001).

11. N. M. Ivochkina, "On approximate solutions to the first initial boundary value problem for the m-Hessian evolution equations," *J. Fixed Point Theory Appl.* **4**, No. .1, 47–56 (2008).

12. N. V. Krylov, "Sequences of convex functions and estimates of the maximum of the solutions of a parabolic equation" [in Russian], *Sib. Mat. Zh.* **17**, 290–303 (1976); English transl.: *Sib. Math. J.* **17**, 226–236 (1976).

13. A. I. Nazarov and N. N. Uraltseva, "Convex-monotone hulls and an estimate of the maximum of the solution of a parabolic equation" [in Russian], *Zap. Nauchn. Sem. LOMI* **147**, 95–109 (1985); English transl.: *J. Sov. Math.* **37**, 851–859 (1987).

14. N. M. Ivochkina, "On classic solvability of the m-Hessian evolution equation," In: *Nonlinear Partial Differential Equations and Related Topics*, pp. 119–129, Am. Math. Soc., Providence, RI (2010).

15. L. Gårding, "An inequality for hyperbolic polynomials," *J. Math. Mech.* **8**, 957–965 (1959).

16. N. V. Filimonenkova and N. M. Ivochkina, "On the backgrounds of the theory of m-Hessian equations," *Commun. Pure Appl. Anal.* **12**, No. 4, 1687–1703 (2013).

17. N. M. Ivochkina, S. I. Prokof'eva, and G. V. Yakunina, "The Gårding cones in the modern theory of fully nonlinear second order differential equations" [in Russian], *Probl. Mat. Anal.* **64**, 63–80 (2012); English transl.: *J. Math. Sci., New York* **184**, No. 3, 295–315 (2012).

18. N. M. Ivochkina, "From Gårding cones to p-convex hypersurfaces" [in Russian], *Sovr. Mat. Fundam. Naprav.* **45** (2012); English transl.: *J. Math. Sci., New York* **201**, No. 5, 634–644 (2014).

Submitted on October 15, 2014