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SOLUTION OF THE AXISYMMETRIC PROBLEM OF CREEP AND DAMAGE FOR A PIECEWISE HOMOGENEOUS BODY WITH MERIDIONAL SECTION OF ANY SHAPE

S. M. Sklepus

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We consider the axisymmetric problem of creep and creep-induced damage for piecewise homogeneous bodies with meridional sections of any shape. We develop a method for the solution of the initial boundary-value problem based on the combined application of the R-function method and the Runge–Kutta–Merson method. The structures of the solution for the main types of boundary conditions are constructed. We present an example of calculation of creep and long-term strength for a three-layer cylinder used as a computational scheme of a solid-oxide fuel element.

State-of-the-Art of the Problem. Statement of the Initial Boundary-Value Problem of Creep

The problems of determination of the stress-strain state and strength of piecewise homogeneous cylindrical bodies are thoroughly described in the Ukrainian and foreign literature. At the same time, the nonlinear deformation of piecewise homogeneous bodies, in particular, the problems of creep and damage have not been adequately studied. This is connected with the complexity of solution of nonlinear initial boundary-value problems for piecewise homogeneous systems and with difficulties connected with the construction of determining relations, which must take into account various effects of deformation of contemporary materials. Moreover, the practical analyses of creep, damage, and long-term strength often require taking into account the complex geometric shape of the body or interfaces of its components.

Consider a body of revolution of finite sizes referred to a cylindrical coordinate system $Orz\varphi$, which consists of M components $V_1, V_2, \ldots, V_M$ $(V = V_1 \cup V_2 \cup \ldots \cup V_M)$ rigidly connected with each other. The body is under the action of external surface loads applied to a part S_p of its surface and a temperature field $T = T(r, z, t)$. The distribution of loads on S_p and given kinematically possible displacements on the surface S_u are such that the desired solution is independent of φ. The external forces and temperature very slowly vary with time that and, hence, the inertial terms in equations of motion can be neglected. The strains in the body remain small in the process of creep.

We denote by ∂V_{ab} the interface of the neighboring parts of the body V_a and V_b. The axis Oz coincides with the axis of revolution. The section of the body in the plane rOz has the shape of the domain Ω with boundary $\partial\Omega$. The domain Ω is the union of constituent domains Ω_k, $k = 1, \ldots, M$, with boundaries $\partial\Omega_k$. The rates of displacements and external loads are given on the parts of the boundary $\partial\Omega_u$ and $\partial\Omega_p$, respectively. We denote by $\partial\Omega_{ab}$ the interface of the neighboring domains Ω_a and Ω_b. By $\partial\Omega^*_{ab}$ and $\partial\Omega^*_{ba}$, we denote the sides of the surface $\partial\Omega_{ab}$ that belong to Ω_a and Ω_b, respectively. Assume that the materials

Podgornyi Institute for Problems in Mechanical Engineering, Ukrainian National Academy of Sciences, Kharkov.

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of the components of the body are isotropic and that the geometric and mechanical characteristics of each part are independent of the angular coordinate φ.

The components of the total strain rate tensor $\dot{\varepsilon}_{ij}$ consist of the components of the elastic strain rate tensors $\dot{\varepsilon}^e_{ij}$, thermal strain rate tensor $\dot{\varepsilon}^T_{ij}$, and irreversible creep strain rate tensor $\dot{p}_{ij}$

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^T_{ij} + \dot{p}_{ij}, \quad i,j = 1,2,3.$$

In the cylindrical coordinate system, we have

$$\dot{\varepsilon}_r(r,z,t) = \dot{\varepsilon}^e_r(r,z,t) + \dot{\varepsilon}^T_r(r,z,t) + \dot{p}_r(r,z,t),$$

$$\dot{\varepsilon}_z(r,z,t) = \dot{\varepsilon}^e_z(r,z,t) + \dot{\varepsilon}^T_z(r,z,t) + p_z(r,z,t),$$

$$\dot{\varepsilon}_\varphi(r,z,t) = \dot{\varepsilon}^e_\varphi(r,z,t) + \dot{\varepsilon}^T_\varphi(r,z,t) + \dot{p}_\varphi(r,z,t),$$

$$\dot{\varepsilon}_{rz}(r,z,t) = \dot{\varepsilon}^e_{rz}(r,z,t) + \dot{\varepsilon}^T_{rz}(r,z,t) + \dot{p}_{rz}(r,z,t).$$

Here, the overdot denotes the total derivative with respect to time t.

Thermal strains are calculated by the formula

$$\varepsilon^T_r = \varepsilon^T_z = \varepsilon^T_\varphi = \alpha(T - T_0), \quad \varepsilon^T_{rz} = 0,$$

where $T = T(r,z,t)$ is the temperature, $\alpha = \alpha(r,z,T)$ is the linear thermal expansion coefficient, and T_0 is the temperature at which stresses and strains are absent. We assume that the temperature distribution $T(r,z,t)$ is given or known from the solution of the problem of nonstationary heat conduction.

We can write the determining relations of creep and kinetic equations for the state parameters within the framework of the Rabotnov theory of structural parameters, in the general case, for the material of the k th component of the body, in the form [3]

$$\dot{p}^k_{ij} = f^k_{ij}(\sigma_{ek}, \psi_k, q_{1k}, q_{2k}, \ldots, q_{Nk}),$$

$$\dot{\psi}_k = g_k(\sigma_{e1k}, \psi_k, q_{1k}, q_{2k}, \ldots, q_{Nk}),$$

$$\dot{q}_{ik} = h_{ik}(\sigma_{e2k}, \psi_k, q_{1k}, q_{2k}, \ldots, q_{Nk}).$$

Here, σ_{ek}, σ_{e1k}, σ_{e2k}, $k = 1,\ldots,M$, are equivalent stresses, and ψ_k, q_{ik}, $i = 1,\ldots,N$, are the scalar damage parameters and structural state parameters.

The boundary-value problem of creep and creep-induced damage for the axisymmetrically loaded body of revolution of finite sizes at an arbitrary moment of time $t \neq 0$ can be reduced to the variational problem of finding the minimum of a functional in the Lagrange form [1]:

$$\Lambda(\dot{\boldsymbol{U}}) = \frac{1}{2}\sum_{i=1}^{M}\iint\limits_{\Omega_i}\Big[\lambda_{1i}(\dot{u}_{r,r}^2+\dot{u}_{z,z}^2+\dot{u}_r^2 r^{-2})+G_i(\dot{u}_{r,z}+\dot{u}_{z,r})^2+2\lambda_i(\dot{u}_{r,r}\dot{u}_{z,z}+\dot{u}_r(\dot{u}_{r,r}+\dot{u}_{z,z})r^{-1})\Big]r\,dr\,dz$$

$$-\sum_{i=1}^{M}\iint\limits_{\Omega_i}\Big[\dot{u}_{r,r}\dot{N}_{ri}^f+\dot{u}_{z,z}\dot{N}_{zi}^f+\dot{u}_r\dot{N}_{\varphi i}^f r^{-1}+\dot{N}_{rzi}^f(\dot{u}_{r,z}+\dot{u}_{z,r})\Big]r\,dr\,dz$$

$$-\int\limits_{\partial\Omega_p}\left(\dot{P}_n^0\dot{u}_n+\dot{P}_\tau^0\dot{u}_\tau\right)d\,\partial\Omega\,, \tag{1}$$

where $\dot{\boldsymbol{U}}(r,z,t)=(\dot{u}_r(r,z,t),\dot{u}_z(r,z,t))$ is the vector of the rates of displacements; $\dot{u}_r(r,z,t)$, $\dot{u}_z(r,z,t)$ are the rates of displacements along the axes Or and Oz, respectively, which are continuous in Ω and satisfy the kinematic boundary conditions in $\partial\Omega_u$; $\dot{P}_n^0(r,z,t)$ and $\dot{P}_\tau^0(r,z,t)$ are the rates of the normal component and tangential component of the external surface forces; $\dot{N}_r^f$, $\dot{N}_z^f$, and $\dot{N}_{rz}^f$ are the velocities of "fictitious" forces; $\boldsymbol{n}$ and $\boldsymbol{\tau}$ are the external normal and tangent to $\partial\Omega$; $\dot{u}_n=\dot{u}_r n_1+\dot{u}_z n_2$ and $\dot{u}_\tau=\dot{u}_z n_1-\dot{u}_r n_2$ are the normal component and tangential component of the vector of displacement rates; n_1 and n_2 are the direction cosines of the normal $\boldsymbol{n}$; $\lambda_i=\lambda_i(r,z,T)=\dfrac{E_i\nu_i}{(1-2\nu_i)(1+\nu_i)}$, $\lambda_{1i}=\lambda_i+2G_i$, $G_i=G_i(r,z,T)=\dfrac{E_i}{2(1+\nu_i)}$, $E_i=E_i(r,z,T)$, and $\nu_i=\nu_i(r,z,T)$, $i=1,\ldots,M$, are the elastic characteristics, which, in the general case, are continuous functions of the coordinates and temperature within the boundaries of each component of the body.

The velocities of the "fictitious" forces induced by creep and thermal strains in the ith component of the body are calculated by the formulas

$$\dot{N}_{ri}^f=\Big[\lambda_{1i}\dot{e}_{ri}+\lambda_i(\dot{e}_{zi}+\dot{e}_{\varphi i})\Big],\qquad \dot{N}_{zi}^f=\Big[\lambda_{1i}\dot{e}_{zi}+\lambda_i(\dot{e}_{ri}+\dot{e}_{\varphi i})\Big],$$

$$\dot{N}_{\varphi i}^f=\Big[\lambda_{1i}\dot{e}_{\varphi i}+\lambda_i(\dot{e}_{ri}+\dot{e}_{zi})\Big],\qquad \dot{N}_{rzi}^f=2G_i\dot{e}_{rzi}\,, \tag{2}$$

where

$$\dot{e}_{ri}=\dot{p}_{ri}+\dot{\varepsilon}_{ri}^T,\qquad \dot{e}_{zi}=\dot{p}_{zi}+\dot{\varepsilon}_{zi}^T,\qquad \dot{e}_{\varphi i}=\dot{p}_{\varphi i}+\dot{\varepsilon}_{\varphi i}^T,$$

$$\dot{e}_{rzi}=\dot{p}_{rzi},\qquad \dot{\varepsilon}_{ri}^T=\dot{\varepsilon}_{zi}^T=\dot{\varepsilon}_{\varphi i}^T=\alpha_i\dot{T}\,,$$

and $\dot{p}_{ri}$, $\dot{p}_{zi}$, $\dot{p}_{\varphi i}$, $2\dot{p}_{rzi}$ depend on the form of the determining relations of creep of the material of the corresponding component.

If the range of variation of the temperature $T\in[T_1,T_2]$ and the elastic characteristics of the materials of the components of the body V_j, $j=1,\ldots,M$, at temperatures T_1 and T_2 are known, then, for Young's moduli E_j, shear moduli G_j, Poisson's ratios ν_j, and linear thermal expansion coefficient α_j, we can take the simplest linear interpolation

$$F_j(r,z,T) = K_{1j}(r,z) + K_{2j}(r,z)T ,$$

where the coefficients K_{1j} and K_{2j} have the form

$$K_{1j}(r,z) = \frac{T_2F_j(r,z,T_1) - T_1F(r,z,T_2)}{T_2 - T_1},$$

$$K_{2j}(r,z) = \frac{F_j(r,z,T_2) - F_j(r,z,T_1)}{T_2 - T_1}.$$

Here, $F_j(r,z,T_1)$ and $F_j(r,z,T_2)$ are the values of any elastic characteristic or the linear thermal expansion coefficient at temperatures T_1 and T_2, respectively.

The creep strain rates in functional (1) are assumed to be given and invariable.

Based on functional (1), we can formulate the functional in the Lagrange form defined in the space of the vectors of displacement rates of the form $\dot{\boldsymbol{U}}_k(r,z,t) = \left(\dot{u}_{rk}(r,z,t), \dot{u}_{zk}(r,z,t)\right)$, $k = 1,\ldots,M$:

$$\begin{aligned}\Pi\left(\dot{\boldsymbol{U}}_1,\dot{\boldsymbol{U}}_2,\ldots,\dot{\boldsymbol{U}}_M\right) = \frac{1}{2}\sum_{k=1}^{M}\iint_{\Omega_k}\Big[\lambda_{1k}\left(\dot{u}_{rk,r}^2 + \dot{u}_{zk,z}^2 + \dot{u}_{rk}^2 r^{-2}\right)\\ + G_k(\dot{u}_{rk,z} + \dot{u}_{zk,r})^2 + 2\lambda_k\left(\dot{u}_{rk,r}\dot{u}_{zk,z} + \dot{u}_{rk}(\dot{u}_{rk,r} + \dot{u}_{zk,z})r^{-1}\right)\Big] r\,dr\,dz\\ - \sum_{k=1}^{M}\iint_{\Omega_k}\left[\dot{u}_{rk,r}\dot{N}_{rk}^f + \dot{u}_{zk,z}\dot{N}_{zk}^f + \dot{u}_{rk}\dot{N}_{\varphi k}^f r^{-1} + \dot{N}_{rzk}^f\left(\dot{u}_{rk,z} + \dot{u}_{zk,r}\right)\right] r\,dr\,dz\\ - \sum_{s}\int_{\partial\Omega_{ps}}\left(\dot{P}_n^0\dot{u}_{ns} + \dot{P}_\tau^0\dot{u}_{\tau s}\right) d\,\partial\Omega .\end{aligned} \tag{3}$$

Here, λ_k, λ_{1k}, and G_k are the functions of the elastic characteristics of the k th component, $\dot{u}_{rk}(r,z,t)$ and $\dot{u}_{zk}(r,z,t)$, $k = 1,\ldots,M$, are the rates of radial and axial displacements in the k th component, and s is the number of the component of the body to which external forces are applied; the velocities of the "fictitious" forces in the k th component of the body are calculated by formulas (2).

The collection of the functions of the rates of displacements $\dot{u}_{rk}(r,z,t)$ and $\dot{u}_{zk}(r,z,t)$ must satisfy the following conditions:

1° be continuous in the corresponding constituent domains Ω_k, $k = 1,\text{K}\,,M$;

2° coincide on the boundaries of neighboring domains

$$\dot{u}_{ra}(r,z,t) = \dot{u}_{rb}(r,z,t), \qquad \dot{u}_{za}(r,z,t) = \dot{u}_{zb}(r,z,t) \quad \text{on} \quad \partial\Omega_{ab};$$

3° satisfy the following kinematic boundary conditions if the corresponding component V_ℓ is adjacent to the boundary of the body

$$\dot{u}_{r\ell}(r,z,t) = \dot{f}^0_{r\ell}, \qquad \dot{u}_{z\ell}(r,z,t) = \dot{f}^0_{z\ell} \quad \text{on} \quad \partial\Omega_{u\ell}.$$

Here, $\dot{f}^0_{r\ell}$ and $\dot{f}^0_{z\ell}$ are given functions.

On the boundaries of the neighboring domains $\partial\Omega_{ab}$, the following conditions of equilibrium must be satisfied:

$$\sigma^a_{n_a}(r,z) - \sigma^b_{n_b}(r,z) = 0, \qquad \tau^a_{n_a}(r,z,t) - \tau^b_{n_b}(r,z) = 0, \tag{4}$$

where $\dot{\sigma}^a_{n_a}(r,z)$, $\dot{\sigma}^b_{n_b}(r,z)$ and $\dot{\tau}^a_{n_a}(r,z,t)$, $\dot{\tau}^b_{n_b}(r,z)$ are the normal and the tangential stresses that act on the surfaces $\partial\Omega^*_{ab}$ and $\partial\Omega^*_{ba}$, respectively; $\boldsymbol{n}_a$ and $\boldsymbol{n}_b$ are the outer normals to the surfaces $\partial\Omega^*_{ab}$ and $\partial\Omega^*_{ba}$, $\boldsymbol{n}_a = -\boldsymbol{n}_b$.

Conditions (4) can be obtained as conditions of stationarity or, in other words, as natural boundary conditions of functional (3) if the functions $\dot{u}_{ra}$, $\dot{u}_{za}$, $\dot{u}_{rb}$, and $\dot{u}_{zb}$ satisfy conditions ***1°*** and ***2°***.

The main unknown problems of creep and creep-induced damage at any point of the body can be found from the solution of the Cauchy problem with respect to time for a system of differential equations, which, for the j th component of the body has the form

$$\frac{du_{rj}}{dt} = \dot{u}_{rj}, \qquad \frac{du_{zj}}{dt} = \dot{u}_{zj},$$

$$\frac{d\varepsilon_{rj}}{dt} = \dot{u}_{rj,r}, \qquad \frac{d\varepsilon_{zj}}{dt} = \dot{u}_{zj,z}$$

$$\frac{d\varepsilon_{\varphi j}}{dt} = \frac{\dot{u}_{rj}}{r}, \qquad \frac{d\gamma_{rzj}}{dt} = 2\frac{d\varepsilon_{rzj}}{dt} = \dot{u}_{rj,z} + \dot{u}_{zj,r},$$

$$\frac{d\sigma_{rj}}{dt} = \lambda_j(\dot{\varepsilon}_{zj} + \dot{\varepsilon}_{\varphi j} - \dot{e}_{zj} - \dot{e}_{\varphi j}) + \lambda_{1j}(\dot{\varepsilon}_{rj} - \dot{e}_{rj}),$$

$$\frac{d\sigma_{zj}}{dt} = \lambda_j(\dot{\varepsilon}_{rj} + \dot{\varepsilon}_{\varphi j} - \dot{e}_{rj} - \dot{e}_{\varphi j}) + \lambda_{1j}(\dot{\varepsilon}_{zj} - \dot{e}_{zj}), \tag{5}$$

$$\frac{d\sigma_{\varphi j}}{dt} = \lambda_j(\dot{\varepsilon}_{rj} + \dot{\varepsilon}_{zj} - \dot{e}_{rj} - \dot{e}_{zj}) + \lambda_{1j}(\dot{\varepsilon}_{\varphi j} - \dot{e}_{\varphi j}),$$

$$\frac{d\sigma_{rzj}}{dt} = G_j(\dot{\gamma}_{rzj} - 2\dot{e}_{rzj}), \qquad \frac{dp_{rj}}{dt} = \dot{p}_{rj}, \qquad \frac{dp_{zj}}{dt} = \dot{p}_{zj},$$

$$\frac{dp_{\varphi j}}{dt} = \dot{p}_{\varphi j}, \qquad \frac{dp_{rzj}}{dt} = \dot{p}_{rzj}, \qquad \frac{d\psi_j}{dt} = \dot{\psi}_j, \qquad \frac{dq_{ij}}{dt} = \dot{q}_{ij}.$$

The initial conditions for the desired functions at the time $t = 0$ are found from the solution of the problem of elastic deformation. For the solution of the elastic problem, a functional of the form of (1) or (3) can be used. In this case, in the formula for the functional, it is necessary to replace the derivatives of the functions with respect to time by these functions and, in calculation of the "fictitious" forces by formulas (2), set creep strains equal to zero.

Method of Solution. Structures of the Solution

We solve the initial problem (5) by the Runge–Kutta–Merson method with automatic choice of the step [2]. The variational problems for functional (1) or (3) at times corresponding to the Runge–Kutta–Merson scheme are solved by the Ritz method in combination with the R-function method [4, 5]. The R-function method enables us to exactly take into account the geometry of the domain and the boundary conditions of the most general form. In this case, the approximate solution of the boundary-value problem is represented in the form of a formula, i.e., of the structure of the solution exactly satisfying either all boundary conditions (the general structure of the solution) or a part of boundary conditions (partial structure of the solution) and is invariant with respect to the shape of the domain Ω. The structures of the solution form a base for the construction of systems of coordinate functions of variational methods.

Consider some conditions of fastening, the corresponding boundary conditions, and the structures of the solution within the framework of the statement of the boundary-value problem on the basis of functional (1).

Let, on the part $\partial\Omega_u$ of the boundary Ω, the following rates of displacements be given:

$$\dot{u}_r(r,z,t) = \dot{f}_r^0, \qquad \dot{u}_z(r,z,t) = \dot{f}_z^0. \tag{6}$$

Assume also that the other part of the boundary $\partial\Omega_p$, where external forces are applied, belongs to the mth component of the body, and, on it, the following normal stress rate and tangential stress rate are given:

$$\begin{aligned} \dot{\sigma}_n &= \dot{\sigma}_r n_1^2 + 2\dot{\sigma}_{rz} n_1 n_2 + \dot{\sigma}_z n_2^2 = \dot{P}_n^0, \\ \dot{\tau}_n &= (\dot{\sigma}_z - \dot{\sigma}_r) n_1 n_2 + \dot{\sigma}_{rz}(n_1^2 - n_2^2) = \dot{P}_\tau^0. \end{aligned} \tag{7}$$

Conditions (7) can be written in terms of the rates of displacements [1]

$$\begin{aligned} &\lambda_{1m}(\dot{u}_{r,n} n_1 + \dot{u}_{z,n} n_2) + \lambda_m(\dot{u}_{z,\tau} n_1 - \dot{u}_{r,\tau} n_2) + \lambda_m \frac{\dot{u}_r}{r} = \dot{P}_n^0 + \dot{P}_n^f, \\ &G_m(\dot{u}_{r,\tau} n_1 - \dot{u}_{r,n} n_2 + \dot{u}_{z,n} n_1 + \dot{u}_{z,\tau} n_2) = \dot{P}_\tau^0 + \dot{P}_\tau^f, \end{aligned} \tag{8}$$

where $\dot{P}_n^f = \dot{N}_r^f n_1^2 + 2\dot{N}_{rz}^f n_1 n_2 + \dot{N}_z^f n_2^2$, $\dot{P}_\tau^f = (\dot{N}_z^f - \dot{N}_r^f) n_1 n_2 + \dot{N}_{rz}^f (n_1^2 - n_2^2)$.

Conditions (8) are natural for functional (1).

It is easy to verify that the partial structure of the solution that satisfies the kinematic boundary conditions (6) can be written in the form

$$\dot{u}_r = \dot{f}_r + \omega_u \Phi_1 + \omega^2 \Phi_{10}, \qquad \dot{u}_z = \dot{f}_z + \omega_u \Phi_2 + \omega^2 \Phi_{20}.$$

Here, $\dot{f}_r$ and $\dot{f}_z$ are the continuations of the functions $\dot{f}_r^0$ and $\dot{f}_z^0$ into the domain $\Omega \bigcup \partial\Omega_p$; $\omega_u(r,z)=0$ are the equations of the area $\partial\Omega_u$ ($\omega_u>0$ beyond $\partial\Omega_u$); $\omega(r,z)=0$ is the equation of the boundary of the domain Ω ($\omega>0$ inside Ω); Φ_i and Φ_{i0}, $i=1,2$, are the undefined components of the structure of the solution [4]. The functions $\dot{f}_r$ and $\dot{f}_z$ can be constructed with the help of the operator of the continuation of the boundary values $EC(\ldots)$ [4]. If we have only two characteristic parts of the boundary of the domain Ω, namely, $\partial\Omega_u$ and $\partial\Omega_p$, then we represent the functions $\dot{f}_r$ and $\dot{f}_z$ in the form

$$\dot{f}_r = EC(\dot{f}_r^0) = \frac{\dot{f}_r^0 \omega_p}{\omega_u+\omega_p}, \qquad \dot{f}_z = EC(\dot{f}_z^0) = \frac{\dot{f}_z^0 \omega_p}{\omega_u+\omega_p},$$

where $\omega_p(r,z)=0$ are the equations of the area $\partial\Omega_p$ ($\omega_p>0$ beyond $\partial\Omega_p$).

The general structure of the solution which exactly satisfies conditions (6) and (8) has the form [1]

$$\dot{u}_r = \dot{u}_{0r}+\dot{u}_{1r}, \qquad \dot{u}_z = \dot{u}_{0z}+\dot{u}_{1z}, \tag{9}$$

where $\dot{u}_{0r}$ and $\dot{u}_{0z}$ satisfy the inhomogeneous boundary conditions, and $\dot{u}_{1r}$, $\dot{u}_{1z}$ satisfy the homogeneous boundary conditions

$$\begin{aligned}
\dot{u}_{0r} &= \frac{1}{\lambda_{1m}}\dot{P}_n\omega\omega_{p,r} - \frac{1}{G_m}\dot{P}_\tau\omega\omega_{p,z} + \dot{f}_r - \omega D_1^{(p)}\dot{f}_r + \omega\left(1+\frac{\lambda_m}{\lambda_{1m}}\right) \\
&\quad \times \omega_{p,r}\omega_{p,z}T_1^{(p)}\dot{f}_r - \omega T_1^{(p)}\dot{f}_z\left(\frac{\lambda_m}{\lambda_{1m}}\omega_{p,r}^2 - \omega_{p,z}^2\right) - \frac{\lambda_m}{\lambda_{1m}}\omega\omega_{p,r}\frac{\dot{f}_r}{r}, \\
\dot{u}_{0z} &= \frac{1}{\lambda_{1m}}\dot{P}_n\omega\omega_{p,z} + \frac{1}{G_m}\dot{P}_\tau\omega\omega_{p,r} + \dot{f}_z - \omega D_1^{(p)}\dot{f}_z - \omega\left(1+\frac{\lambda_m}{\lambda_{1m}}\right) \\
&\quad \times \omega_{p,r}\omega_{p,z}T_1^{(p)}\dot{f}_z - \omega T_1^{(p)}\dot{f}_r\left(\frac{\lambda_m}{\lambda_{1m}}\omega_{p,z}^2 - \omega_{p,r}^2\right) - \frac{\lambda_m}{\lambda_{1m}}\omega\omega_{p,z}\frac{\dot{f}_r}{r}, \\
\dot{u}_{1r} &= \Phi_1\omega_u - \omega\left[D_1^{(p)}(\Phi_1\omega_u) - T_1^{(p)}(\Phi_1\omega_u)\omega_{p,r}\omega_{p,z}\left(1+\frac{\lambda_m}{\lambda_{1m}}\right)\right. \\
&\quad \left. + \frac{\lambda_m}{\lambda_{1m}}\frac{\Phi_1\omega_u}{r}\omega_{2,r} - T_1^{(p)}(\Phi_2\omega_u)\left(\omega_{p,z}^2 - \frac{\lambda_m}{\lambda_{1m}}\omega_{p,r}^2\right)\right] + \omega\omega_p\Phi_3, \\
\dot{u}_{1z} &= \Phi_2\omega_u - \omega\left[D_1^{(p)}(\Phi_2\omega_u) + T_1^{(p)}(\Phi_2\omega_u)\omega_{p,r}\omega_{p,z}\left(1+\frac{\lambda_m}{\lambda_{1m}}\right)\right. \\
&\quad \left. + \frac{\lambda_m}{\lambda_{1m}}\frac{\Phi_1\omega_u}{r}\omega_{p,z} + T_1^{(p)}(\Phi_1\omega_u)\left(\omega_{p,r}^2 - \frac{\lambda_m}{\lambda_{1m}}\omega_{p,z}^2\right)\right] + \omega\omega_p\Phi_4.
\end{aligned}$$

Here, $\omega(r,z)=0$ is the equation of the boundary of the domain Ω normalized to the first order ($\omega_{,n}=-1$ on $\partial\Omega$, $\omega>0$ inside Ω); $\omega_u(r,z)=0$ and $\omega_p(r,z)=0$ are the equations of the regions $\partial\Omega_u$ and $\partial\Omega_p$ normalized to the first order; Φ_i, $i=1,\ldots,4$, are undefined components of the structure of the solution; $D_1^{(p)}=\dfrac{\partial\omega_p}{\partial r}\dfrac{\partial}{\partial r}+\dfrac{\partial\omega_p}{\partial z}\dfrac{\partial}{\partial z}$ and $T_1^{(p)}=-\dfrac{\partial\omega_p}{\partial z}\dfrac{\partial}{\partial r}+\dfrac{\partial\omega_p}{\partial r}\dfrac{\partial}{\partial z}$ are differential operators [4, 5]; $\dot P_n$ and $\dot P_\tau$ are the continuations of the function of contour forces into the domain $\Omega\cup\partial\Omega_u$

$$\dot P_n = EC(\dot P_n^0)+EC(\dot P_n^f), \qquad \dot P_\tau = EC(\dot P_\tau^0)+EC(\dot P_\tau^f)$$

or

$$\dot P_n = \frac{\dot P_n^0\omega_u}{\omega_u+\omega_p}+\dot N_r^f\omega_{p,r}^2+2\dot N_{rz}^f\omega_{p,r}\omega_{p,z}+\dot N_z^f\omega_{p,z}^2,$$

$$\dot P_\tau = \frac{\dot P_\tau^0\omega_u}{\omega_u+\omega_p}+(\dot N_z^f-\dot N_r^f)\omega_{p,r}\omega_{p,z}+\dot N_{rz}^f(\omega_{p,r}^2-\omega_{p,z}^2).$$

Note that the structures of the solution for the main axisymmetric problems of the theory of elasticity were obtained in the monograph [5]. They can be used for the solution of the problem of elastic deformation, in finding the initial conditions for the main unknowns of the initial boundary-value problem of creep and creep-induced damage.

Further, we consider the main boundary conditions, conditions of joining of the parts of the body, and the corresponding structures of the solution within the framework of the statement of the problem on the basis of functional (3). In this case, the problem is reduced to the determination of the rates of the radial displacements $\dot u_{ri}$ and axial displacements $\dot u_{zi}$ in each of the domains Ω_i, $i=1,\ldots,M$. Then the functions $\dot u_{ri}$ and $\dot u_{zi}$ must satisfy the equations of equilibrium written for the rates, conditions ***2°*** on the boundaries of neighboring components, and the kinematic boundary conditions ***3°*** if the corresponding component is adjacent to the boundary of the body.

Denote by Ω_n, $n=1,\ldots,N$, the domains adjacent to the boundary of the body. We set the rates of displacements on parts of the boundaries $\partial\Omega_{un}$ of the domains Ω_n, the stress rates on parts $\partial\Omega_{pn}$, and conditions of joining of neighboring parts on parts $\partial\Omega_{cn}$. The entire boundary of the domain Ω_n can be represented as the union of boundaries $\partial\Omega_n=\partial\Omega_{un}\cup\partial\Omega_{pn}\cup\partial\Omega_{cn}$. If only the rates of displacements or only the rates of stresses are given, then the corresponding part ($\partial\Omega_{un}$ or $\partial\Omega_{pn}$) of the boundary can be absent.

Let the boundary conditions for the rates of displacements on the part $\partial\Omega_{un}$ have the form

$$\dot u_{rn}(r,z,t) = \dot f_{rn}^0, \qquad \dot u_{zn}(r,z,t) = \dot f_{zn}^0. \tag{10}$$

Then the partial structures of the solution for the rates of displacements $\dot u_{rn}$ and $\dot u_{zn}$ can be written in the form

$$\dot u_{rn} = \dot u_{0rn}+\dot u_{1rn}, \qquad \dot u_{zn} = \dot u_{0zn}+\dot u_{1zn}, \tag{11}$$

where u_{0rn} and u_{0zn} satisfy the inhomogeneous boundary conditions, and u_{1rn}, u_{1zn} are homogeneous

conditions and are determined by the following formulas:

$$\dot{u}_{0rn} = \dot{f}_{rn}, \qquad \dot{u}_{0zn} = \dot{f}_{zn}, \tag{12}$$

$$\dot{u}_{1rn} = \omega_u \Phi_1 + \omega^2 \Phi_{10} + \omega_{ucn}^2 \Phi_{1n}, \qquad \dot{u}_{1zn} = \omega_u \Phi_2 + \omega^2 \Phi_{20} + \omega_{ucn}^2 \Phi_{2n}. \tag{13}$$

Here, $\omega(r,z) = 0$ is the equation of the boundary of the body $\partial\Omega$ ($\omega > 0$ inside Ω); $\omega_u(r,z) = 0$ is the equation of the area $\partial\Omega_u$ ($\omega_u > 0$ beyond $\partial\Omega_u$); $\omega_{ucn}(r,z) = 0$ is the equation of the area $\partial\Omega_{ucn} = \partial\Omega_{un} \bigcup \partial\Omega_{cn}$ ($\omega_{ucn} > 0$ beyond $\partial\Omega_{ucn}$). The functions $\dot{f}_{rn}$ and $\dot{f}_{zn}$ can be written in the form

$$\dot{f}_{rn} = EC(\dot{f}_{rn}^0) = \frac{\dot{f}_{rn}^0 \omega_{pcn}}{\omega_{un} + \omega_{pcn}}, \qquad \dot{f}_{zn} = EC(\dot{f}_{zn}^0) = \frac{\dot{f}_{zn}^0 \omega_{pcn}}{\omega_{un} + \omega_{pcn}},$$

where $\omega_{pcn}(r,z) = 0$ is the equation of the area $\partial\Omega_{pcn} = \partial\Omega_{pn} \bigcup \partial\Omega_{cn}$ ($\omega_{pcn} > 0$ beyond $\partial\Omega_{pcn}$) and $\omega_{un(r,z)} = 0$ is the equation of the area $\partial\Omega_{un}$ ($\omega_{un} > 0$ beyond $\partial\Omega_{un}$).

The structures of the solution represented by formulas (11)–(13), satisfy exactly the kinematic boundary conditions on $\partial\Omega_u$ and the conditions of joining **2°** for any choice of the undefined components Φ_1, Φ_2, Φ_{10}, Φ_{20}, Φ_{1i}, and Φ_{2i} $i = 1,\ldots,M$.

If on the external boundary of some domain Ω_j, only the rates of stresses are given, i.e., in other words, the boundary $\partial\Omega_{uj}$ is absent, then, in this case, in formulas (12) and (13), we must take $\dot{u}_{0rj} \equiv 0$, $\dot{u}_{0zj} \equiv 0$, $\omega_{ucj} = \omega_{cj}$, where $\omega_{cj}(r,z) = 0$ is the equation of area $\partial\Omega_{cj}$ ($\omega_{cj} > 0$ beyond $\partial\Omega_{cj}$).

For the internal components Ω_ℓ, $\ell = 1,\ldots,L$, the domains Ω, on the boundaries of which only the condition of joining with the neighboring components must be satisfied, in the structures of the solution, we must take $\dot{u}_{0r\ell} \equiv 0$, $\dot{u}_{0z\ell} \equiv 0$, and $\omega_{uc\ell} = \omega_\ell$, where $\omega_\ell(r,z) = 0$ is the equation of the boundary of the domain Ω_ℓ ($\omega_\ell > 0$ inside Ω_ℓ).

The equations of the boundary of the domain Ω and its components in the structures of the solution are constructed with the help of R-functions [4].

In discretization of the boundary-value problem, the undefined components of the structures of the solution are represented in the form [4]

$$\Phi_\ell(\boldsymbol{x}) \approx \Phi_{\ell N}(\boldsymbol{x}) = \sum_{k=1}^{N_\ell} C_k^\ell \varphi_k^\ell, \qquad \boldsymbol{x} = (r,z),$$

where $\{\varphi_k^\ell\}$ are the known elements of some functional space $\boldsymbol{M}^\ell$, in which Φ_ℓ are contained and form a certain sequence in it, and C_k^ℓ are unknown coefficients. As $\{\varphi_k^\ell\}$, we can choose ordinary power polynomials, Chebyshev polynomials, splines, etc. [4].

Numerical Results

As an example, we consider the processes of creep and damage of a three-layer cylinder used as a computational scheme of a solid-oxide fuel cell (Fig. 1) developed by the Siemens-Westinghouse Power Corp. [8].

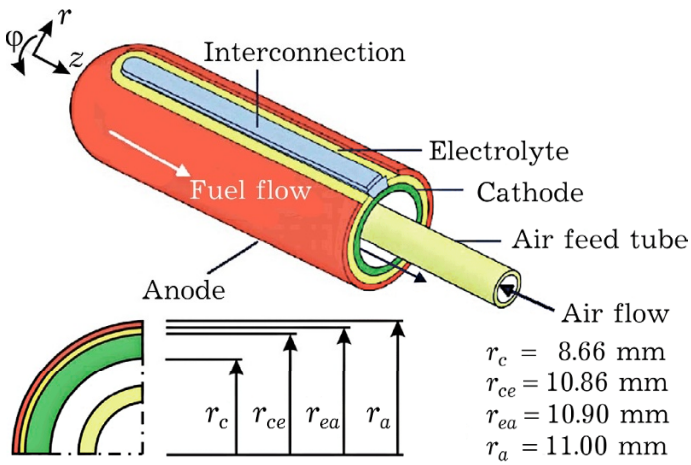

Fig. 1

In the contemporary engineering, the solid-oxide fuel cells (SOFC) are among the most efficient power-generating systems converting chemical energy into electric energy. Solid-oxide fuel cells consist of an electrolyte, a cathode, and an anode. The electrolyte layer is laid between two thin porous electrodes. The cathode, anode, and electrolyte are made of ceramic materials. The theory and technology of production of SOFC are now extensively developed and investigated [7–10].

One of the central problems in the design and operation of solid-oxide fuel cells is to guarantee their long-term strength and increase the period of failure-free operation.

The sizes of the fuel cell in the radial direction are shown in Fig. 1. The layers of the cylinder are rigidly connected with each other. The length of the electrochemically active part is $\ell = 1.5$ m. The thickness of the cathode is $h_1 = 2.2 \cdot 10^{-3}$ m, the thickness of the electrolyte is $h_2 = 0.04 \cdot 10^{-3}$ m, and the thickness of the anode is $h_3 = 0.1 \cdot 10^{-3}$ m. The cathode was made of WPC3 material, the electrolyte was made of 8YSZ material (zirconium dioxide stabilized by yttrium oxide), and the anode was made of Ni+YSZ material (nickel cermet) [8]. The working temperature was $T = 900°$ C. Young's moduli and Poisson's ratios of the materials of the cathode, electrolyte, and anode at a temperature of $900°$ C are as follows [8, 13]: $E_1 = 58.0$ GPa, $E_2 = 155.0$ GPa, $E_3 = 64.0$ GPa, $\nu_1 = 0.25$, $\nu_2 = 0.316$, and $\nu_3 = 0.25$. The linear thermal expansion coefficients are $\alpha_1 = 10.7 \cdot 10^{-6}\,\mathrm{K}^{-1}$, $\alpha_2 = 10.3 \cdot 10^{-6}\,\mathrm{K}^{-1}$, and $\alpha_3 = 12.0 \cdot 10^{-6}\,\mathrm{K}^{-1}$. The temperature at which stresses and strains are absent is $T_0 = 1250°$ C. External surface forces are absent, i.e., $P_n^0 = 0$ and $P_\tau^0 = 0$.

For the existing level of stress intensity, the material of the cathode does not reveal creep. The law of uniaxial creep and the kinetic equation of damage for the materials of electrolyte and anode have the form

$$\dot{p} = A \exp\left(-\frac{Q}{RT}\right) t^{-k} \frac{\sigma^n}{(1-\psi)^n}, \qquad \frac{d\psi}{dt} = B \exp\left(-\frac{\Delta}{RT}\right) \frac{\sigma^m}{(1-\psi)^m}.$$

Here, ψ is the scalar damage parameter, which at the initial moment of time $t = 0$, is equal to zero and, at the moment $t = t_*$ of fracture, is $\psi = 1$; $R = 0.0083144$ kJ/(mole · K) is a universal gas constant; T is absolute temperature. The creep constants of the electrolyte material are as follows [12]: $A = 704.5\,(\mathrm{MPa})^{-n} \cdot \mathrm{h}^{-1}$, $B = 1.1722\,(\mathrm{MPa})^{-m} \cdot \mathrm{h}^{-1}$, $m = 1.7$, $n = 1$, $k = 0$, $Q = 320.0$ kJ/mole, and $\Delta = 44.23$ kJ/mole. The creep

constants of the anode material are the following [11]: $A = 9.145 \cdot 10^{-13}\,(\text{MPa})^{-n} \cdot \text{h}^{-1}$, $B = 0$, $m = 0$, $n = 7.181$, $k = 0.943$, and $Q = 72.5\,\text{kJ/mole}$.

For the complex stressed state, we write the determining relations of creep and the kinetic equation of damage in the form

$$\dot{p}_r = \frac{3}{2} A \exp\left(-\frac{Q}{RT}\right) t^{-k} \frac{\sigma_{\text{i}}^{n-1}}{(1-\psi)^n}\left(\sigma_r - \frac{1}{3} I_1\right),$$

$$\dot{p}_z = \frac{3}{2} A \exp\left(-\frac{Q}{RT}\right) t^{-k} \frac{\sigma_{\text{i}}^{n-1}}{(1-\psi)^n}\left(\sigma_z - \frac{1}{3} I_1\right),$$

$$\dot{p}_\varphi = \frac{3}{2} A \exp\left(-\frac{Q}{RT}\right) t^{-k} \frac{\sigma_{\text{i}}^{n-1}}{(1-\psi)^n}\left(\sigma_\varphi - \frac{1}{3} I_1\right),$$

$$\dot{p}_{rz} = \frac{3}{2} A \exp\left(-\frac{Q}{RT}\right) t^{-k} \frac{\sigma_{\text{i}}^{n-1}}{(1-\psi)^n}\sigma_{rz},$$

$$\frac{d\psi}{dt} = B \exp\left(-\frac{\Delta}{RT}\right)\frac{\sigma_{\text{i}}^{m}}{(1-\psi)^m},$$

where

$$\sigma_{\text{i}} = \sqrt{\sigma_r^2 + \sigma_z^2 + \sigma_\varphi^2 - \sigma_r\sigma_z - \sigma_r\sigma_\varphi - \sigma_\varphi\sigma_z + 3\sigma_{rz}^2}$$

is the intensity of stresses and $I_1 = \sigma_r + \sigma_z + \sigma_\varphi$.

Further, we consider the boundary conditions and the corresponding structures of the solution. The lateral boundaries of the cylinder $r = r_c = 8.66 \cdot 10^{-3}\,\text{m}$ and $r = r_a = 11.0 \cdot 10^{-3}\,\text{m}$ are free from fastening and external loads. The ends $z = \pm \ell/2$ of the fuel element can move freely along the Oz-axis.

The boundary conditions on the ends of the cylinder $z = \pm \ell/2$ are

$$\dot{\sigma}_{zi}(\dot{u}_{ri}, \dot{u}_{zi}) = 0, \qquad \dot{\sigma}_{zri}(\dot{u}_{ri}, \dot{u}_{zi}) = 0, \quad i = 1,2,3.$$

On the internal surface $r = r_c$ and external surface $r = r_a$ of the cylinder,

$$\dot{\sigma}_{rj}(\dot{u}_{rj}, \dot{u}_{zj}) = 0, \qquad \dot{\sigma}_{rzj}(\dot{u}_{rj}, \dot{u}_{zj}) = 0, \quad j = 1,3.$$

On the boundaries of the layers, the following conditions of joining must be satisfied:

$$\dot{u}_{rk} = \dot{u}_{rk+1}, \qquad \dot{u}_{zk} = \dot{u}_{zk+1}, \quad k = 1,2. \tag{14}$$

In numerical realization, we use the symmetry of the problem. In this case, on the axis of symmetry, in the section $z = 0$, we give the following conditions:

$$\dot{u}_{zi} = 0, \qquad \frac{\partial \dot{u}_{ri}}{\partial z} = 0, \quad i = 1,2,3. \tag{15}$$

Using the methods of construction of the structures of the solution described in the foregoing, we obtain structures satisfying conditions (14) and (15)

$$\dot{u}_{ri} = \Phi_1 - \omega_u D_1^{(u)} \Phi_1 + \omega^2 \Phi_{10} + \omega_{uci}^2 \Phi_{1i},$$

$$\dot{u}_{zi} = \omega_u \Phi_2 + \omega^2 \Phi_{20} + \omega_{uci}^2 \Phi_{2i}.$$

Here, $\omega_u = z$, $\omega_{uci} = \omega_u \wedge_0 \omega_{ci} = \omega_u + \omega_{ci} - \sqrt{\omega_u^2 + \omega_{ci}^2}$, and $\omega_{ci} = 0$ are the equations of the interfaces of the layers of the cylinder

$$\omega_{c1} = r_2 - r, \qquad \omega_{c2} = \frac{(r - r_2)(r_3 - r)}{r_3 - r_2}, \qquad \omega_{c3} = r - r_3,$$

$\wedge_0$ is the symbol of the R-conjunction operation [4]; $r_1 = r_c$, $r_2 = r_{ce}$, $r_3 = r_{ea}$, $r_4 = r_a$; $D_1^{(u)} = \frac{\partial \omega_u}{\partial r}\frac{\partial}{\partial r} + \frac{\partial \omega_u}{\partial z}\frac{\partial}{\partial z}$ is a differential operator; Φ_1, Φ_2, Φ_{10}, Φ_{20}, Φ_{1i}, and Φ_{2i}, $i = 1,2,3$, are the undefined components of the structure of the solution. Note that here, the functions ω_{uci} can also be obtained as ordinary products $\omega_{uci} = \omega_u \cdot \omega_{ci}$.

The equation of the boundary of the domain Ω ($\omega = 0$, $\omega_{,n} = -1$ on $\partial\Omega$, $\omega > 0$ inside Ω) has the form

$$\omega = \omega_r \wedge_0 \omega_z = \omega_r + \omega_z - \sqrt{\omega_r^2 + \omega_z^2} = 0,$$

where $\omega_r = \frac{(r - r_1)(r_4 - r)}{r_4 - r_1}$, $\omega_z = \frac{2z}{\ell}\left(\frac{\ell}{2} - z\right)$.

In the numerical realization, the functions Φ_1, Φ_2, Φ_{10}, Φ_{20}, Φ_{1i}, and Φ_{2i} are present in the form of linear combinations of Schoenberg bicubic splines [6].

Systems of spline functions were constructed on uniform rectilinear meshes. In this case, Φ_1, Φ_2, Φ_{10}, and Φ_{20} were given in the whole domain Ω, and Φ_{1i}, Φ_{2i}, $i = 1,2,3$, were given only in the corresponding subdomains Ω_i with equations of the boundaries ($\omega_i = 0$, $\omega_{i,n} = 1$ on $\partial\Omega_i$, and $\omega_i > 0$ inside Ω_i)

$$\omega_i = \omega_{ri} \wedge_0 \omega_z = \omega_{ri} + \omega_z - \sqrt{\omega_{ri}^2 + \omega_z^2} = 0,$$

where $\omega_{ri} = \frac{(r - r_i)(r_{i+1} - r)}{r_{i+1} - r_i}$.

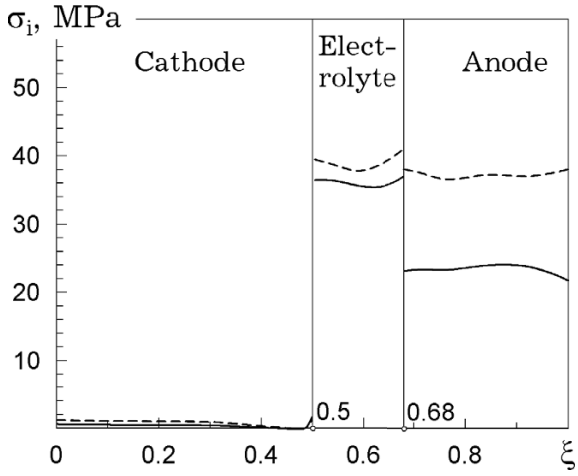

Fig. 2

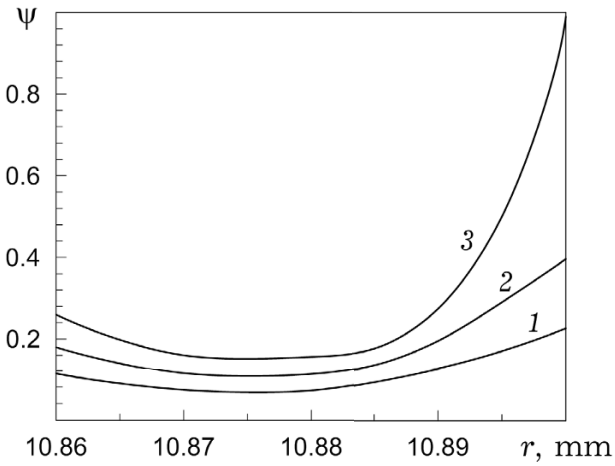

Fig. 3

As a result of calculations, we established that the time to fracture is $t_* = 4360\,\text{h}$. Fracture begins in the electrolyte layer at the points lying on the boundary with the anode. Figures 2–4 show the results of calculations of stresses and damage in the section $z = 0$.

In Fig. 2, we show the results on the intensity of stresses at different moments of time, where the dashed line corresponds to $t = 0$ and the solid line corresponds to $t = t_*$. For the better visualization of the results, along the abscissa axis, we plot the dimensionless radial coordinate ξ, which, in the corresponding layers, was calculated by the formula

$$\xi_n = \alpha_n \frac{r - r_{n-1}}{r_n - r_{n-1}} + \sum_{k=0}^{n-1} \alpha_k, \qquad n = 1, 2, 3,$$

where $r_0 = 8.66\,\text{mm}$, $r_1 = 10.86\,\text{mm}$, $r_2 = 10.9\,\text{mm}$, and $r_3 = 11.0\,\text{mm}$; α_k are weight coefficients: $\alpha_0 = 0$, $\alpha_1 = 0.5$, $\alpha_2 = 0.18$, and $\alpha_3 = 0.32$. Figure 3 shows the distribution of damage in the electrolyte layer at dif-

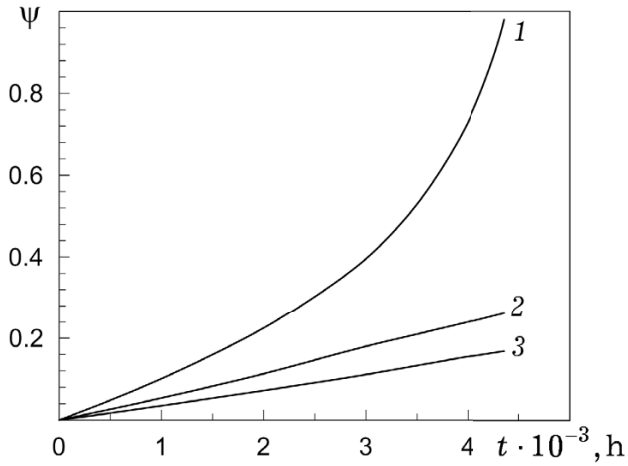

Fig. 4

ferent moments of time: curve *1* corresponds to $t = 2000$ h, curve *2* corresponds to $t = 3000$ h, and curve *3* corresponds to $t = t_* = 4360$ h.

In Fig. 4, we show the plots of increase in the degree of damage with time at different points of the electrolyte: curve *1* corresponds to the degree of damage on the boundary of the electrolyte and anode ($r = 10.8999$ mm), curve *2* corresponds to the degree of damage on the boundary of the electrolyte and cathode ($r = 10.8601$ mm), and curve *3* corresponds to the degree of damage inside the electrolyte ($r = 10.88$ mm).

Conclusions

We develop a numerical-analytic method for the solution of the axisymmetric initial boundary-value problem of creep and creep-induced damage for a piecewise homogeneous body of revolution with meridional section of any shape subjected to the action of force and temperature loads. The method is based on the combined use of the R-function method and the Runge–Kutta–Merson method. We construct the structures of the solutions for the main types of boundary conditions. As an example, we solved the problem of creep, damage, and long-term strength for a solid-oxide tubular fuel cell by using a three-layer cylinder as its computational scheme. The results of our investigations enable us to conclude that, during operation, the redistribution of mechanical stresses and accumulation of damage occur in solid-oxide fuel cells as a result of creep. The levels of stresses in the electrolyte and anode remain high for the whole time. This eventually leads to the fracture of the fuel cell. To extend the failure-free period of operation of fuel cells, it is necessary to decrease the level of stresses, e.g., as a result of the decrease in the difference between the linear thermal expansion coefficients of the materials of the layers.

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