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ON THE INSTABILITY OF A ROTATING ELASTOPLASTIC COMPOSITE FLAT ANNULAR DISK

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We propose a procedure for the investigation of possible loss of stability by a rotating composite flat annular circular disk by the method of small parameter. We deduce a characteristic equation for the critical radius of the plastic zone as the first approximation. The critical angular rotational velocity is numerically found for various parameters of the disk.

Introduction

The "state beyond the elasticity limit" in rapidly rotating flat disks overloaded by centrifugal tensile forces can be regarded as the source of a plane elastoplastic problem [21, 28, 29]. Its solution is connected with finding the boundary between the elastic and plastic domains [14] and the stressed state formed in the elastic domain by the forces applied to the contour of the disk. The critical values of radius of the plastic zone and rotational velocity accompanying the transition of the solid circular disk free of contour forces to one of its unstable states [4] were determined in [7, 9] by the approximate method of small parameter described later in [10] and characterized by a certain similarity to the second version of the method of perturbation of boundary shape in continuum mechanics [5]. Moreover, the loss of stability and the exhaustion of the load-carrying ability [6, 17, 22–24, 27, 30] of solid disks made of perfectly plastic materials [15] were studied for the case where the components of stresses in these materials satisfy the equilibrium equations of plane problem [3], the compatibility condition [2] in the elastic domain, and the Saint-Venant plasticity condition [8] in the plastic region.

These investigations were continued in a series of works [11–13, 18, 19] devoted to the instability of flat homogeneous and inhomogeneous solid and annular circular disks, stepwise disks, and disks of any profile, including the disks subjected to the action of radial contour pressure depending on the rotational velocity. The efficiency of the analytic method of perturbation of boundary shapes was demonstrated in finding the parameters of superhigh-speed operation of some disk systems, which is especially important for the problems of stability and strength of turbines and other heavy disks [16, 20, 25, 26, 31–33].

In the present work, we describe a procedure of evaluation (by the method of small parameter) of the characteristic critical values [1] corresponding to the loss of stability of a simple radially inhomogeneous flat annular circular disk loaded in its plane by given radial contour forces. The material of the disk is perfectly plastic and satisfies the plasticity condition $\sigma_{\theta\theta} = \sigma_s$.

1. Statement of the Problem

The investigated disk $\mathcal{D}$ consists of two homogeneous and isotropic flat disks $\mathcal{D}_1$ and $\mathcal{D}_2$. The inner radius of the annular circular disk $\mathcal{D}_1$ is equal to a and its outer radius coincides with the inner radius of the

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annular circular disk $\mathcal{D}_2$ and is equal to c. The outer radius of the disk $\mathcal{D}_2$ is equal to b. Along the circle $r = c$, the disks $\mathcal{D}_1$ and $\mathcal{D}_2$ made of different materials are rigidly joined into a single disk $\mathcal{D}$. The yield point of the material of the disk $\mathcal{D}_1$ is denoted by σ_{s1}, its modulus of elasticity by E_1, density by γ_1, and Poisson's ratio by ν_1. The same parameters for the material of the disk $\mathcal{D}_2$ are denoted by σ_{s2}, E_2, γ_2, and ν_2, respectively. In the notation σ_s, E, and ν used without indication of the number of disk section the subscripts are omitted. The constant angular rotational velocity of the disk $\mathcal{D}$ is equal to ω.

We consider the shape of the loss of stability of the disk $\mathcal{D}$, which is self-balanced and slightly differs from circular, where the equation of its outer boundary accurate to first-order infinitesimals can be represented in the form

$$r = b + d\cos n\theta, \qquad d = \text{const}, \qquad n \geq 2, \qquad n \in \mathbb{N},$$

or

$$\rho = 1 + \delta\cos n\theta, \tag{1}$$

where $\rho = r/b$ is a dimensionless current radius, δ is a small parameter, and θ is a polar angle. We denote by r_{01} or/and r_{02} (Figs. 1–5) the current radius of plastic zone of the undisturbed disk. It is necessary to obtain, in the first approximation, the characteristic equation for the critical radius of plastic zone $r_0 = r_{0*}$ and to determine the corresponding value of critical angular rotational velocity $\omega = \omega_*$. Recall that, for this purpose, one should establish the condition of the existence of nontrivial solutions of the system of linear homogeneous equations

$$\sigma'_{rr} + \frac{d\sigma^0_{rr}}{dr}u' = 0, \qquad \sigma'_{r\theta} - \frac{\sigma^0_{\theta\theta} - \sigma^0_{rr}}{b}\frac{du'}{d\theta} = 0, \qquad r = b,$$

$$\sigma'_{rr} = 0, \qquad \sigma'_{r\theta} = 0, \qquad r = r_0,$$

for arbitrary constants appearing in the expressions for components of the stresses and displacements σ'_{rr}, $\sigma'_{r\theta}$, and u', which determine the disturbed stress-strain state of the rotating disk $\mathcal{D}$. These linearized disturbances of the first order of smallness satisfy the differential equilibrium equations of plane problem and the partial differential equations of constraint between the stresses and displacements, whereas the undisturbed stressed state (with superscript "0") is determined by the ordinary differential equations of quasistatic equilibrium and constraint equations in the elastic zone or Saint-Venant yield condition in the plastic zone.

2. Solution in the Case $\mathcal{D}_{1(pe)}\mathcal{D}_{2(e)}$

The undisturbed stressed state of plastic domain $\mathcal{D}_{1(p)}$ (Fig. 1) is determined by the initial problem

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = -\frac{\sigma_1}{b^2}r,$$

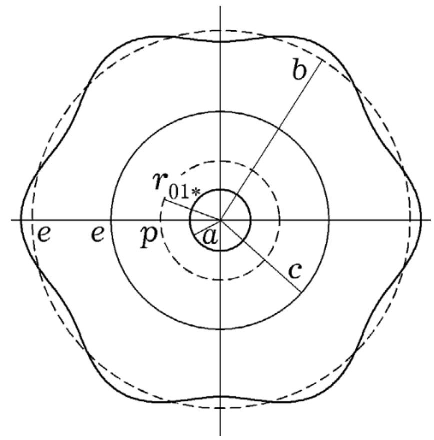

Fig. 1

$$\sigma_1 = \gamma_1 b^2 \omega^2 = \text{const}, \qquad \sigma_{rr}(a) = -p_i \tag{2}$$

at $\sigma_{\theta\theta} = \sigma_{s1}$. Therefore, the tangential and radial stresses, related to the yield point σ_{s2}, are

$$\sigma_{\theta\theta} = \frac{\sigma_{s1}}{\sigma_{s2}} = s, \tag{3}$$

$$\sigma_{rr} = s - \frac{\sigma_1}{3\sigma_{s2}}\rho^2 + \frac{\beta}{\rho}\left(-s - \frac{p_i}{\sigma_{s2}} + \frac{\sigma_1}{3\sigma_{s2}}\beta^2\right), \tag{4}$$

if one uses the dimensionless polar radius (here, $\beta = a/b$).

In the elastic domain $\mathcal{D}_{1(e)}$, the corresponding relations can be written as

$$\sigma_{\theta\theta} = C_1 + \frac{C_2}{\rho^2} - \frac{\sigma_1(3\nu_1 + 1)}{8\sigma_{s2}}\rho^2, \tag{5}$$

$$\sigma_{rr} = C_1 - \frac{C_2}{\rho^2} - \frac{\sigma_1(\nu_1 + 3)}{8\sigma_{s2}}\rho^2, \tag{6}$$

where C_1 and C_2 are some constants.

The undisturbed stressed state of elastic domain $\mathcal{D}_{2(e)}$ is determined by the initial problem

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = -\frac{\sigma_2}{b^2}r, \qquad \sigma_2 = \gamma_2 b^2 \omega^2 = \text{const}, \qquad \sigma_{rr}(b) = p_e, \tag{7}$$

and, hence, we have in this domain

$$\sigma_{\theta\theta} = \frac{p_e}{\sigma_{s2}} + C_3\left(1+\frac{1}{\rho^2}\right) + \frac{\sigma_2(\nu_2+3)}{8\sigma_{s2}} - \frac{\sigma_2(3\nu_2+1)}{8\sigma_{s2}}\rho^2 , \tag{8}$$

$$\sigma_{rr} = \frac{p_e}{\sigma_{s2}} + C_3\left(1-\frac{1}{\rho^2}\right) + \frac{\sigma_2(\nu_2+3)}{8\sigma_{s2}}(1-\rho^2) , \tag{9}$$

where C_3 is an unknown constant.

To find C_1, C_2, C_3, and ω depending on the radius of the plastic zone $\beta_0 = r_{01}/b$, it is necessary to take into account the continuity of radial stresses σ_{rr} and radial displacements $u = \sigma_s\rho(\sigma_{\theta\theta} - \nu\sigma_{rr})/E$ (related to b) for $\rho = \beta_0$ and $\rho = c/b = \bar\beta$. The corresponding rules of conjugation for stresses have the form

$$\sigma_{rr}(\beta_0+0) = \sigma_{rr}(\beta_0-0), \qquad \sigma_{\theta\theta}(\beta_0+0) = \sigma_{\theta\theta}(\beta_0-0) , \tag{10}$$

$$\sigma_{rr}(\bar\beta+0) = \sigma_{rr}(\bar\beta-0), \qquad \sigma_{\theta\theta}(\bar\beta+0) = \varepsilon\sigma_{\theta\theta}(\bar\beta-0) + k\sigma_{rr}(\bar\beta-0) , \tag{11}$$

where $\varepsilon = E_2/E_1$ and $k = \nu_2 - \varepsilon\nu_1$. Applying relations (10) and (11) to solutions (3)–(6), (8), and (9), we obtain the following system of four linear equations for C_1, C_2, C_3, and $x = \sigma_2/(24\sigma_{s2})$:

$$C_1 + \beta_0^{-2}C_2 - 3\Gamma(3\nu_1+1)\beta_0^2 x = s ,$$

$$C_1 - \beta_0^{-2}C_2 - \Gamma[(3\nu_1+1)\beta_0^2 + 8\beta^3\beta_0^{-1}]x = s + \beta\beta_0^{-1}\left(-s - \frac{p_i}{\sigma_{s2}}\right) ,$$

$$(\varepsilon+k)C_1 + \bar\beta^{-2}(\varepsilon-k)C_2 - (1+\bar\beta^{-2})C_3 + \ell x = \frac{p_e}{\sigma_{s2}} ,$$

$$C_1 - \bar\beta^{-2}C_2 - (1-\bar\beta^{-2})C_3 + mx = \frac{p_e}{\sigma_{s2}} , \tag{12}$$

where

$$\Gamma = \frac{\gamma_1}{\gamma_2}, \qquad m = -3\{\nu_2 + 3 + \bar\beta^2[\Gamma(\nu_1+3) - (\nu_2+3)]\} ,$$

$$\ell = -3\{\nu_2 + 3 + \bar\beta^2\{\Gamma[\varepsilon(3\nu_1+1) + k(\nu_1+3)] - (3\nu_2+1)\}\} .$$

The solution of system (12) is given by

$$C_1 = s + 0.5\beta\beta_0^{-1}\left(-s - \frac{p_i}{\sigma_{s2}}\right) + 2\Gamma[(3\nu_1+1)\beta_0^2 + 2\beta^3\beta_0^{-1}]x , \tag{13}$$

$$C_2 = -0.5\beta\beta_0\left(-s-\frac{p_i}{\sigma_{s2}}\right)+\Gamma\beta_0^2[(3\nu_1+1)\beta_0^2-4\beta^3\beta_0^{-1}]x\,, \tag{14}$$

$$C_3 = G\left\{(1-m\ell^{-1})\frac{p_e}{\sigma_{s2}}-(1-m\ell^{-1}(\varepsilon+k))C_1+\bar{\beta}^{-2}(1+m\ell^{-1}(\varepsilon-k))C_2\right\}, \tag{15}$$

$$x = \frac{1}{24}\frac{\omega^2}{q_2^2} = \frac{-Hs+K\dfrac{p_e}{\sigma_{s2}}-0.5\beta\beta_0^{-1}(H-J\bar{\beta}^{-2}\beta_0^2)\left(-s-\dfrac{p_i}{\sigma_{s2}}\right)}{\ell+\Gamma(3\nu_1+1)(2H+J\bar{\beta}^{-2}\beta_0^2)\beta_0^2+4\Gamma\beta^3\beta_0^{-1}(H-J\bar{\beta}^{-2}\beta_0^2)}\,, \tag{16}$$

where

$$G = \frac{1}{m\ell^{-1}(1+\bar{\beta}^{-2})-(1-\bar{\beta}^{-2})}, \qquad q_2 = \frac{1}{b}\sqrt{\frac{1}{\gamma_2}\sigma_{s2}}\,,$$

$$H = \varepsilon+k+G(1+\bar{\beta}^{-2})(1-m\ell^{-1}(\varepsilon+k)), \qquad K = 1+G(1+\bar{\beta}^{-2})(1-m\ell^{-1})\,,$$

$$J = \varepsilon-k-G(1+\bar{\beta}^{-2})(1+m\ell^{-1}(\varepsilon-k))\,.$$

Thus, in view of relations (13)–(16), dependences (3)–(6), (8), and (9) assign the components σ_{rr}^0 and $\sigma_{\theta\theta}^0$ of zero-order approximation to the solution

$$\sigma_{rr} = \sigma_{rr}^0+\delta\sigma'_{rr}+\delta^2\ldots, \qquad \sigma_{\theta\theta} = \sigma_{\theta\theta}^0+\delta\sigma'_{\theta\theta}+\delta^2\ldots,$$

$$\sigma_{r\theta} = \sigma_{r\theta}^0+\delta\sigma'_{r\theta}+\delta^2\ldots,$$

$$u = u^0+\delta u'+\delta^2\ldots, \qquad v = v^0+\delta v'+\delta^2\ldots$$

of the problem of plastic equilibrium, determining the location of elastoplastic boundary.

Remark 1. Setting $\varepsilon = s = \Gamma = 1$, $\nu_1 = \nu_2 = \nu$, and $p_e = 0$, we obtain the following known [13] relation from (16):

$$\frac{\omega^2}{q^2} = \frac{24+12\beta\beta_0^{-1}(1+\beta_0^2)\left(-1-\dfrac{p_i}{\sigma_s}\right)}{3(\nu+3)-(3\nu+1)(2-\beta_0^2)\beta_0^2-4\beta^3\beta_0^{-1}(1+\beta_0^2)}$$

between the angular rotational velocity and radius of the plastic zone of a flat homogeneous annular circular disk, subjected to the action of a given internal radial pressure p_i.

In order to use the boundary and conjugation conditions

$$\sigma'^e_{rr} + A_1 u'^e = 0, \qquad \rho = 1, \tag{17}$$

$$\sigma'^e_{r\theta} - A_2 \frac{du'^e}{d\theta} = 0, \qquad \rho = 1, \tag{18}$$

$$\sigma'^e_{rr} = 0, \qquad \rho = \beta_0, \tag{19}$$

$$\sigma'^e_{r\theta} = 0, \qquad \rho = \beta_0, \tag{20}$$

we recall the form of disturbances of the first order of smallness σ'^e_{rr}, $\sigma'^e_{r\theta}$, and u'^e. For the radial and tangential stresses, related to the yield point σ_{s2}, and radial displacement, related to b, in the domain $\mathcal{D}_{2(e)}$, we have [12]

$$\sigma'^e_{rr} = [a_{\mathrm{I}}(\bar{\beta},\rho)a_2 + a_{\mathrm{II}}(\bar{\beta},\rho)a_1 + a_{\mathrm{III}}(\bar{\beta},\rho)b_2 + a_{\mathrm{IV}}(\bar{\beta},\rho)b_1]\cos n\theta,$$

$$\sigma'^e_{r\theta} = [c_{\mathrm{I}}(\bar{\beta},\rho)a_2 + c_{\mathrm{II}}(\bar{\beta},\rho)a_1 + c_{\mathrm{III}}(\bar{\beta},\rho)b_2 + c_{\mathrm{IV}}(\bar{\beta},\rho)b_1]\sin n\theta,$$

$$u'^e = \frac{\sigma_{s2}}{E_2}[d^{**}_{\mathrm{I}}(\bar{\beta},\rho)a_2 + d^{**}_{\mathrm{II}}(\bar{\beta},\rho)a_1 + d^{**}_{\mathrm{III}}(\bar{\beta},\rho)b_2 + d^{**}_{\mathrm{IV}}(\bar{\beta},\rho)b_1]\cos n\theta$$

and, in $\mathcal{D}_{1(e)}$,

$$\sigma'^e_{rr} = \left[a_{\mathrm{I}}\left(\frac{\beta_0}{\bar{\beta}},\frac{\rho}{\bar{\beta}}\right)a_1 + a_{\mathrm{II}}\left(\frac{\beta_0}{\bar{\beta}},\frac{\rho}{\bar{\beta}}\right)a_0 + a_{\mathrm{III}}\left(\frac{\beta_0}{\bar{\beta}},\frac{\rho}{\bar{\beta}}\right)b_1 + a_{\mathrm{IV}}\left(\frac{\beta_0}{\bar{\beta}},\frac{\rho}{\bar{\beta}}\right)b_0\right]\cos n\theta,$$

$$\sigma'^e_{r\theta} = \left[c_{\mathrm{I}}\left(\frac{\beta_0}{\bar{\beta}},\frac{\rho}{\bar{\beta}}\right)a_1 + c_{\mathrm{II}}\left(\frac{\beta_0}{\bar{\beta}},\frac{\rho}{\bar{\beta}}\right)a_0 + c_{\mathrm{III}}\left(\frac{\beta_0}{\bar{\beta}},\frac{\rho}{\bar{\beta}}\right)b_1 + c_{\mathrm{IV}}\left(\frac{\beta_0}{\bar{\beta}},\frac{\rho}{\bar{\beta}}\right)b_0\right]\sin n\theta,$$

where

$$a_0 = q'_1 a_2 + q'_2 a_1 + q'_3 b_2 + q'_4 b_1, \qquad b_0 = q'_5 a_2 + q'_6 a_1 + q'_7 b_2 + q'_8 b_1.$$

Here, a_2, a_1, b_2, and b_1 are undetermined coefficients; $q'_1,\ldots,q'_8$ are coefficients that can be expressed via n, β_0, $\bar{\beta}$, ν_1, ν_2, and ε, and $a_{\mathrm{I}}(\cdot),\ldots,a_{\mathrm{IV}}(\cdot)$, $c_{\mathrm{I}}(\cdot),\ldots,c_{\mathrm{IV}}(\cdot)$, $d^{**}_{\mathrm{I}}(\cdot),\ldots,d^{**}_{\mathrm{IV}}(\cdot)$ are known functions [2]. In addition,

$$A_1 = 2C_3 - 6(\nu_2 + 3)x, \qquad A_2 = A_1 + 24x; \tag{21}$$

in expressions (15) and (16) in this case here, the forces p_i/σ_{s2} and p_e/σ_{s2} must be connected with β_0. Suppose that

$$p_i = \varepsilon_i\sigma_{s2} + \kappa_i\gamma_2 b^2\omega^2, \quad p_e = \varepsilon_e\sigma_{s2} + \kappa_e\gamma_2 b^2\omega^2, \tag{22}$$

where, with regard for (2) and (7), the coefficient $\varepsilon_i > 0$ corresponds, for example, to the interference fit of the disk $\mathcal{D}$ under study on a shaft; $\kappa_i > 0$ determines the compressive action of the rotating shaft on the rotating disk; $\varepsilon_e > 0$ can mean the appearing compression of the disk under study in the course of mounting blades over its outer contour, and $\kappa_e < 0$ the stretching action of the blades at disk rotation. Then we obtain from (16) and (22) for $\kappa_i \neq 0$

$$\frac{p_e}{\sigma_{s2}} = \varepsilon_e - \mu(s + \varepsilon_i + \xi(\beta_0)), \tag{23}$$

$$-s - \frac{p_i}{\sigma_{s2}} = \xi(\beta_0) = -\frac{(s+\varepsilon_i)\tau + 24\kappa_i(-Hs + K[\varepsilon_e - \mu(s+\varepsilon_i)])}{\tau - 24\kappa_i(K\mu + 0.5\beta\beta_0^{-1}(H - J\bar{\beta}^{-2}\beta_0^2))} \tag{24}$$

and for $\kappa_e \neq 0$

$$-s - \frac{p_i}{\sigma_{s2}} = -s - \varepsilon_i - \mu^{-1}(\eta(\beta_0) - \varepsilon_e), \tag{25}$$

$$\frac{p_e}{\sigma_{s2}} = \eta(\beta_0) = \frac{\varepsilon_e\tau + 24\kappa_e(-Hs - 0.5\beta\beta_0^{-1}(H - J\bar{\beta}^{-2}\beta_0^2)[-s - \varepsilon_i + \mu^{-1}\varepsilon_e])}{\tau - 24\kappa_e(K + 0.5\beta\mu^{-1}\beta_0^{-1}(H - J\bar{\beta}^{-2}\beta_0^2))}, \tag{26}$$

where

$$\mu = \frac{\kappa_e}{\kappa_i}, \quad \tau = \ell + \Gamma(3\nu_1 + 1)(2H + J\bar{\beta}^{-2}\beta_0^2)\beta_0^2 + 4\Gamma\beta^3\beta_0^{-1}(H - J\bar{\beta}^{-2}\beta_0^2).$$

Rewriting now conditions (17)–(20) in detailed form in view of relations (21)–(26), we arrive at a system of linear homogeneous equations for a_2, a_1, b_2, and b_1. The required characteristic equation has the form

$$\tilde{\Delta}(\beta_0) = 0, \tag{27}$$

where $\tilde{\Delta}(\beta_0)$ is the determinant of the matrix $(a_{ij})_{i,j=1}^4$ of this system, in which

$$a_{11} = 1 + A_1 d_{\mathrm{I}}^{**}(\bar{\beta},1)\sigma_{s2}\frac{1}{E_2}, \quad a_{12} = A_1 d_{\mathrm{II}}^{**}(\bar{\beta},1)\sigma_{s2}\frac{1}{E_2},$$

$$a_{13} = A_1 d_{\mathrm{III}}^{**}(\bar{\beta},1)\sigma_{s2}\frac{1}{E_2}, \quad a_{14} = A_1 d_{\mathrm{IV}}^{**}(\bar{\beta},1)\sigma_{s2}\frac{1}{E_2},$$

$$a_{21} = nA_2 d_{\mathrm{I}}^{**}(\bar{\beta},1)\sigma_{s2}\frac{1}{E_2}, \quad a_{22} = nA_2 d_{\mathrm{II}}^{**}(\bar{\beta},1)\sigma_{s2}\frac{1}{E_2},$$

$$a_{23} = 1 + nA_2 d_{\mathrm{III}}^{**}(\bar{\beta},1)\sigma_{s2}\frac{1}{E_2}, \quad a_{24} = nA_2 d_{\mathrm{IV}}^{**}(\bar{\beta},1)\sigma_{s2}\frac{1}{E_2},$$

$$a_{31} = q_1', \quad a_{32} = q_2', \quad a_{33} = q_3', \quad a_{34} = q_4',$$

$$a_{41} = q_5', \quad a_{42} = q_6', \quad a_{43} = q_7', \quad a_{44} = q_8'.$$

We obtain the critical value of the square of angular velocity, corresponding to the critical radius of plastic domain β_{0*}, $\beta_{0*} \in [\beta, \bar{\beta}]$, from relation (16):

$$\omega_*^2 = \frac{24 q_2^2 [-Hs + K[\varepsilon_e - \mu(s + \varepsilon_i + \xi(\beta_{0*}))] - 0.5\beta\beta_{0*}^{-1}(H - J\bar{\beta}^{-2}\beta_{0*}^2)\xi(\beta_{0*})]}{\ell + \Gamma(3\nu_1 + 1)(2H + J\bar{\beta}^{-2}\beta_{0*}^2)\beta_{0*}^2 + 4\Gamma\beta^3\beta_{0*}^{-1}(H - J\bar{\beta}^{-2}\beta_{0*}^2)}$$

for $\kappa_i \neq 0$;

$$\omega_*^2 = \frac{24 q_2^2 (-Hs + K\eta(\beta_{0*}) - 0.5\beta\beta_{0*}^{-1}(H - J\bar{\beta}^{-2}\beta_{0*}^2)[-s - \varepsilon_i - \mu^{-1}(\eta(\beta_{0*}) - \varepsilon_e)])}{\ell + \Gamma(3\nu_1 + 1)(2H + J\bar{\beta}^{-2}\beta_{0*}^2)\beta_{0*}^2 + 4\Gamma\beta^3\beta_{0*}^{-1}(H - J\bar{\beta}^{-2}\beta_{0*}^2)}$$

for $\kappa_e \neq 0$, and

$$\omega_*^2 = \frac{24 q_2^2 [-Hs + K\varepsilon_e - 0.5\beta\beta_{0*}^{-1}(H - J\bar{\beta}^{-2}\beta_{0*}^2)(-s - \varepsilon_i)]}{\ell + \Gamma(3\nu_1 + 1)(2H + J\bar{\beta}^{-2}\beta_{0*}^2)\beta_{0*}^2 + 4\Gamma\beta^3\beta_{0*}^{-1}(H - J\bar{\beta}^{-2}\beta_{0*}^2)}$$

for $\kappa_i = \kappa_e = 0$.

3. Solution in the Case $\mathcal{D}_{1(p)}\mathcal{D}_{2(e)}$

The mechanism of the loss of stability of the disk $\mathcal{D}$ whose plastic state is spread to the entire annular domain $\mathcal{D}_1$ for the elastic annular domain $\mathcal{D}_2$ (Fig. 2) is called special (***01°***) in [11, 12]. Since the plasticity of materials of the sections $\mathcal{D}_1$ and $\mathcal{D}_2$ is different, and there is a discontinuity of the tangential stress, $\beta_0 = \bar{\beta}$ and ω in a certain range of angular velocities remain independent quantities, and further increase in the rotational velocity not necessarily leads to the appearance of plastic state along the circle $\rho = \bar{\beta}$ in $\mathcal{D}_2$. At the same time, the values of coefficients of the characteristic equation change till, at a certain (required) value $\omega = \omega_*$, it turns into identity $\tilde{\Delta}(\bar{\beta}) \equiv 0$.

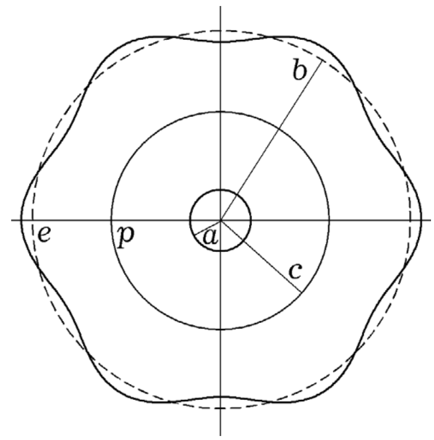

Fig. 2

To obtain these coefficients, we must first conjugate solutions (4) and (9) on the circle $\rho = \overline{\beta}$. We have

$$(1-\overline{\beta}^{-2})C_3 + \frac{p_e}{\sigma_{s2}} + 3(\nu_2+3)(1-\overline{\beta}^2)x = s - 8\Gamma\overline{\beta}^2 x + \beta\overline{\beta}^{-1}\left(-s - \frac{p_i}{\sigma_{s2}} + 8\Gamma\beta^2 x\right),$$

whence

$$C_3 = \{s - \varepsilon_e + \beta\overline{\beta}^{-1}(-s-\varepsilon_i) - [24\kappa_e + 3(\nu_2+3) + (8\Gamma - 3(\nu_2+3))\overline{\beta}^2$$

$$+ 8\beta\overline{\beta}^{-1}(3\kappa_i - \beta^2\Gamma)]x\}\{1-\overline{\beta}^{-2}\}^{-1}. \qquad (28)$$

Since the elastic domain is homogeneous in the case under consideration, we have in the characteristic equation (27)

$$a_{31} = 0, \quad a_{32} = 1, \quad a_{33} = 0, \quad a_{34} = 0,$$

$$a_{41} = 0, \quad a_{42} = 0, \quad a_{43} = 0, \quad a_{44} = 1.$$

Furthermore, A_1 and A_2, determined from relations (21) and (28), depend on the variable x. The solution of Eq. (27) on the segment $[\underline{x}, \overline{x}]$, where $\underline{x}$ is determined from (16) for $\beta_0 = \overline{\beta}$:

$$\underline{x} = \frac{-Hs + K\varepsilon_e + 0.5\beta\overline{\beta}^{-1}(H-J)(s+\varepsilon_i)}{-24K\kappa_e + \ell + \Gamma(3\nu_1+1)(2H+J)\overline{\beta}^2 - 4\beta\overline{\beta}^{-1}(H-J)(3\kappa_i - \beta^2\Gamma)},$$

and $\overline{x}$ is determined from the condition $\sigma_{\theta\theta}(\overline{\beta}+0) = 1$ [see (8)]:

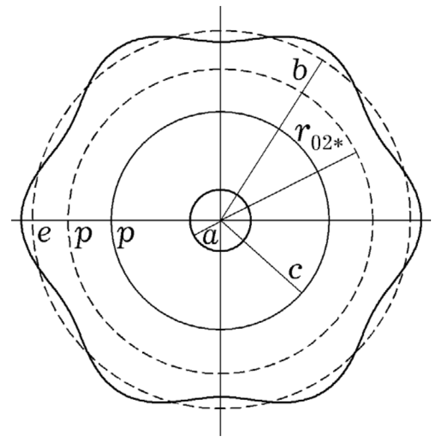

Fig. 3

$$\bar{x} = \{[1-\varepsilon_e](1-\bar{\beta}^{-2})-[s-\varepsilon_e+\beta\bar{\beta}^{-1}(-s-\varepsilon_i)](1+\bar{\beta}^{-2})\}\{[24\kappa_e + 3(\nu_2+3)-3(3\nu_2+1)\bar{\beta}^2](1-\bar{\beta}^{-2})-[24\kappa_e+3(\nu_2+3) + (8\Gamma-3(\nu_2+3))\bar{\beta}^2+8\beta\bar{\beta}^{-1}(3\kappa_i-\beta^2\Gamma)](1+\bar{\beta}^{-2})\}^{-1},$$

gives the required critical value ω_*.

4. Solution in the Case $\mathcal{D}_{1(p)}\mathcal{D}_{2(pe)}$

Suppose that, prior to the loss of stability of the disk $\mathcal{D}$, the plastic state had time to spread over a certain annular domain $\mathcal{D}_{2(p)}$ in $\mathcal{D}_2$ preliminarily covering the entire ring $\mathcal{D}_1$ (Fig. 3). The undisturbed stressed state of the disk $\mathcal{D}$ under study is determined by relations (3) and (4) in the domain $\mathcal{D}_{1(p)}$; in the domain $\mathcal{D}_{2(p)}$, we get

$$\sigma_{\theta\theta} = 1, \tag{29}$$

$$\sigma_{rr} = 1-\frac{\sigma_2}{3\sigma_{s2}}\rho^2+\frac{C_2}{\rho}, \tag{30}$$

and the stresses in $\mathcal{D}_{2(e)}$ are given by relations (8) and (9). The conjugation of solutions (4) and (30) on the circle $\rho=\bar{\beta}$ and the conjugation of (29), (8) and (30), (9) on the circle $\rho=\beta_0=r_{02}/b$ lead to a system of equations for C_2, C_3, and x:

$$\bar{\beta}^{-1}C_2+8[(\Gamma-1)\bar{\beta}^2-\Gamma\beta^3\bar{\beta}^{-1}]x = s-1+\beta\bar{\beta}^{-1}\left(-s-\frac{p_i}{\sigma_{s2}}\right),$$

$$(1+\beta_0^{-2})C_3+3[\nu_2+3-(3\nu_2+1)\beta_0^2]x = 1-\frac{p_e}{\sigma_{s2}},$$

$$-\beta_0^{-1}C_2+(1-\beta_0^{-2})C_3+[3(\nu_2+3)-(3\nu_2+1)\beta_0^2]x = 1-\frac{p_e}{\sigma_{s2}}.$$

Its solution has the form

$$C_2 = \bar{\beta}(s-1)+\beta\left(-s-\frac{p_i}{\sigma_{s2}}\right)-8[(\Gamma-1)\bar{\beta}^3-\Gamma\beta^3]x,$$

$$C_3 = \frac{1-\frac{p_e}{\sigma_{s2}}-3[\nu_2+3-(3\nu_2+1)\beta_0^2]x}{1+\beta_0^{-2}},$$

$$x = \frac{2\left(1-\frac{p_e}{\sigma_{s2}}\right)+\beta_0^{-1}(1+\beta_0^2)\left(\bar{\beta}(s-1)+\beta\left[-s-\frac{p_i}{\sigma_{s2}}\right]\right)}{\tau}, \tag{31}$$

where [with regard for (22), (23), and (25)]

$$\xi(\beta_0) = -\frac{(s+\varepsilon_i)\tau+24\kappa_i[2(1-[\varepsilon_e-\mu(s+\varepsilon_i)])+\bar{\beta}\beta_0^{-1}(s-1)(1+\beta_0^2)]}{\tau+24\kappa_i(2\mu+\beta\beta_0^{-1}(1+\beta_0^2))},$$

$$\eta(\beta_0) = \frac{\varepsilon_e\tau+24\kappa_e[2+\beta_0^{-1}(1+\beta_0^2)(\bar{\beta}(s-1)+\beta[-s-\varepsilon_i+\mu^{-1}\varepsilon_e])]}{\tau+24\kappa_e(2+\beta\mu^{-1}\beta_0^{-1}(1+\beta_0^2))},$$

$$\tau = (1+\beta_0^2)[3(\nu_2+3)-(3\nu_2+1)\beta_0^2+8\beta_0^{-1}((\Gamma-1)\bar{\beta}^3-\Gamma\beta^3)]+3(1-\beta_0^2)[\nu_2+3-(3\nu_2+1)\beta_0^2].$$

Thus, taking in the characteristic equation (27)

$$a_{11} = 1+A_1d_{\mathrm{I}}^{**}(\beta_0,1)\sigma_{s2}\frac{1}{E_2}, \quad a_{12} = A_1d_{\mathrm{II}}^{**}(\beta_0,1)\sigma_{s2}\frac{1}{E_2},$$

$$a_{13} = A_1d_{\mathrm{III}}^{**}(\beta_0,1)\sigma_{s2}\frac{1}{E_2}, \quad a_{14} = A_1d_{\mathrm{IV}}^{**}(\beta_0,1)\sigma_{s2}\frac{1}{E_2},$$

$$a_{21} = nA_2d_{\mathrm{I}}^{**}(\beta_0,1)\sigma_{s2}\frac{1}{E_2}, \quad a_{22} = nA_2d_{\mathrm{II}}^{**}(\beta_0,1)\sigma_{s2}\frac{1}{E_2},$$

$$a_{23} = 1+nA_2d_{\mathrm{III}}^{**}(\beta_0,1)\sigma_{s2}\frac{1}{E_2}, \quad a_{24} = nA_2d_{\mathrm{IV}}^{**}(\beta_0,1)\sigma_{s2}\frac{1}{E_2},$$

$$a_{31} = 0, \quad a_{32} = 1, \quad a_{33} = 0, \quad a_{34} = 0,$$

$$a_{41} = 0, \quad a_{42} = 0, \quad a_{43} = 0, \quad a_{44} = 1$$

or

$$a_{11} = n + A_1 \frac{\sigma_{s2}}{E_2} \frac{(\nu_2+1)n}{n-1}, \quad a_{12} = n - A_1 \frac{\sigma_{s2}}{E_2} \frac{(\nu_2+1)n}{n+1},$$

$$a_{13} = n - 2 + A_1 \frac{\sigma_{s2}}{E_2} \frac{n-2+\nu_2(n+2)}{n+1},$$

$$a_{14} = n + 2 - A_1 \frac{\sigma_{s2}}{E_2} \frac{n+2+\nu_2(n-2)}{n-1},$$

$$a_{21} = -1 + A_2 \frac{\sigma_{s2}}{E_2} \frac{(\nu_2+1)n}{n-1}, \quad a_{22} = 1 - A_2 \frac{\sigma_{s2}}{E_2} \frac{(\nu_2+1)n}{n+1},$$

$$a_{23} = -1 + A_2 \frac{\sigma_{s2}}{E_2} \frac{n-2+\nu_2(n+2)}{n+1}, \quad a_{24} = 1 - A_2 \frac{\sigma_{s2}}{E_2} \frac{n+2+\nu_2(n-2)}{n-1},$$

$$a_{31} = n\beta_0^{n-2}, \quad a_{32} = n\beta_0^{-n-2}, \quad a_{33} = (n-2)\beta_0^{n}, \quad a_{34} = (n+2)\beta_0^{-n},$$

$$a_{41} = -\beta_0^{n-2}, \quad a_{42} = \beta_0^{-n-2}, \quad a_{43} = -\beta_0^{n}, \quad a_{44} = \beta_0^{-n},$$

we now see that it remains to take into account relations (21) and (31).

5. Solution in the Case $\mathcal{D}_{1(e)}\mathcal{D}_{2(pe)}$

The plastic zone $\mathcal{D}_{2(p)}$ has the form of an annular domain $\overline{\beta} \le \rho \le \beta_0$ in $\mathcal{D}_2$. The state of $\mathcal{D}_1$ is elastic (Fig. 4). The functions

$$\sigma_{\theta\theta} = -\frac{p_i}{\sigma_{s2}} + C_1(\beta^{-2} + \rho^{-2}) + \frac{\sigma_1}{8\sigma_{s2}}((\nu_1 + 3)\beta^2 - (3\nu_1 + 1)\rho^2), \tag{32}$$

$$\sigma_{rr} = -\frac{p_i}{\sigma_{s2}} + C_1(\beta^{-2} - \rho^{-2}) + \frac{\sigma_1(\nu_1 + 3)}{8\sigma_{s2}}(\beta^2 - \rho^2) \tag{33}$$

and (29), (30) are subjected to conjugation at $\rho = \overline{\beta}$, and (29), (30) and (8), (9) at $\rho = \beta_0$.

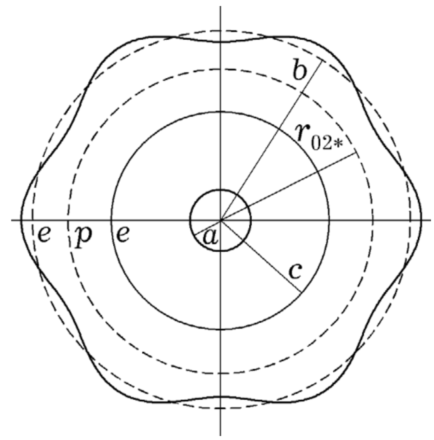

Fig. 4

From the corresponding system of equations, we find

$$C_1 = \frac{1+(\varepsilon+k)\dfrac{p_i}{\sigma_{s2}} - 3\Gamma[(\varepsilon+k)(\nu_1+3)\beta^2 - \{\varepsilon(3\nu_1+1)+k(\nu_1+3)\}\bar{\beta}^2]x}{(\varepsilon+k)\beta^{-2}+(\varepsilon-k)\bar{\beta}^{-2}},$$

$$C_2 = \frac{2\beta_0\left\{-1+\dfrac{p_e}{\sigma_{s2}}+[3(\nu_2+3)-(3\nu_2+1)(2-\beta_0^2)\beta_0^2]x\right\}}{1+\beta_0^2},$$

$$C_3 = -\frac{-1+\dfrac{p_e}{\sigma_{s2}}+3[\nu_2+3-(3\nu_2+1)\beta_0^2]x}{1+\beta_0^{-2}},$$

$$x = \left\{(1+\beta_0^2)\left[(\varepsilon+k-1)\beta^{-2}+(\varepsilon-k+1)\bar{\beta}^{-2}+2\varepsilon\bar{\beta}^{-2}\frac{p_i}{\sigma_{s2}}\right]\right.$$

$$\left.+2\bar{\beta}^{-1}\beta_0[(\varepsilon+k)\beta^{-2}+(\varepsilon-k)\bar{\beta}^{-2}]\left(-1+\frac{p_e}{\sigma_{s2}}\right)\right\}\tau^{-1} \quad (34)$$

Here,

$$\tau = 2\{(1+\beta_0^2)[3\Gamma\varepsilon(-2(\nu_1+1)+(\nu_1+3)\beta^2\bar{\beta}^{-2}+(\nu_1-1)\beta^{-2}\bar{\beta}^2)+4(\varepsilon-k)$$

$$+4(\varepsilon+k)\beta^{-2}\bar{\beta}^2]-\bar{\beta}^{-1}\beta_0[(\varepsilon+k)\beta^{-2}+(\varepsilon-k)\bar{\beta}^{-2}][3(\nu_2+3)-(3\nu_2+1)(2-\beta_0^2)\beta_0^2]\}.$$

For $\kappa_i \neq 0$, it is necessary to take

$$-1+\frac{p_e}{\sigma_{s2}} = -1+\varepsilon_e-\mu(\varepsilon_i-\xi(\beta_0)), \tag{35}$$

$$\frac{p_i}{\sigma_{s2}} = \xi(\beta_0) = \{\varepsilon_i\tau+24\kappa_i[(1+\beta_0^2)[(\varepsilon+k-1)\beta^{-2}+(\varepsilon-k+1)\bar{\beta}^{-2}]$$

$$+2\bar{\beta}^{-1}\beta_0[(\varepsilon+k)\beta^{-2}+(\varepsilon-k)\bar{\beta}^{-2}](-1+\varepsilon_e-\mu\varepsilon_i)]\}\{\tau-48\kappa_i$$

$$\times(\varepsilon\bar{\beta}^{-2}(1+\beta_0^2)+\bar{\beta}^{-1}\mu\beta_0[(\varepsilon+k)\beta^{-2}+(\varepsilon-k)\bar{\beta}^{-2}])\}^{-1}, \tag{36}$$

and, for $\kappa_e \neq 0$,

$$\frac{p_i}{\sigma_{s2}} = \varepsilon_i+\mu^{-1}(1-\varepsilon_e+\eta(\beta_0)), \tag{37}$$

$$-1+\frac{p_e}{\sigma_{s2}} = \eta(\beta_0) = \{(-1+\varepsilon_e)\tau+24\kappa_e(1+\beta_0^2)[(\varepsilon+k-1)\beta^{-2}$$

$$+(\varepsilon-k+1)\bar{\beta}^{-2}+2\varepsilon\bar{\beta}^{-2}(\varepsilon_i+\mu^{-1}(1-\varepsilon_e))]\}\{\tau-48\kappa_e$$

$$\times(\varepsilon\bar{\beta}^{-2}\mu^{-1}(1+\beta_0^2)+\bar{\beta}^{-1}\beta_0[(\varepsilon+k)\beta^{-2}+(\varepsilon-k)\bar{\beta}^{-2}])\}^{-1}. \tag{38}$$

In this case, in view of relations (21) and (34)–(38), we get the same characteristic equation as in Sec. 4.

6. Solution in the Case $\mathcal{D}_{1(pe)}\mathcal{D}_{2(pe)}$

The annular subdomains $\mathcal{D}_{1(p)}$ and $\mathcal{D}_{2(p)}$ of partial disks $\mathcal{D}_1$ and $\mathcal{D}_2$ of the analyzed composite annular disk $\mathcal{D}$ are simultaneously in the plastic state (Fig. 5). The undisturbed stressed state of $\mathcal{D}$ is described by four pairs of relations: (3), (4) in $\mathcal{D}_{1(p)}$, (5), (6) in $\mathcal{D}_{1(e)}$, (29), (30) (with $C_2 := C_2^*$) in $\mathcal{D}_{2(p)}$, and (8), (9) in $\mathcal{D}_{2(e)}$. The conjugation of the components of stresses (3) and (5), (4) and (6) on the circle $\rho=\beta_{01}=r_{01}/b$, (5) and (29), (6) and (30) on the circle $\rho=\bar{\beta}$, and also (29) and (8), (30) and (9) on $\rho=\beta_0=r_{02}/b$ leads to the system of equations

$$C_1+\beta_{01}^{-2}C_2-3\Gamma(3\nu_1+1)\beta_{01}^2x = s \text{ ,s}$$

$$C_1-\beta_{01}^{-2}C_2-\Gamma[(3\nu_1+1)\beta_{01}^2+8\beta^3\beta_{01}^{-1}]x = s+\beta\beta_{01}^{-1}\left(-s-\frac{p_i}{\sigma_{s2}}\right),$$

$$(\varepsilon+k)C_1+\bar{\beta}^{-2}(\varepsilon-k)C_2-3\bar{\beta}^2\Gamma(\varepsilon(3\nu_1+1)+k(\nu_1+3))x = 1,$$

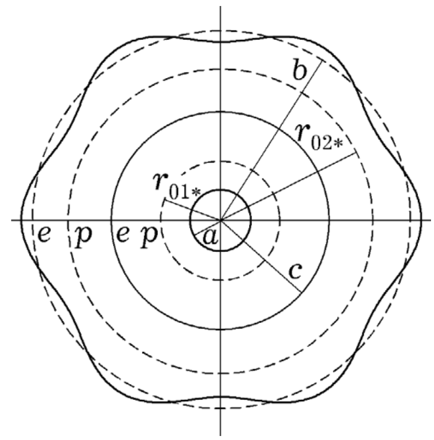

Fig. 5

$$C_1 - \bar{\beta}^{-2}C_2 - \bar{\beta}^{-1}C_2^* - \bar{\beta}^2(3\Gamma(\nu_1+3)-8)x = 1,$$

$$(1+\beta_0^{-2})C_3 + 3(\nu_2+3-(3\nu_2+1)\beta_0^2)x = 1-\frac{p_e}{\sigma_{s2}},$$

$$-\beta_0^{-1}C_2^* + (1-\beta_0^{-2})C_3 + (3(\nu_2+3)-(3\nu_2+1)\beta_0^2)x = 1-\frac{p_e}{\sigma_{s2}},$$

which gives

$$C_1 = \Big\{0.5(1+\varepsilon-k)(1+\beta_0^2) + \bar{\beta}^{-1}\beta_0(\varepsilon-k)\left(-1+\frac{p_e}{\sigma_{s2}}\right) + [2\bar{\beta}^2(1+\beta_0^2)$$
$$\times(3\Gamma\varepsilon(\nu_1+1)-2(\varepsilon-k)) + \bar{\beta}^{-1}\beta_0(\varepsilon-k)(3(\nu_2+3)$$
$$-(3\nu_2+1)(2-\beta_0^2)\beta_0^2)]x\Big\}\{\varepsilon(1+\beta_0^2)\}^{-1},$$

$$C_2 = \Big\{0.5(1-(\varepsilon+k))(1+\beta_0^2) - \bar{\beta}^{-1}\beta_0(\varepsilon+k)\left(-1+\frac{p_e}{\sigma_{s2}}\right) + [\bar{\beta}^2(1+\beta_0^2)$$
$$\times(3\Gamma\varepsilon(\nu_1-1)+4(\varepsilon+k)) - \bar{\beta}^{-1}\beta_0(\varepsilon+k)(3(\nu_2+3)$$
$$-(3\nu_2+1)(2-\beta_0^2)\beta_0^2)]x\Big\}\{\bar{\beta}^{-2}\varepsilon(1+\beta_0^2)\}^{-1},$$

$$C_2^* = \frac{2\beta_0 \left\{ -1 + \frac{p_e}{\sigma_{s2}} + (3(\nu_2+3) - (3\nu_2+1)(2-\beta_0^2)\beta_0^2)x \right\}}{1+\beta_0^2},$$

$$C_3 = -\frac{-1 + \frac{p_e}{\sigma_{s2}} + 3(\nu_2 + 3 - (3\nu_2+1)\beta_0^2)x}{1+\beta_0^{-2}},$$

$$x = \tau_1 + \tau_2 \left(-1 + \frac{p_e}{\sigma_{s2}} \right) \tag{39}$$

and the rational equation for β_{01} with a parameter β_0

$$f(\beta_{01}) \equiv C_1 - \beta_{01}^{-2} C_2 - \Gamma[(3\nu_1+1)\beta_{01}^2 + 8\beta^3\beta_{01}^{-1}]x - s - \beta\beta_{01}^{-1}\left(-s - \frac{p_i}{\sigma_{s2}} \right) = 0, \tag{40}$$

where

$$\tau_1 = \frac{0.5(\beta_{01}^2[1+\varepsilon-k-2\varepsilon s] + \bar{\beta}^2[1-(\varepsilon+k)])(1+\beta_0^2)}{\tau},$$

$$\tau_2 = \frac{\bar{\beta}^{-1}\beta_0(\beta_{01}^2(\varepsilon-k) - \bar{\beta}^2(\varepsilon+k))}{\tau},$$

$$\tau = 3\Gamma\varepsilon\beta_{01}^4(3\nu_1+1)(1+\beta_0^2) - \tau^*,$$

$$\tau^* = \bar{\beta}^2(1+\beta_0^2)(2\beta_{01}^2[3\Gamma\varepsilon(\nu_1+1) - 2(\varepsilon-k)] + \bar{\beta}^2[3\Gamma\varepsilon(\nu_1-1) + 4(\varepsilon+k)])$$

$$+ \bar{\beta}^{-1}\beta_0(\beta_{01}^2(\varepsilon-k) - \bar{\beta}^2(\varepsilon+k))(3(\nu_2+3) - (3\nu_2+1)(2-\beta_0^2)\beta_0^2),$$

$$-1 + \frac{p_e}{\sigma_{s2}} = \frac{-1+\varepsilon_e + 24\kappa_e\tau_1}{1-24\kappa_e\tau_2},$$

$$-s - \frac{p_i}{\sigma_{s2}} = -s - \varepsilon_i - \frac{24\kappa_i(\tau_1 + \tau_2(-1+\varepsilon_e))}{1-24\kappa_e\tau_2}.$$

Combining Eq. (40) with the characteristic equation $\det(a_{ij}) = 0$ [see Sec. 4 and (21), (39)], we obtain, a system of equations for determining the critical value $(\beta_{01*}, \beta_{0*}) \in [\beta, \bar{\beta}) \times [\bar{\beta}, 1)$. On finding the critical radii of the domains $\mathcal{D}_{1(p)}$ and $\mathcal{D}_{2(p)}$, we also find the critical rotational velocity ω_*.

7. Numerical Examples and Discussion

For a composite disk with parameters $\beta = 0.2$, $\bar{\beta} = 0.94$, $\nu_1 = 0.31$, $\nu_2 = 0.3$, $\varepsilon = 1$, $\Gamma = 0.99$, $s = 0.99$, $\sigma_{s2}/E_2 = 0.01$, and $\varepsilon_i = \varepsilon_e = \kappa_e = 0$, which loses, according to our assumption, its stability by the scenario $\mathcal{D}_{1(pe)}\mathcal{D}_{2(e)}$, the values of the critical radius β_{0*} of the plastic zone $\mathcal{D}_{1(p)}$ and the relative critical rotational velocity ω_*/q_2 calculated for different n [see (1)] are presented in Table 1.

Table 1

	n	2	3	4	5
$\kappa_i = 0$	β_{0*}	0.7351	0.8560	0.9052	0.9314
	ω_*/q_2	1.5068	1.5385	1.5470	1.5502
$\kappa_i = 0.01$	β_{0*}	0.7356	0.8564	0.9055	0.9316
	ω_*/q_2	1.5024	1.5339	1.5423	1.5456
$\kappa_i = 0.1$	β_{0*}	0.7401	0.8592	0.9075	0.9331
	ω_*/q_2	1.4646	1.4944	1.5023	1.50533

The calculated values of the critical radius β_{0*} of the plastic zone $\mathcal{D}_{2(p)}$ and relative critical rotational velocity ω_*/q_2 in the case of realization of the mechanism $\mathcal{D}_{1(p)}\mathcal{D}_{2(pe)}$ of the loss of stability of a composite disk with parameters $\beta = 0.2$, $\bar{\beta} = 0.5$, $\nu_1 = 0.3$, $\nu_2 = 0.2$, $\varepsilon = 1.2$, $\Gamma = 0.9$, $s = 1.1$, $\sigma_{s2}/E_2 = 0.01$, and $\varepsilon_i = \varepsilon_e = \kappa_e = 0$ are summarized in Table 2.

Table 2

	n	2	3	4	5
$\kappa_i = 0$	β_{0*}	0.7308	0.8537	0.9036	0.9302
	ω_*/q_2	1.5523	1.5800	1.5875	1.5904
$\kappa_i = 0.01$	β_{0*}	0.7313	0.8540	0.9038	0.9303
	ω_*/q_2	1.5477	1.5753	1.5827	1.5855
$\kappa_i = 0.1$	β_{0*}	0.7359	0.8569	0.9059	0.9319
	ω_*/q_2	1.5080	1.5344	1.5414	1.5440

As follows from these examples, the described method for the investigation of the instability of composite annular disks in the elastoplastic state enables one to find the critical sizes of plastic domain and the critical rotational velocity caused by the geometry of the disks, their determining physical properties, and the parameters of loading. The corresponding results were obtained as applied to the self-balanced form of instability, which is natural just for annular disks.

In view of the results presented in [11, 12], the proposed method for finding the time of transition of the overloaded rapidly rotating disk into the unstable state enables us to determine the approximate solution of the plane elastoplastic problem corresponding to the realization of all possible mechanisms of the loss of stability typical of the disks with radially inhomogeneous materials. This plays a decisive role in the problem of avoiding the overloading of disk parts especially susceptible to the loss of strength.

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