DOI 10.1007/s10958-015-2250-7
Journal of Mathematical Sciences, Vol. 205, No. 2, February, 2015

NONSMOOTH BOUNDARY VALUE PROBLEMS IN THEORY OF RODS, PLATES, AND SHELLS

V. N. Tarasov

Komi Scientific Center of Ural Branch of RAS
4 24, Kommunisticheskaya st., Syktyvkar 167982, Russia
vntarasov@dm.komisc.ru

UDC 539.3

We study the stability of elastic systems with one-sided constraints on displacements. We analytically solve the contact problem for rods and propose a new method for solving contact problems with a free boundary. We present numerical results concerning the stability of elastic systems with one-sided constraints on displacements. Bibliography: 10 *titles.*

We consider problems related to the following topics:

- free contact between thin-wall elements of constructions,
- stability of elastic systems with one-sided constraints on displacements.

In Section 1, we consider the contact problem for flexible elements of constructions. We obtain an analytic solution to the contact problem for rods. We also propose a new method (of generalized reaction) for solving contact problems with a free boundary. This method can be applied to a large class of problems [1]. Formally, the method is an iteration scheme of the gradient projection method applied to the energy functional of an elastic system expressed in terms of the reaction function of obstacle. The contact problem can be formulated as a problem of convex programming. Using the saddle point theorem for the Lagrange function [2], we consider the dual problem which can be solved by the gradient projection method [3]. In mechanics, the variables of the dual problem are interpreted as the reaction forces of contact. To apply this method, it is necessary to find the inverse operators of the equilibrium equations for contacting elements. The solution of equations for rods, plates, and shells is a rather complicated task. However, all classical methods (grids, finite elements, boundary elements and so on) are available for solving such problems.

The classical stability problem for elastic systems is reduced to finding eigenvalues of linear operators. The study of stability and supercritical behavior of elastic systems with one-sided constraints on displacements is reduced to finding bifurcation points of nonsmooth equations or parameters for which the corresponding variational problems with one-sided constraints can have not necessarily one solution.

In Section 2, we obtain new results concerning the stability of elastic systems with one-sided

Translated from *Problemy Matematicheskogo Analiza* **77**, December 2014, pp. 163-186.

constraints. We analytically solve the stability problem for rods in an elastic medium provided that the rods are compressed by a longitudnal force and their bendings are bounded from one side by a rigid obstacle. We clarify how boundary conditions affect the critical value of the force. We also study the stability of an annulus supported by elastic threads under the action of the normal pressure force and a torus-like shell containing an elastic filler such that there is a one-sided contact between the shell and the filler.

1 Contact Problem for Flexible Elements of Construction

1.1. Contact problem for two parallel beams. We consider the contact problem for two parallel beams of length ℓ. Denote by $w_i(\xi)$, $i = 1, 2$, the beam deflections and by $x(\xi)$ the contact reaction force. Assume that the first beam is under the action of a normal load $q(\xi)$ and the second one is under the action of the first beam, when the deflection of the first beam exceeds the initial clearance between the beams $\rho = \text{const}$. In this case, the equilibrium equation takes the form

$$\begin{aligned} d_1 \frac{d^4 w_1}{d\xi^4} &= q(\xi) - x(\xi), \\ d_2 \frac{d^4 w_2}{d\xi^4} &= x(\xi), \end{aligned} \tag{1.1}$$

where $d_1 = EJ_1$, $d_2 = EJ_2$, J_1, and J_2 are moments of inertia of cross-sections, and E is the Young modulus.

The deflections $w_1(\xi)$ and $w_2(\xi)$ and the contact reaction force $x(\xi)$ satisfy the inequalities

$$w_1(\xi) \leqslant w_2(\xi) + \rho, \quad x(\xi) \geqslant 0, \tag{1.2}$$

and the equation

$$x(\xi)[w_1(\xi) - w_2(\xi) - \rho] = 0. \tag{1.3}$$

The first inequality in (1.2) is the inpenetrability condition, and the second one is a one-sided constraint. Equation (1.3) is the complementary slackness condition which means that $x(\xi) > 0$ implies $w_1(\xi) - w_2(\xi) - \rho = 0$ (there is contact at the point ξ). Conversely, $w_1(\xi) - w_2(\xi) - \rho < 0$ implies $x(\xi) = 0$.

The inequalities (1.2) and Equation (1.3) hold if for any $\alpha > 0$

$$x(\xi) = [x(\xi) + \alpha(w_1(\xi) - w_2(\xi) - \rho)]_+, \tag{1.4}$$

where the subscript "+" means the positive part of a function, i.e.,

$$\varphi_+(\xi) = \max\{0, \varphi(\xi)\} = \frac{1}{2}(\varphi(\xi) + |\varphi(\xi)|).$$

Indeed, from (1.4) if follows that $x(\xi) \geqslant 0$. If the expression in the square brackets is negative, then $x(\xi) = 0$ which implies $w_1 - w_2 - \rho \leqslant 0$. In the case $x(\xi) > 0$, the subscript "+" can be omitted and (1.4) implies $w_1 - w_2 - \rho = 0$. For the sake of definiteness we impose the boundary conditions of hinged support

$$w_i(0) = w_i''(0) = w_i(\ell) = w_i''(\ell) = 0, \quad i = 1, 2. \tag{1.5}$$

The variational statement of this problem has the form

$$J(w_1, w_2) = \int_0^\ell \left(\frac{EJ_1}{2} w_1''^2 + \frac{EJ_2}{2} w_2''^2 + q w_1 \right) d\xi \to \min_{w_1, w_2} \tag{1.6}$$

with the condition (1.2).

Introducing the Green function for the boundary value problems (1.1), (1.5),

$$G_i(\xi, t) = \frac{1}{6d_i} \left[(\xi - t)_+^3 + \frac{(t-\ell)}{\ell} \xi^3 + \frac{t(\ell - t)(2\ell - t)}{\ell} \xi \right],$$

we can write solutions to Equation (1.1) as $w_i(\xi) = G_i(q - x)$, where

$$G_i \varphi = \int_0^\ell G_i(\xi, t) \varphi(t) dt, \tag{1.7}$$

and Equation (1.4) as

$$x(\xi) = [x(\xi) - \alpha(G_2 x + \rho - G_1(q - x))]_+. \tag{1.8}$$

To solve (1.8), one can use the method of successive approximations:

$$x_{n+1} = [x_n - \alpha(G_2 x_n + \rho - G_1(q - x_n))]_+, \quad n = 0, 1, \ldots, \tag{1.9}$$

where $x_0(\xi) \equiv 0$ can be taken for the initial approximation.

To find a function class where we look for a solution to Equation (1.8), we analytically solve the above problem in the case $q(\xi) = q = \text{const}$, $d_1 = d_2 = d_0$.

Assume that $[\xi_0, \xi_1]$ $(\xi_0 < \xi_1)$ is the contact zone. By symmetry of the problem, $\xi_1 = \ell - \xi_0$. The function $u(\xi) = w_1(\xi) - w_2(\xi)$ satisfies

$$d_0 \frac{d^4 u}{d\xi^4} = q - 2x(\xi), \tag{1.10}$$

and $u(\xi) = \rho$ for $\xi \in [\xi_0, \xi_1]$. For $0 < \xi < \xi_0$ we have $x(\xi) \equiv 0$. Therefore,

$$d_0 \frac{d^4 u}{d\xi^4} = q. \tag{1.11}$$

The general solution to Equation (1.11) has the form

$$u(\xi) = \frac{q\xi^4}{24 d_0} + c_0 + c_1 \xi + c_2 \xi^2 + c_3 \xi^3.$$

By the boundary condition, $c_0 = c_2 = 0$. Since the function $u(\xi)$, together with its first and second order derivatives, is continuous at the point ξ_0, we have

$$\begin{aligned} & c_1 \xi_0 + c_3 \xi_0^3 + \frac{q}{24 d_0} \xi_0^4 = \rho, \\ & c_1 + 3c_3 \xi_0^2 + \frac{q}{6d_0} \xi_0^3 = 0, \\ & 6c_3 \xi_0 + \frac{q}{2d_0} \xi_0^2 = 0 \end{aligned} \tag{1.12}$$

which implies

$$c_1 = \frac{q}{12d_0}\xi_0^3, \quad c_3 = -\frac{q}{12d_0}\xi_0, \quad \xi_0 = \left(\frac{24d_0\rho}{q}\right)^{1/4}, \tag{1.13}$$

and the function $u(\xi)$ is represented by

$$u(\xi) = \begin{cases} \frac{q\xi}{12d_0}[\xi_0^3 - \xi_0\xi^2 + \xi^3], & 0 \leqslant \xi \leqslant \xi_0, \\ \rho, & \xi \in [\xi_0, \ell - \xi_0], \\ \frac{q(\ell-\xi)}{12d_0}[\xi_0^3 - \xi_0(\ell-\xi)^2 + (\ell-\xi)^3], & \xi \in [\ell - \xi_0, \ell]. \end{cases} \tag{1.14}$$

From (1.14) it follows that

$$u'''(\xi_0 - 0) = \frac{q}{2d_0}\xi_0, \quad u'''(\xi_0 + 0) = 0. \tag{1.15}$$

From (1.10) and (1.15) we find

$$x(\xi) = \frac{q}{4}\xi_0\delta(\xi - \xi_0) + \frac{q}{4}\xi_0\delta(\xi - \ell + \xi_0) + +q[H(\xi - \xi_0) - H(\xi - \ell + \xi_0)], \tag{1.16}$$

where $\delta(\xi)$ is the generalized Dirac δ-function. We set $v(\xi) = w_1(\xi) + w_2(\xi)$. Then the function $v(\xi)$ satisfies the equation

$$d_0\frac{d^4v}{d\xi^4} = q$$

which implies

$$v(\xi) = \frac{q}{24d_0}[\ell^3\xi - 2\ell\xi^3 + \xi^4]. \tag{1.17}$$

The deflections are expressed as follows:

$$w_1(\xi) = \frac{1}{2}(u(\xi) + v(\xi)), \quad w_2(\xi) = \frac{1}{2}(v(\xi) - u(\xi)).$$

Thus, if the contact domain is a segment of nonzero length, then $w_i''(\xi)$, $i = 1, 2$, are continuous, $w_i'''(\xi)$ are piecewise continuous, whereas the 4th order derivatives contain terms of Dirac δ-function type (cf. formula (1.16)). From (1.13) we find the existence condition in the case of a contact domain of nonzero length ($\xi_0 < \ell/2$)

$$q \geqslant \frac{384d_0\rho}{\ell^4}. \tag{1.18}$$

For

$$0 < q < \frac{384d_0\rho}{5\ell^4}$$

there is no contact between the beams, i.e., $w_1(\xi) < \rho$ for all $\xi \in [0, \ell]$, and for

$$\frac{384d_0\rho}{5\ell^4} \leqslant q \leqslant \frac{384d_0\rho}{\ell^4} \tag{1.19}$$

the contact domain consists of a single point $\xi_0 = \ell/2$. Then (1.12) takes the form

$$c_1 \frac{\ell}{2} + c_3 \frac{\ell^3}{8} + \frac{q}{d_0} \frac{\ell^4}{384} = \rho,$$
$$c_1 + 3c_3 \frac{\ell^2}{4} + \frac{q}{d_0} \frac{\ell^3}{48} = 0$$

which implies

$$c_1 = \frac{ql^3}{384 d_0} + \frac{3\rho}{\ell}, \quad c_3 = -\frac{q\ell}{32 d_0} - \frac{4\rho}{l^3}. \tag{1.20}$$

In this case, the function $u(\xi)$ is defined by

$$u(\xi) = \begin{cases} c_1 \xi + c_3 \xi^3 + \dfrac{\xi^4 q}{24 d_0}, & \xi \in [0, \ell/2], \\ c_1(\ell - \xi) + c_3(\ell - \xi)^3 + \dfrac{(\ell - \xi)^4 q}{24 d_0}, & \xi \in [\ell/2, \ell], \end{cases} \tag{1.21}$$

and $v(\xi)$ is computed by formula (1.17).

From (1.20) and (1.21) we find

$$u'''(\ell/2 + 0) - u'''(\ell/2 - 0) = -\frac{5q\ell}{8d_0} + \frac{48\rho}{\ell^3}$$

which implies

$$x(\xi) = q - d_0 \frac{d^4 w}{d\xi^4} = \left(\frac{5q\ell}{16} - \frac{24\rho d_0}{\ell^3} \right) \delta(\xi - \ell/2). \tag{1.22}$$

For anchorage boundary conditions an analytic solution is found in a similar way.

1.2. Method of generalized reaction. Assume that displacements u_1 and u_2 of two elastic elements of constructions (rods, plates, and shells) under certain loads cause contact between these elements. We assume that a deformation of every element is described by the linear equations for u_1 and u_2:

$$\begin{aligned} &L_i u_i(\xi) = f_i(\xi), \quad \xi \in \Omega_i \subset R^{n_i}, \\ &\Gamma_{ij} u_i(\xi) = 0, \quad \xi \in \partial\Omega_i, \quad j \in 1 : r_i, \quad i = 1, 2. \end{aligned} \tag{1.23}$$

With the boundary value problems (1.23) one can associate (nonlinear) operators A_i (the boundary conditions are included in the definition of operators) in Hilbert spaces.

Thus, instead of the problem (1.23), we can study the operator equations

$$A_i u_i = f_i(\xi), \quad \xi \in \Omega_i, \ i \in 1 : 2.$$

Suppose that the initial clearance between thin-wall elements of construction in the initial undeformed state in a domain of possible contact $\Omega_0 \subset \Omega_1 \cap \Omega_2$ is determined by a function $\rho(\xi)$,

$\xi \in \Omega_0$. Then the contact problem under consideration can be formulated as follows:

$$\begin{cases} A_1 u_1 = f_1(\xi) - x(\xi)\widetilde{H}(\xi), \\ A_2 u_2 = f_2(\xi) + x(\xi)\widetilde{H}(\xi), \end{cases} \tag{1.24}$$

$$\begin{cases} u_1(\xi) - u_2(\xi) - \rho(\xi) \leqslant 0, \quad \xi \in \Omega_0, \\ x(\xi) \geqslant 0, \quad \xi \in \Omega_0, \\ x(\xi)[u_1(\xi) - u_2(\xi) - \rho(\xi)] = 0. \end{cases} \tag{1.25}$$

Equations (1.24) are the equilibrium equations, $f_i(\xi)$, $i = 1, 2$, are the external forces acting on contacting elements, $x(\xi)$ is the contact reaction force, and

$$\widetilde{H}(\xi) = \begin{cases} 1, & \xi \in \Omega_0, \\ 0, & \xi \notin \Omega_0. \end{cases}$$

The second inequality in (1.25) is a one-sided constraint, the third one is the complementary slackness condition, i.e., $x(\xi) > 0$ implies $u_1(\xi) - u_2(\xi) - \rho(\xi) = 0$ and, conversely, $u_1(\xi) - u_2(\xi) - \rho(\xi) < 0$ implies $x(\xi) = 0$. From (1.24) we find

$$x(\xi) = \frac{1}{2}(A_2 u_2 - A_1 u_1 + f_1 - f_2). \tag{1.26}$$

As above, we prove (1.25) if

$$x(\xi) = [x(\xi) + \alpha(u_1(\xi) - u_2(\xi) - \rho(\xi))]_+, \quad \xi \in \Omega_0, \tag{1.27}$$

for any $\alpha > 0$. Thus, instead of (1.24) and (1.25) we can consider the system

$$A_1 u_1 = f_1 - x, \quad A_2 u_2 = f_2 + x, \quad x = [x - \alpha(u_2 + \rho - u_1)]_+ \widetilde{H}(\xi). \tag{1.28}$$

If the inverses of A_1 and A_2 exist, then from the first two equations of (1.28) we have

$$u_1 = A_1^{-1}(f_1 - x), \quad u_2 = A_2^{-1}(f_2 + x). \tag{1.29}$$

Substituting these expressions into the third equation, we find

$$x = [x - \alpha(A_2^{-1}(f_2 + x) - A_1^{-1}(f_1 - x) + \rho)]_+, \quad \xi \in \Omega_0. \tag{1.30}$$

Equation (1.30) can be solved by the method of successive approximations:

$$x_{k+1} = [x_k - \alpha(A_2^{-1}(f_2 + x_k) - A_1^{-1}(f_1 - x_k) + \rho)]_+, \quad \xi \in \Omega_0. \tag{1.31}$$

To clarify the character of convergence of (1.31), we formulate some assertions related to the theory of extremum problems (cf., for example, [3]).

1.2.1. Necessary extremum conditions and the gradient projection method. It is required to find $u_* \in M$ such that

$$f(u_*) = \min_{u \in M} f(u). \tag{1.32}$$

Assume that $M \in R^n$ is a convex set and $f(u)$ is continuously differentiable in R^n.

Proposition 1.1. *For a point $u_* \in M$ to be a minimum point of a functional $f(u)$ over the set M it is necessary and, if $f(u)$ is convex, sufficient that*

$$\left(\frac{\partial f(u_*)}{\partial u}, u - u_*\right) \geqslant 0 \quad \forall u \in M. \tag{1.33}$$

Let $v \in R^n$. We denote by $P_M(v)$ the nearest point of the set M to the point v. The point $P_M(v)$, called the *projection* of v onto the convex set M, is a minimum point of the functional $\varphi(u) = 1/2\|u - v\|^2$ over the set M. From (1.33) it follows that

$$(P_M(v) - v, u - P_M(v)) \geqslant 0 \quad \forall u \in M. \tag{1.34}$$

Denote $z(y) = P_M(y)$. Then

$$(z(y) - y, x - z(y)) \geqslant 0 \quad \forall x \in M. \tag{1.35}$$

Let $z(y_1)$ and $z(y_2)$ be the projections of two points on the set M. Then

$$||z(y_1) - z(y_2)|| \leqslant ||y_1 - y_2|| \tag{1.36}$$

(the distance between the projections of points does not exceed the distance between the points).

Let $\alpha > 0$. Denote by $\omega(u)$ the projection of $u - \alpha\frac{\partial f(u)}{\partial u}$ on the set M:

$$\omega(u) = P_M\left(u - \alpha\frac{\partial f(u)}{\partial u}\right).$$

We assume that $u \in M$. By (1.33),

$$(\omega(u) - u + \alpha\frac{\partial f(u)}{\partial u}, u - \omega(u)) \geqslant 0$$

which implies

$$\left(\frac{\partial f(u)}{\partial u}, \omega(u) - u\right) \leqslant -\frac{1}{\alpha}\|\omega(u) - u\|^2. \tag{1.37}$$

Proposition 1.2. *The condition* (1.33) *is equivalent to the condition*

$$u_* = P_M\left(u_* - \alpha\frac{\partial f(u_*)}{\partial u}\right) \quad \forall\ \alpha > 0. \tag{1.38}$$

Proof. From (1.34) we find

$$(\omega(u_*) - u_* + \alpha\frac{\partial f(u_*)}{\partial u}, u - \omega(u_*)) \geqslant 0 \quad \forall u \in M. \tag{1.39}$$

Let (1.38) hold. Then

$$\left(\alpha\frac{\partial f(u_*)}{\partial u}, u - u_*\right) \geqslant 0 \quad \forall u \in M,$$

i.e., (1.33) holds.

Let (1.33) hold. From (1.38) we find

$$(\omega(u_*) - u_*, \omega(u_*) - u_*) \leqslant \alpha\left(\frac{\partial f(u_*)}{\partial u}, u_* - \omega(u_*)\right).$$

From (1.33) it follows that $(\omega(u_*) - u_*, \omega(u_*) - u_*) \leqslant 0$, i.e., $u_* = \omega(u_*)$. □

A point $u_* \in M$ satisfying the necessary extremum condition (for example, (1.33) or (1.38)) is said to be *stationary.* We refer the reader to [3] for details.

We formulate the problem of minimizing $f(u)$ over the set M.

1.2.2. The gradient projection method. We fix a number $\alpha > 0$ and an initial approximation $u_0 \in M$. Suppose that $u_k \in M$ is already obtained. Then $u_{k+1} = \omega(u_k)$.

Proposition 1.3. *Let $D = \{u \in M : f(u) \leqslant f(u_0)\}$ be bounded. Then there exists $\alpha_0 > 0$ such that for all $0 < \alpha \leqslant \alpha_0$ any limit point of the sequence $\{u_k\}$, $k = 0, 1, 2\ldots$, is a stationary point of the functional $f(u)$ over the set M.*

Let $M = \{u \in R^n : \ h_j(u) \leqslant 0, \ j \in J = 1 : N\}$, where $h_j(u)$ are convex continuously differentiable functions, and the set M satisfies the Slater condition, i.e., there exists a point $\overline{u}$ such that $h_j(\overline{u}) < 0$, $j \in J$. We also assume that the function $f(u)$ is strongly convex, i.e., there exists a constant $m > 0$ such that

$$f(u) \geqslant f(v) + \left(\frac{\partial f(v)}{\partial u}, u - v\right) + m||u - v||^2 \quad \forall u, v \in R^n.$$

We consider the Lagrange function

$$L(u, x) = f(u) + \sum_{i=1}^{N} x_i h_i(u). \tag{1.40}$$

Under the above assumptions, there exists a unique solution to the problem (1.32).

1.2.3. The saddle point theorem. A point u_* is a solution to the problem (1.32) if and only if there exists a Lagrange multiplier $x^* = (x_1^*, x_2^*, \ldots, x_N^*)$ such that u_* and x^* are saddle points of the Lagrange function on the set $R^n \times X$

$$L(u_*, x) \leqslant L(u_*, x^*) \leqslant L(u, x^*) \quad \forall u \in R^n, x \in X,$$

where $X = \{x \in R^N : \ x_j \geqslant 0, \ j \in J\}$. If a saddle point exists, the operations of taking maximum and minimum commute:

$$\min_{u\in R^n} \max_{x\in X} L(u, x) = \max_{x\in X} \min_{u\in R^n} L(u, x) = L(u_*, x^*).$$

We consider two extremum problems

$$\psi(u) \to \min_{u}, \quad \psi(u) = \max_{x\in X} L(u, x), \tag{1.41}$$

$$\varphi(x) \to \sup_{x\in X}, \quad \varphi(x) = \min_{u} L(u, x). \tag{1.42}$$

The problems (1.41) and (1.42) are dual problems of nonlinear programming. If $u \notin M$ in (1.41), then we set $\psi(u) = \infty$, i.e., (1.41) is exactly the problem (1.32). We refer the reader to [2] for details.

We return to the problem (1.24), (1.25). We assume that the operators A_i in (1.28) are positive definite operators defined on the same Hilbert space H of functions in the domain Ω $(\Omega = \Omega_1 = \Omega_2 = \Omega_0)$

$$A_i u = \sum_{|\alpha|\leqslant m} D^\alpha (C_{\alpha i}(\xi) D^\alpha u), \tag{1.43}$$

where

$$D^{\alpha}(\ \ ) = \frac{\partial^{|\alpha|}(\ \ )}{\partial\xi_1^{\alpha_1}\partial\xi_2^{\alpha_2}\dots\partial\xi_n^{\alpha_n}},$$

$\alpha = (\alpha_1, \alpha_2 \dots \alpha_n)$ is a multi-index with length $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. With the operators A_i we associate the bilinear forms

$$a_i(u, v) = \int\limits_{\Omega} \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} C_{\alpha i}(\xi) D^{\alpha}u \cdot D^{\alpha}v d\xi. \tag{1.44}$$

The positive definiteness means that

$$a_i(u, u) \ \geqslant\ \gamma_i^2 \|u\|_H^2, \tag{1.45}$$

where $\|u\|_H$ is the H-norm. We set

$$J(u_1, u_2) = \frac{1}{2}a_1(u_1, u_1) + \frac{1}{2}a_2(u_2, u_2) - (f_1, u_1) - (f_2, u_2).$$

The contact problem can be formulated as the extremum problem

$$J(u_1, u_2) \to \min_{u_1,\ u_2} \tag{1.46}$$

with the condition

$$u_1 - u_2 - \rho \leqslant 0. \tag{1.47}$$

For the problem (1.46), (1.47) we introduce the Lagrangian

$$L(u_1, u_2, x) = J(u_1, u_2) + (x, u_1 - u_2 - \rho). \tag{1.48}$$

We also introduce the functionals

$$\widetilde{\Phi}(x) = \min_{u_1,u_2} L(u_1, u_2, x), \quad \Phi(x) = -\widetilde{\Phi}(x). \tag{1.49}$$

The functional $\Phi(x)$ is expressed by

$$\Phi(x) = \frac{1}{2}A_1^{-1}(f_1 - x), f_1 - x) + \frac{1}{2}A_2^{-1}(f_2 + x), f_2 + x) + (x, \rho).$$

We consider the problem

$$\Phi(x) \to \inf_{x\in M}, \tag{1.50}$$

where $M = \{x \in L_2(\Omega)\,|\, x \geqslant 0\}$.

The functional $\Phi(x)$ is bounded from below on M. We set $\Phi^* = \inf_{x\in M} \Phi(x)$.

The quadratic functional $\Phi(x)$ is continuously differentiable, and its derivative takes the form

$$\Phi'(x) = A_2^{-1}(f_2 + x) - A_1^{-1}(f_1 - x) + \rho. \tag{1.51}$$

We temporarily assume that the infimum is attained at a point x^*. We write the necessary minimum condition for the problem (1.50) in the form (1.38)

$$x^* = P_M(x_* - \alpha\Phi'(x)) = \left[x^* - \alpha\left(A_2^{-1}(f_2 + x) - A_1^{-1}(f_1 - x) + \rho\right)\right]_+ \tag{1.52}$$

(it is obvious that the projection of $\Phi(x)$ onto the set M of nonnegative functions is the "cut-off function" of $\Phi(x)$). The last equation coincides with (1.30).

In infinite-dimensional spaces, the point x^* does not necessarily exist, i.e., there is no square summable function $x(\xi)$ such that $\Phi(x) = \Phi^*$ (cf. 1.16) and (1.22), where the contact force contains terms of Dirack δ-function type).

However, the following assertion holds [1].

Theorem 1.1. *There exists* $\alpha_0 > 0$ *such that for any* $0 < \alpha \leqslant \alpha_0$ *the sequence* $\{x_n\}$ *defined by* (1.31) *is minimizing, i.e.,*

$$\lim_{n\to\infty} \Phi(x_n) = \Phi_*, \tag{1.53}$$

and the sequence of functions

$$u_{1k} = A_1^{-1}(f_1 - x_k), \quad u_{2k} = A_2^{-1}(f_2 + x_k) \tag{1.54}$$

converge to the solution to the problem (1.46), (1.47).

2 Stability of Elastic Systems under One-Sided Constraints on Displacements

2.1. Statement of the problem. To study the stability of elastic systems, variational methods are often used. Assume that the potential energy is expressed by the functional

$$\Phi(\lambda, w) = F(w) - \lambda G(w),$$

where $F(w)$ is the elastic energy and $G(w)$ is the external force work. It is required to find λ (interpreted as the critical load in practice) such that the variational problem

$$\Phi(\lambda, w) \to \min_w$$

has a nontrivial solution. To study the stability of the problem, one usually uses the quadratic approximation of the potential energy functional, whereas an exact expression for the full energy is used only for determining the supercritical behavior of the system after loss of stability. In the case of quadratic approximations, the parameter λ can be found by solving the problem

$$F(w) \to \min_w \tag{2.1}$$

with the condition

$$G(w) = 1. \tag{2.2}$$

However, for engineering design, it is actual to study the stability of elastic systems with one-sided constraints on displacements. The influence of one-sided constraints is expressed by the fact that the function w in the problem (2.1), (2.2) satisfies some inequalities. Since we deal with quadratic functionals, we can formulate the variational problem as follows. Let A and Q be selfadjoint bounded linear operators in a separable Hilbert space H. Assume that A is positive definite and Q is nonnegative definite and compact. Thus, for any $w \in H$

$$(Aw, w) \geqslant c_0\|w\|^2, \quad c_0 > 0, \tag{2.3}$$

$$(Qw, w) \geqslant 0. \tag{2.4}$$

We consider the problem

$$f(w) = \frac{1}{2}(Aw, w) \to \min_w \tag{2.5}$$

with the condition

$$g(w) = \frac{1}{2}(Qw, w) = 1, \quad w \in K, \tag{2.6}$$

where K is a convex closed cone in H.

Proposition 2.1. *The problem* (2.5), (2.6) *has a solution.*

Proof. The set $\{w \in H : g(w) = 1\}$ is weakly closed in the space H. Indeed, let $w_n \in H$ weakly converge to $w \in H$. Then $Qw_n \to Qw$ strongly as $n \to \infty$. Further,

$$g(w_n) - g(w) = \frac{1}{2}(Qw_n, w_n) - \frac{1}{2}(Qw, w) = \frac{1}{2}(Q(w_n - w), w) - \frac{1}{2}(w_n, Qw - Qw_n).$$

Using the Schwarz inequality, from the weak convergence of w_n it follows that

$$|g(w_n) - g(w)| \leqslant \frac{1}{2}|(Q(w_n - w), w)| + \frac{1}{2}\|w_n\|\|Qw - Qw_n\| \to 0, \quad n \to \infty,$$

which implies $g(w) = 1$ because of the continuity of Q. Since a convex closed cone K is weakly closed in H, the set of elements $w \in H$ satisfying (2.6) is weakly closed, whereas the intersection of this set with any ball in H is weakly compact. The functional $f(w)$ is convex and continuous and, consequently, weakly lower semicontinuous. Since $f(w) \to \infty$ as $\|w\| \to \infty$ in view of (2.3), the functional f attains the minimum. □

We consider the problem of nonlinear programming

$$f(u) = \frac{1}{2}(Au, u) \to \min_u \tag{2.7}$$

with the constraints

$$g(u) = \frac{1}{2}(Qu, u) = 1, \tag{2.8}$$

$$(b_j, u) \leqslant 0, \quad j \in J = 1 : m, \tag{2.9}$$

where A is a positive definite matrix of order n, Q is a nonnegative definite matrix of order n, and $b_j \in R^n$. The problem (2.7)–(2.9) can be obtained by finite-dimensional approximation of the problem (2.5), (2.6). Denote by Γ the cone defined by the inequalities (2.9). Let u^* be a solution to the problem (2.7)–(2.9). By the Kuhn–Tucker theorem, there exist Lagrange multipliers $\mu_j \geqslant 0$, $j \in 1 : m$, λ^* such that

$$\begin{aligned} &Au^* - \lambda^* Qu^* + \sum_{j=1}^{m} \mu_j b_j = 0, \\ &\frac{1}{2}(Qu^*, u^*) = 1, \\ &\mu_j(b_j, u) = 0, \quad j \in 1 : m. \end{aligned} \tag{2.10}$$

A point u^* satisfying (2.10) is said to be *stationary.*

We describe an algorithm for solving the problem (2.7)–(2.9). Suppose that $u_0 \in \Gamma$ and $g(u_0) = 1$ is an initial approximation. Assume that $u_k \in \Gamma$, $g(u_k) = 1$, is already known. We set

$$\widetilde{\Gamma}_k = \{u \in \Gamma | (Qu_k, u - u_k) = 0\}. \tag{2.11}$$

We find $\widetilde{u}_k \in \widetilde{\Gamma}_k$ such that

$$\frac{1}{2}(A\widetilde{u}_k, \widetilde{u}_k) = \min_{u \in \widetilde{\Gamma}_k} \frac{1}{2}(Au, u). \tag{2.12}$$

Then we set

$$u_{k+1} = s_k^{-1} \widetilde{u}_k, \tag{2.13}$$

where $s_k = \sqrt{g(\widetilde{u}_k)}$. Since $\widetilde{u}_k$ is a solution to the problem (2.12), there exist Lagrange multipliers $\mu_{k,j} \geqslant 0$ and λ_k such that

$$\begin{aligned} &A\widetilde{u}_k - \lambda_k Q u_k + \sum_{j=1}^{m} \mu_{k,j} b_j = 0, \\ &(Qu_k, \widetilde{u}_k - u_k) = 0, \\ &\mu_{k,j}(b_j, \widetilde{u}_k) = 0, \quad j \in 1 : m. \end{aligned} \tag{2.14}$$

It is possible to show [4] that the sequence $\{\lambda_k\}$ is monotone decreasing and bounded from below. Consequently, $\{\lambda_k\}$ has limit λ_* and any limit point of $\{u_k\}$ is stationary.

Remark 2.1. The auxiliary problem (2.12) is a problem of convex quadratic programming. Methods for solving such problems are well developed.

Remark 2.2. If $J = \varnothing$, i.e., $\Gamma = R^n$, and Q is the identity matrix, then we deal with the known Kellogg method for determining a minimal eigenvalue of the matrix A.

Remark 2.3. The problem (2.7)–(2.9) can have several local minima. Therefore, it can turn out that $\lambda_* > \lambda^*$ and the algorithm converges to one of the local minima.

We introduce the matrix $B(\lambda) = A - \lambda Q$. It is easy to see that if $\lambda < \lambda^*$, then $(B(\lambda)u, u) \geqslant \mu||u||$, where $\mu > 0$, for all $u \in \Gamma$. Conversely, for $\lambda > \lambda^*$ there is a vector $u \in \Gamma$ such that $(B(\lambda)u, u) < 0$. Thus, the problem (2.7)–(2.9) is reduced to identifying the conditional positive definiteness of quadratic forms on cones.

2.2. One-sided constraints on displacements. Consider a rod of length ℓ in an elastic medium with rigidity C under the action of a longitudnal force P. We denote by D the rod rigidity under bending. We consider the functional

$$J(w) = \frac{1}{2} \int_0^{\ell} (Dw''^2 + Cw^2 - Pw'^2) dx. \tag{2.15}$$

We assume that the rod deflection w is bounded from one side by a rigid obstacle so that

$$w(x) \geqslant 0, \quad x \in [0, \ell]. \tag{2.16}$$

Computation of the stability of a rod is reduced to finding a minimal value of P such that the variational problem

$$J(w) \to \min_w \tag{2.17}$$

with the condition (2.16) has a nontrivial solution. We consider boundary conditions of three types:

- boundary conditions of anchorage

$$w(0) = w(\ell) = 0, \quad w'(0) = w'(\ell) = 0, \tag{2.18}$$

- boundary conditions of hinged support

$$w(0) = w(\ell) = 0, \quad w''(0) = w''(\ell) = 0, \tag{2.19}$$

- boundary condition of anchorage at $x = 0$ and free boundary condition at $x = \ell$

$$w(0) = 0, \quad w'(0) = 0, \quad w^{''}(\ell) = 0, \quad Dw^{'''}(\ell) + Pw^{'}(\ell) = 0. \tag{2.20}$$

2.2.1. Boundary conditions of anchorage. The problem of finding a critical force is reduced to the following isoperimetric type problem

$$\widetilde{J}(w) = \frac{1}{2}\int_0^\ell (Dw''^2 + Cw^2)dx \to \min \tag{2.21}$$

with the constraint

$$J_1(w) = \frac{1}{2}\int_0^\ell w'^2 dx = 1 \tag{2.22}$$

and the conditions (2.16) and (2.18).

The extremum problem (2.21), (2.22) with (2.16), (2.18) has a solution since the set of functions $w \in W_2^2[0, \ell]$ satisfying (2.16) and (2.22) is weakly compact and the functional $\widetilde{J}(w)$ is convex. It is known that a continuous convex functional attains the minimum on a weakly compact set. Here, $W_2^2[0, \ell]$ is the Sobolev space of functions that are square integrable, together with the first and second order derivatives, on $[0, \ell]$ (the first order derivative is absolutely continuous).

We look for a solution to the problem (2.21), (2.22), (2.16), (2.18) among functions that are strictly positive on an interval $(0, \ell_1)$, $0 < \ell_1 \leqslant \ell$, and vanish outside this interval [5].

Since $w > 0$ for $x \in (0, \ell_1)$, it satisfies the Euler equation on $(0, \ell_1)$:

$$w^{IV} + \omega w + \rho^2 w^{''} = 0, \tag{2.23}$$

where $\omega = C/D$, $\rho^2 = \lambda/D$, λ is the Lagrange multiplier for the constraint (2.22).

In this case, Equation (2.23) is the equilibrium equation for a rod compressed by a longitudnal force in an elastic medium. We note that (2.23) coincides with the equilibrium equation for a cylindrical shell compressed by a longitudnal force in the axisymmetric case. It is possible to show that for the existence of a nontrivial solution to Equation (2.23) with the boundary condition (2.18) or (2.19) the following condition is necessary:

$$\rho^2 \geqslant 2\sqrt{\omega}. \tag{2.24}$$

Under this condition, the general solution to Equation (2.23) has the form

$$w(x) = c_1 \sin(m_1 x) + c_2 \sin(m_2 x) + +c_3 \cos(m_1 x) + c_4 \cos(m_2 x), \tag{2.25}$$

where

$$m_1 = \sqrt{\frac{\rho^2}{2} + \sqrt{\frac{\rho^4}{4} - \omega}}, \quad m_2 = \sqrt{\frac{\rho^2}{2} - \sqrt{\frac{\rho^4}{4} - \omega}}. \tag{2.26}$$

Since $w(x) > 0$ for $x \in (0, \ell_1)$ and $w(x) \equiv 0$ for $x \in (\ell_1, \ell)$, we conclude that either ℓ_1 coincides with ℓ or ℓ_1 is found by solving the problem

$$J(w) = \frac{1}{2} \int_0^{\ell_1} (w''^2 + \omega w^2) dx \to \min_{w, \ell_1} \tag{2.27}$$

with the constraint

$$J_1(w) = \frac{1}{2} \int_0^{\ell_1} w'^2 dx = 1. \tag{2.28}$$

By the minimum condition on ℓ_1 in the problem (2.27), (2.28) and the conditions $w(\ell_1) = 0$, $w'(\ell_1) = 0$, from (2.27) we obtain the boundary condition $w''(\ell_1) = 0$.

Thus, the function $w(x)$ is twice continuously differentiable on the entire interval $[0, \ell]$ and satisfies the conditions

$$w(0) = w(\ell_1) = 0, \quad w'(0) = w'(\ell_1) = 0, \quad w''(\ell_1) = 0. \tag{2.29}$$

Substituting (2.25) into (2.18), we obtain the system of equations for arbitrary constants $c_1, \ldots, c_4$ and ℓ_1

$$\begin{aligned} & c_3 + c_4 = 0, \\ & m_1 c_1 + m_2 c_2 = 0, \\ & c_1 \sin y + c_2 \sin z + c_3 \cos y + c_4 \cos z = 0, \\ & c_1 m_1 \cos y + c_2 m_2 \cos z - c_3 m_1 \sin y - c_4 m_2 \sin z = 0, \\ & c_1 m_1^2 \sin y + c_2 m_2^2 \sin z + c_3 m_1^2 \cos y + c_4 m_2^2 \cos z = 0, \end{aligned} \tag{2.30}$$

where $y = m_1 \ell_1$ and $z = m_2 \ell_1$. Considering the first four equations for the unknowns $c_1, \ldots, c_4$ and equating to zero the determinant of the coefficient matrix, we see that for the existence of a nontrivial solution it is necessary that

$$2zy(1 - \cos z \cos y) - (z^2 + y^2) \sin z \sin y = 0. \tag{2.31}$$

Considering the 1st, 2nd, 3rd, and 5th equations of the system (2.30), we obtain the equation

$$z \cos z \sin y - y \sin z \cos y = 0. \tag{2.32}$$

With the minimal critical force it is associated the solution $y = 3\pi$, $z = \pi$ to Equations (2.31), (2.32), i.e., $3\pi = m_1 \ell_1$, $\pi = m_2 \ell_1$. Using (2.26), we find

$$\rho^2 = \frac{10}{3} \sqrt{\omega}, \quad \ell_1 = \frac{\sqrt{3}\pi}{\sqrt[4]{\omega}}. \tag{2.33}$$

If $\ell_1 < \ell$, then

$$w(x) = c \cdot \sin^3(m_2 x) H(\ell_1 - x), \quad x \in [0, \ell], \tag{2.34}$$

where $m_2 = \sqrt[4]{\omega}/\sqrt{3}$ and $H(t)$ is the Heaviside function.

2.2.2. Boundary condition of hinged support. In this case,

$$w(0) = w(\ell_1) = 0, \quad w''(0) = w''(\ell_1) = 0, \quad w'(\ell_1) = 0. \tag{2.35}$$

Then it is necessary to replace the second equation in (2.30) with the equation $m_1^2 c_3 + m_2^2 c_4 = 0$, which implies $c_3 = c_4 = 0$ in view of the first equation, and (2.30) is replaced with the system

$$\begin{aligned} &c_1 \sin y + c_2 \sin z = 0,\\ &c_1 m_1 \cos y + c_2 m_2 \cos z = 0,\\ &c_1 m_1^2 \sin y + c_2 m_2^2 \sin z = 0. \end{aligned} \tag{2.36}$$

For the existence of a nontrivial solution to the system (2.36) it is necessary that

$$\det \begin{pmatrix} \sin y & \sin z \\ m_1 \cos y & m_2 \cos z \end{pmatrix} = 0, \tag{2.37}$$

$$\det \begin{pmatrix} \sin y & \sin z \\ m_1^2 \sin y & m_2^2 \sin z \end{pmatrix} = 0, \tag{2.38}$$

which yields

$$\begin{aligned} &m_2 \cos z \sin y = m_1 \cos y \sin z,\\ &m_2^2 \sin y \sin z = m_1^2 \sin y \sin z. \end{aligned}$$

The second equation implies $\sin y = 0$ or $\sin z = 0$. If $\sin y = 0$, then from the first equation we find $\sin z = 0$ (otherwise, $\cos y = 0$, which is impossible). Hence

$$y = m_1 \ell_1 = \pi i, \quad z = m_2 \ell_1 = \pi j, \quad i, j = 1, 2, \ldots. \tag{2.39}$$

From (2.39) and the second equation in (2.36) we find

$$c_2 = -c_1 \beta \frac{m_1}{m_2}, \quad \beta = \begin{cases} 1, & (i-j) \text{ is even,} \\ -1, & (i-j) \text{ is odd.} \end{cases}$$

From (2.25) it follows that

$$w(x) = c_1 \Big(\sin m_1 x - \beta \frac{m_1}{m_2} \sin m_2 x \Big), \quad 0 \leqslant m_2 x \leqslant \pi j, \ j = 1, 2, \ldots. \tag{2.40}$$

Denote $\alpha = m_1 m_2^{-1} = i \cdot j^{-1}$. Then from (2.26) and

$$\rho^2 = \frac{1 + \alpha^2}{\alpha} \sqrt{\omega}$$

we obtain the value of the critical force. Choosing i and j such that $\rho^2 = \lambda/D$ is minimal and $w(x)$ in (2.40) is nonnegative, we find

$$\rho^2 = \frac{5}{2} \sqrt{\omega}, \quad \ell_1 = \frac{\sqrt{2}\pi}{\sqrt[4]{\omega}}.$$

If $\ell_1 < \ell$, then

$$w(x) = c\left(2\sin\frac{\pi x}{\ell_1} + \sin\frac{2\pi x}{\ell_1}\right) H(\ell_1 - x), \quad c > 0. \tag{2.41}$$

2.2.3. Free boundary condition. In the case of the boundary condition of anchorage at $x = 0$ and the free boundary condition at $x = \ell$, we have (2.20). Then the inequality (2.24) is replaced with $\rho^2 < 2\sqrt{\omega}$ and the general solution to Equation (2.23) takes the form

$$w(x) = c_1 e^{\alpha x}\sin(\beta x) + c_2 e^{\alpha x}\cos(\beta x) + c_3 e^{-\alpha x}\sin(\beta x) + c_4 e^{-\alpha x}\cos(\beta x), \tag{2.42}$$

where

$$\alpha = \frac{1}{2}\sqrt{2\sqrt{\omega} - \rho^2}, \quad \beta = \frac{1}{2}\sqrt{2\sqrt{\omega} + \rho^2}. \tag{2.43}$$

Assume that there exists a region that is closely adjacent to the wall, i.e.,

$$\begin{aligned} w(x) &= 0, \quad x \in [0, \ell_1], \\ w(x) &> 0, \quad x \in (\ell_1, \ell]. \end{aligned} \tag{2.44}$$

As above, $w = 0$, $w' = 0$, and $w'' = 0$ for $x = \ell_1$. Thus, we have two systems of equations

$$w(\ell_1) = 0, \quad w'(\ell_1) = 0, \quad w^{''}(\ell) = 0, \quad w^{'''}(\ell) + \rho^2 w^{'}(\ell) = 0, \quad \rho^2 = \frac{P}{D}, \tag{2.45}$$

$$w(\ell_1) = 0, \quad w'(\ell_1) = 0, \quad w^{''}(\ell) = 0, \quad w^{'''}(\ell) + \rho^2 w^{'}(\ell) = 0. \tag{2.46}$$

For the existence of a nontrivial solution it is necessary that the determinants of the coefficient matrices at $c_1, \ldots, c_4$ vanish. It is clear that we can put $\ell_1 = 0$ in Equations (2.45) and (2.46) (it suffices to replace x with $x - \ell_1$). Then ℓ is the unknown which should be found. We set $\widetilde{\ell} = \ell - \ell_1$. The determinant of the system (2.45) takes the form

$$\begin{aligned} \triangle_1(\omega, \widetilde{\ell}, \rho) = {} & \cos^2(\beta\widetilde{\ell})(\omega\rho^2 - \sqrt{\omega}\rho^4 + 2\sqrt{\omega^3}) + \frac{1}{2}e^{\alpha\widetilde{\ell}}\Big(\sqrt{\omega^3} - \frac{1}{2}\omega\rho^2 - \frac{1}{2}\sqrt{\omega}\rho^4\Big) \\ & + \frac{1}{2}e^{-\alpha\widetilde{\ell}}\Big(\sqrt{\omega^3} - \frac{1}{2}\omega\rho^2 - \frac{1}{2}\sqrt{\omega}\rho^4\Big) + \sqrt{\omega^3} - \frac{1}{2}\omega\rho^2 + \frac{1}{2}\sqrt{\omega}\rho^4, \end{aligned}$$

whereas the determinant of the system (2.46) is equal to

$$\begin{aligned} \triangle_2(\omega, \widetilde{\ell}, \rho) = {} & \frac{1}{2}\sin(\beta\widetilde{\ell})\beta(\rho^2\omega - \rho^4\sqrt{\omega} + 2\sqrt{\omega^3}) \\ & + \frac{1}{4}e^{\alpha\widetilde{\ell}}\alpha(\rho^2\omega + \rho^4\sqrt{\omega} - 2\sqrt{\omega^3}) - \frac{1}{4}e^{-\alpha\widetilde{\ell}}\alpha(\rho^2\omega + \rho^4\sqrt{\omega} - 2\sqrt{\omega^3}). \end{aligned}$$

The determinants $\triangle_1(\omega, \widetilde{\ell}, \rho)$ and $\triangle_2(\omega, \widetilde{\ell}, \rho)$ were computed with the help of the system MAPLE. Thus, for $\widetilde{\ell}$ and ρ^2 we have the system of two nonlinear equations

$$\triangle_1(\omega, \widetilde{\ell}, \rho) = 0, \quad \triangle_2(\omega, \widetilde{\ell}, \rho) = 0. \tag{2.47}$$

The system (2.47) was solved by the Newton method. The numerical results are given in Table 1.

TABLE 1. The values of the critical force depending on the rigidity ω.

N	1	2	3	4	5	6
ω	100	200	350	450	550	800
$\widetilde{\ell}$	0.745	0.627	0.545	0.512	0.487	0.443
ρ^2	12.6	17.8	23.5	26.7	29.5	35.6
ρ_*^2	11.9	15.6	19.5	21.8	23.8	28.5

In Table 1, the values of ρ^2 correspond to the critical load on the rod under one-sided constraints on displacements with different rigidity ω. The last row contains the values of $P = \rho_*^2$ in the case where no one-sided constraints are imposed on displacements.

2.3. Stability of annuli under one-sided constraints. We study the stability of an elastic annulus reinforced by elastic threads which do not react on compression loads. Suppose that one thread end is fixed at the center and the other is attached at some point of the annulus. Assume that the thread is inextensible, i.e., the distance between the annulus center and the attachment point does not increase under deformations. Denote by ϑ the central angle, by $w(\vartheta)$ the radial displacement (deflection), and by $v(\vartheta)$ the tangent displacement of annulus points. Since the annulus axis is incompressible, we have

$$v' = -w. \tag{2.48}$$

Furthermore, the threads are placed so frequently that we can assume that they are continuously distributed along the annulus. Then the stability problem is reduced to finding values of P such that the variational problem

$$J(w) = \frac{B}{2R^3}\int\limits_0^{2\pi}(w''+w)^2 d\vartheta - \frac{P}{2}\int\limits_0^{2\pi}(w'^2 - w^2)d\vartheta \to \min_w \tag{2.49}$$

has a nontrivial solution such that

$$w(\vartheta) \leqslant 0, \tag{2.50}$$

where B is the bending rigidity of the annulus plane and R is the annulus radius. The first integral in (2.49) is the elastic energy and the second one is the work of the normal pressure forces.

We write the Euler equation for the functional (2.49)

$$w^{IV} + (2+k^2)w'' + (1+k^2)w = 0, \tag{2.51}$$

where $k^2 = \dfrac{PR^3}{B}$. The corresponding characteristic equation

$$\lambda^4 + (2+k^2)\lambda^2 + (1+k^2) = 0$$

has a solution $\lambda_{1,2} = \pm i$, $\lambda_{3,4} = \pm\sqrt{1+k^2}\,i$. Then the bending function is written as

$$w = A_1 \sin\vartheta + A_2\cos\vartheta + A_3\sin\alpha\vartheta + A_4\cos\alpha\vartheta, \tag{2.52}$$

where $\alpha = \sqrt{1+k^2}$.

We fix an angle $\beta > 0$. As in the case of rods, we assume that $w(\vartheta) < 0$ for $\vartheta \in (0, \beta)$ and $w(\vartheta) \equiv 0$ for $\vartheta \in (\beta, 2\pi)$. The first order derivative $w'(\vartheta)$ is continuous for $\vartheta \in (0, 2\pi)$. Therefore, w satisfies the boundary conditions

$$w(0) = 0, \quad w'(0) = 0, \quad w(\beta) = 0, \quad w'(\beta) = 0. \tag{2.53}$$

Substituting (2.52) into (2.53), we obtain the system of linear equations

$$\begin{aligned} &A_2 + A_4 = 0, \\ &A_1 + \alpha A_3 = 0, \\ &A_1 \sin\beta + A_2 \cos\beta + A_3 \sin(\alpha\beta) + A_4 \cos(\alpha\beta) = 0, \\ &A_1 \cos\beta - A_2 \sin\beta + \alpha A_3 \cos(\alpha\beta) - \alpha A_4 \sin(\alpha\beta) = 0. \end{aligned} \tag{2.54}$$

Simplifying, we find

$$\begin{aligned} &A_3(\sin(\alpha\beta) - \alpha\sin\beta) + A_4(\cos(\alpha\beta) - \cos\beta) = 0, \\ &A_3(\alpha\cos(\alpha\beta) - \alpha\cos\beta) + A_4(\sin\beta - \alpha\sin(\alpha\beta)) = 0. \end{aligned} \tag{2.55}$$

The system of equations has a nontrivial solution provided that its determinant vanishes, i.e.,

$$d(\alpha) = -2\alpha\cos(\alpha\beta)\cos\beta + 2\alpha - \sin(\alpha\beta)\sin\beta - \alpha^2\sin(\alpha\beta)\sin\beta = 0. \tag{2.56}$$

Solving Equation (2.56) with respect to α, we obtain the function $\alpha = \alpha(\beta)$. For a given β the equation has infinitely many roots. It is obvious that $\alpha = 1$ is a root of the equation for any β. Note that $\alpha = 1$ corresponds to $P = 0$. Further, we find the bending shape by formula (2.52). It is easy to see that formula (2.52) with $\alpha = 1$ means that the annulus moves like a rigid body. Consequently, we need to find the minimal root of Equation (2.56) such that $\alpha > 1$. The condition (2.50) is also necessary. We see that the greater the angle β the less k^2 and, consequently, the force P. The values of the critical parameter α depending on β are presented in Table 2.

TABLE 2. The values of the critical parameter α depending on β.

β	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$
α	4.9801	4.2915	3.2136	3	2.4841

In the case $\beta > \pi$, numerical experiments show that the graph of w changes the sign on $(0, \beta)$, i.e., w does not satisfy the nonnegativity condition.

Thus, the minimal critical value of α is 3, which implies $k^2 = 8$, which corresponds to the equality $P = 8B/R^3$. Note that the critical value of P is equal to $3B/R^3$ for a nonreinforced annulus.

We consider arbitrarily located and absolutely rigid threads. Suppose that an annulus is reinforced by M absolutely rigid threads. One thread end is fixed at the annulus point A corresponding to the angle $\vartheta = \varepsilon_{1j}$ and the second end is fixed at the point B corresponding to the angle $\vartheta = \varepsilon_{2j}$, $j \in 1 : M$. Displacements of annulus points are denoted by $w_i = w(\varepsilon_{ij})$, $v_i = v(\varepsilon_{ij})$, $i = 1, 2$, $j \in 1 : M$.

We set $\alpha_j = \varepsilon_{2j} - \varepsilon_{1j}$. Denote by ϱ_j the distance between the points A and B before deformation. Let us compute the distance ϱ_j^* between A and B after deformation. We introduce the unit vectors $\mathbf{i}$, j and $\mathbf{i}'$, j' such that $\mathbf{i}$ is directed along the annulus radius at the point $\vartheta = \varepsilon_{1j}$, $\mathbf{j}$ is obtained by the anticlockwise rotation of $\mathbf{i}$ by the angle $\pi/2$, $\mathbf{i}'$ is directed along the annulus radius at the point $\vartheta = \varepsilon_{2j}$, and $\mathbf{j}'$ is obtained by the anticlockwise rotation of $\mathbf{i}'$ by the angle $\pi/2$. The angle between $\mathbf{i}$ and $\mathbf{i}'$ is equal to α_j. The matrix of transition from ($\mathbf{i}$,$\mathbf{j}$) to ($\mathbf{i}'$,$\mathbf{j}'$) has the form

$$\begin{pmatrix} \cos\alpha_j - \sin\alpha_j \\ \sin\alpha_j \cos\alpha_j \end{pmatrix}.$$

The displacement vector at $\vartheta = \varepsilon_{2j}$ can be written as

$$w_2\mathbf{i}' + v_2\mathbf{j}' = (w_2\cos\alpha_j - v_2\sin\alpha_j)\mathbf{i} + (w_2\sin\alpha_j + v_2\cos\alpha_j)\mathbf{j},$$

whereas the distance between the points A and B before deformation is $\varrho_j = 2R\sin\dfrac{\alpha_j}{2}$. Then the distance between these points after deformation is equal to

$$\varrho_j^* = \Big[\Big(-\varrho_j\sin\frac{\alpha_j}{2} + w_2\cos\alpha_j - v_2\sin\alpha_j - w_1\Big)^2 + + \Big(\varrho_j\cos\frac{\alpha_j}{2} + w_2\sin\alpha_j + v_2\cos\alpha_j - v_1\Big)^2\Big]^{1/2}$$

or (after suitable transformations)

$$\begin{aligned}\varrho_j^* = \Big[&\varrho_j^2 + 2\varrho_j(w_2 + w_1)\sin\frac{\alpha_j}{2} + (v_2 - v_1)\cos\frac{\alpha_j}{2} \\ &+ (w_2\cos\alpha_j - v_2\sin\alpha_j - w_1)^2 + (w_2\sin\alpha_j + v_2\cos\alpha_j - v_1)^2\Big]^{1/2}.\end{aligned}$$

Under small deformations of the annulus, we can approximately set

$$\varrho_j^* = \varrho_j + (w_2 + w_1)\sin\frac{\alpha_j}{2} + (v_2 - v_1)\cos\frac{\alpha_j}{2}. \tag{2.57}$$

Denote by $\Delta\varrho_j = \varrho_j^* - \varrho_j$ the lengthening of the jth thread.

Using (2.57), we formulate the stability problem: Find the minimal value of P such that the variational problem

$$V = \frac{B}{2R^3}\int_0^{2\pi}(w'' + w)^2 d\vartheta + \frac{P}{2}\int_0^{2\pi} w'(v - w')d\vartheta \to \min_{w,v} \tag{2.58}$$

with the condition

$$\Delta\varrho_j = (w_2 + w_1)\sin\frac{\alpha_j}{2} + (v_2 - v_1)\cos\frac{\alpha_j}{2} \leqslant 0, \quad j \in 1 : M, \tag{2.59}$$

has a nontrivial solution.

The first integral in (2.58) is the elastic energy of the deformed annulus, and the second integral is the work of the normal pressure force. We look for $v(\vartheta)$ in (2.58), (2.59) as the partial sum of the Fourier series

$$v(\vartheta) = \sum_{k=2}^{n}(x_k\sin k\vartheta + y_k\cos k\vartheta) \tag{2.60}$$

(the harmonics with $k = 0$ and $k = 1$ correspond to displacements of the annulus regarded as a rigid body). Taking into account the incompressibility condition (2.48), we obtain, instead of (2.58), (2.59), the problem

$$\widetilde{V} = \frac{1}{2}\sum_{k=2}^{n} k^2(k^2-1)\left[\frac{B}{R^3}(k^2-1) - P\right](x_k^2 + y_k^2) \to \min, \tag{2.61}$$

$$\Delta\widetilde{\varrho}_j = \sum_{k=2}^{n}(c_{kj}x_k + d_{kj}y_k) \leqslant 0, \quad j \in 1:M, \tag{2.62}$$

where

$$c_{kj} = \cos\frac{\alpha_j}{2}(\sin k\varepsilon_{2j} - \sin k\varepsilon_{1j}) - \sin\frac{\alpha_j}{2}(k\cos k\varepsilon_{2j} + k\cos k\varepsilon_{1j}),$$
$$d_{kj} = \cos\frac{\alpha_j}{2}(\cos k\varepsilon_{2j} - \cos k\varepsilon_{1j}) + \sin\frac{\alpha_j}{2}(k\sin k\varepsilon_{2j} + k\sin k\varepsilon_{1j}).$$

In the case where the threads are located along the sides of a regular M-gon ($\alpha_j = 2\pi/M$, $\varepsilon_{1j} = \alpha_j \cdot (j-1)$, $\varepsilon_{2j} = \alpha_j \cdot j$), for solving the problem (2.61), (2.62) the method of branches and boundaries was used.

TABLE 3. The values of the critical pressure depending on the number of threads.

M	3	4	5	6	7	8	9	10
P^*R^3/B	4.32	3.00	4.57	5.27	6.28	6.50	7.26	7.37

The results of numerical experiments are presented in Table 3. If the annulus is reinforced by threads fixed at the vertices of the square ($M = 4$), then the value of the critical force coincides with the critical pressure for the annulus without reinforcements ($P = 3B/R^3$), i.e., in this case, the annulus is transformed to an ellipse without increasing the distance between points at the square vertices.

2.4. Stability of rectangular plates. Suppose that a rectangular plate is loaded by the normal force σ for $x = 0$, $x = a$, $0 \leqslant y \leqslant b$, and by the tangent force τ on the all boundaries. Denote by $w(x, y)$, $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$, the plate deflection. The potential strain energy of the plate has the form [6]

$$U(w) = \frac{D}{2}\int_0^a\int_0^b \big((\triangle w)^2 - (1-\nu)L(w,w)\big)\, dxdy, \tag{2.63}$$

where

$$\triangle w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}, \quad L(w,w) = 2\left(\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right).$$

The work of the external force is expressed by [6]

$$V(w) = \frac{1}{2}\int_0^a\int_0^b \left(\sigma\left(\frac{\partial w}{\partial x}\right)^2 + 2\tau\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) dxdy. \tag{2.64}$$

We assume that the plate is located over two rigid edges along the x-axis in such a way that

$$w(x, b_0) \geqslant 0, \quad w(x, b_1) \geqslant 0, \tag{2.65}$$

where $0 < b_0 < b_1 < b$. The study of the plate stability is reduced to finding the forces σ and τ such that the variational problem

$$U - V \to \min_{w \in \mathscr{P}} \tag{2.66}$$

has a nontrivial solution. Here, $\mathscr{P}$ is the set of functions $w(x, y)$ satisfying (2.65). Assume that the boundary conditions of hinged support

$$w(x, 0) = w(x, b) = 0, \quad \frac{\partial^2 w(x, 0)}{\partial y^2} = \frac{\partial^2 w(x, b)}{\partial y^2} = 0, \tag{2.67}$$

$$w(0, y) = w(a, y) = 0, \quad \frac{\partial^2 w(0, y)}{\partial x^2} = \frac{\partial^2 w(a, y)}{\partial x^2} = 0 \tag{2.68}$$

or the boundary conditions of anchorage

$$w(x, 0) = w(x, b) = 0, \quad \frac{\partial w(x, 0)}{\partial y} = \frac{\partial w(x, b)}{\partial y} = 0, \tag{2.69}$$

$$w(0, y) = w(a, y) = 0, \quad \frac{\partial w(0, y)}{\partial x} = \frac{\partial w(a, y)}{\partial x} = 0 \tag{2.70}$$

hold. Then the expression for the plate energy is simplified because, in this case,

$$\int_0^a \int_0^b L(w, w)\, dxdy = 0.$$

The critical loads σ^* and τ^* can be found as follows. We fix σ and τ and solve the problem

$$U = \frac{D}{2} \int_0^a \int_0^b (\triangle w)^2 \, dxdy \to \min_{w \in \mathscr{P}} \tag{2.71}$$

with the constraint

$$V = \frac{1}{2} \int_0^a \int_0^b \left(\sigma \left(\frac{\partial w}{\partial x} \right)^2 + 2\tau \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) dxdy = 1. \tag{2.72}$$

Let w^* be a solution to the problem (2.71), (2.72), and let $\lambda = U(w^*)$. Then the critical loads are computed by

$$\sigma^* = \lambda \sigma, \quad \tau^* = \lambda \tau. \tag{2.73}$$

In the domain $[0, a] \times [0, b]$, we introduce the uniform grid $T = T_x \times T_y$ and denote by N and K the number of grid points along the x– and y–axes respectively. In the case of arbitrary boundary conditions for the finite-dimensional approximations of the problem (2.63)–(2.65), the plate deflection $w(x, y)$ will be approximated by interpolation cubic splines in two variables.

Let $w_{ij} = w(x_i, y_j)$, $i \in 0 : N$, $j \in 0 : K$, be the values of w at grid points. Replacing $w(x, y)$ with an interpolation cubic spline and computing the integrals (2.63) and (2.64), we obtain the potential energy $f(w_{ij})$ and the work of the external forces $g(w_{ij})$. The functions $f(w_{ij})$ and $g(w_{ij})$, are quadratic forms of the variables w_{ij}. Thus, instead of the problem (2.71), we obtain the problem of nonlinear programming (2.7)–(2.9). Table 4 contains the numerical results for $a = 1$, $b = 0.2$, $N = 20$, $K = 10$, $D = 1$. The 3rd row contains the values of λ for the plate that is hinged around the contour without any restrictions on the plate deflection. The 4th row contains the values of λ_* under the restrictions on deflection given by two rigid edges ($b_0 = b/3$ and $b_1 = 2b/3$). The 6th and 7th rows present the corresponding results for a plate which is rigidly fixed over all the edges.

TABLE 4.

τ	1	1	0	1	2	3	4	5
σ	3	2	1	1	1	1	1	1
λ	313.0	444.6	988.4	723.6	492.1	366.1	290.3	240.1
λ_*	357.4	515.7	1107	871.0	599.6	443.2	351.0	288.7
λ	551.0	775.7	1757	1237	827	611.5	482.5	399.2
λ^*	750.2	1049	2384	1640	1079	790.5	623.6	513.7

2.5. Stability of a torus-like shell with one-sided reinforcement.

2.5.1. Definition of the elastic energy and external force work. Assume that a shell with a median surface S takes the shape $\widetilde{S}$ under a deformation. We denote by g_{ij}, h_{ij}, $\widetilde{g}_{ij}$, $\widetilde{h}_{ij}$, $i, j = 1, 2$, the coefficients of the first and second quadratic forms of nondeformed and deformed surfaces respectively.

We assume that a deformation is axisymmetric. By [7], the strain energy caused by transition from a state S to a state $\widetilde{S}$ is expressed as

$$U = \iint \Phi_1(\varepsilon_1, \varepsilon_2, \varkappa_1, \varkappa_2)ds, \tag{2.74}$$

where

$$\Phi_1 = \frac{Eh^3}{24(1-\nu^2)}(\varkappa_1^2 + \varkappa_2^2 + 2\nu\varkappa_1\varkappa_2) + \frac{Eh}{2(1-\nu^2)}(\varepsilon_1^2 + \varepsilon_2^2 + 2\nu\varepsilon_1\varepsilon_2), \tag{2.75}$$

E is the Young modulus, ν is the Poisson ratio, ε_1 and ε_2 are the extremum values of the ratio

$$\frac{\sum\limits_{i,j=1}^{2}(\widetilde{g}_{ij} - g_{ij})du_i du_j}{\sum\limits_{i,j=1}^{2} g_{ij} du_i du_j}, \tag{2.76}$$

whereas $\varkappa_1$ and $\varkappa_2$ are the extremum values of the ratio

$$\frac{\sum\limits_{i,j=1}^{2}(\widetilde{h}_{ij} - h_{ij})du_i du_j}{\sum\limits_{i,j=1}^{2} g_{ij} du_i du_j}. \tag{2.77}$$

As a result, under an axisymmetric deformation, the surface $\widetilde{S}$ becomes a surface of revolution of the curve γ about the z-axis in the XOZ-plane, given by the equations $x = \varphi(\theta)$, $z = \psi(\theta)$. The point $(\varphi(\theta),\ 0,\ \psi(\theta))$ of the curve γ goes to the point $(\varphi(\theta)\cos\lambda,\ \varphi(\theta)\sin\lambda,\ \psi(\theta))$ under rotation by the angle λ. Hence the equations of the surface of revolution take the form [8]

$$\begin{aligned} x &= \varphi(\theta)\cos\lambda, \\ y &= \varphi(\theta)\sin\lambda, \\ z &= \psi(\theta). \end{aligned} \tag{2.78}$$

We consider a torus under the action of a normal pressure force. Then, in (2.76) and (2.77), $u_1 = \theta$ is the polar angle in the meridian plane and $u_2 = \lambda$ is the angle in the plane of parallel disc. Denote by $w(\theta)$ and $u(\theta)$ the normal and tangent displacements of points on the torus surface. The Cartesian coordinates of points on the torus-like surface before deformation are as follows:

$$\begin{aligned} x &= (R + a\cos\theta)\cos\lambda, \\ y &= (R + a\cos\theta)\sin\lambda, \quad 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant \lambda \leqslant 2\pi, \\ z &= a\sin\theta, \end{aligned} \tag{2.79}$$

i.e., $\varphi = R + a\cos\theta$, $\psi = a\sin\theta$ in the case of an undeformed torus. After deformation, the equations take the form (2.78), where

$$\begin{aligned} \varphi(\theta) &= R + (a + w(\theta))\cos\theta - u(\theta)\sin\theta, \\ \psi(\theta) &= (a + w(\theta))\sin\theta - u(\theta) cos \theta. \end{aligned} \tag{2.80}$$

We study the case where the axisymmetric form loses stability in small when the obtained buckles have the form of annulus folds directed along the λ-coordinate (displacements are independent of λ). For surfaces of revolution the first and second quadratic forms are written as [8]

$$\begin{aligned} I &= \left(\varphi'^2 + \psi'^2\right) d\theta^2 + \varphi^2 d\lambda^2, \\ II &= \left(\frac{\psi''\varphi' - \psi'\varphi''}{\sqrt{\varphi'^2 + \psi'^2}}\right) d\theta^2 + \frac{\psi'\varphi}{\sqrt{\varphi'^2 + \psi'^2}} d\lambda^2. \end{aligned} \tag{2.81}$$

For the undeformed surface we have

$$\begin{aligned} I_0 &= a^2 d\theta^2 + (R + a\cos\theta)^2 d\lambda^2, \\ II_0 &= a d\theta^2 + \cos\theta(R + a\cos\theta) d\lambda^2. \end{aligned} \tag{2.82}$$

Using (2.76), (2.77), (2.79), (2.81), and (2.82), it is possible to express ε_1, ε_2 and $\varkappa_1$, $\varkappa_2$. Under an axisymmetric deformation, the quadratic forms I and II take the diagonal form. Therefore,

$$\begin{aligned} \varepsilon_1 &= \frac{\varphi'^2 + \psi'^2 - a^2}{a^2}, \\ \varepsilon_2 &= \frac{\varphi^2 - (R + a\cos\theta)^2}{(R + a\cos\theta)^2}, \\ \varkappa_1 &= \frac{\psi''\varphi' - \psi'\varphi''}{a^2\sqrt{\varphi'^2 + \psi'^2}} - \frac{1}{a}, \\ \varkappa_2 &= \frac{\psi'\varphi - \cos\theta(R + a\cos\theta)\sqrt{\varphi'^2 + \psi'^2}}{\cos^2\theta(R + a\cos\theta)^2\sqrt{\varphi'^2 + \psi'^2}}. \end{aligned} \tag{2.83}$$

By the Euler–Bernoulli theorem, in the case of the external normal pressure force, the external force work is equal to $A = P\Delta V$, where ΔV is the change of shell volume under deformation. It is known [9] that the volume of a body with surface $x = x(\theta,\lambda)$, $y = y(\theta,\lambda)$, $z = z(\theta,\lambda)$ is found (up to the sign) by

$$V = \frac{1}{3}\int_0^{2\pi}\int_0^{2\pi} \det\begin{bmatrix} x & y & z \\ x_\theta^{'} & y_\theta^{'} & z_\theta^{'} \\ x_\lambda^{'} & y_\lambda^{'} & z_\lambda^{'} \end{bmatrix} d\theta d\lambda\,. \tag{2.84}$$

In the case of an axisymmetric deformation, the determinant in (2.84) is independent of λ.

Using (2.78)–(2.80), it is possible to compute the shell volume under deformation by

$$V = \frac{2\pi}{3}\int_0^{2\pi} \Phi_2\left(w, u, w^{'}, u^{'}\right) d\theta, \tag{2.85}$$

where

$$\begin{aligned}
a_{11} &= R + a\cos\theta + w(\theta)\cos\theta - u(\theta)\sin\theta,\\
a_{13} &= a\sin\theta + w(\theta)\sin\theta + u(\theta)\cos\theta,\\
a_{21} &= -a\sin\theta + w^{'}(\theta)\cos\theta - w(\theta)\sin\theta - u^{'}(\theta)\sin\theta - u(\theta)\cos\theta,\\
a_{23} &= a\cos\theta + w^{'}(\theta)\sin\theta + w(\theta)\cos\theta + u^{'}(\theta)\cos\theta - u(\theta)\sin\theta,\\
a_{32} &= R + a\cos\theta + w(\theta)\cos\theta - u(\theta)\sin\theta,\\
a_{32} &= R + a\cos\theta + w(\theta)\cos\theta - u(\theta)\sin\theta,\\
a_{12} &= 0, \quad a_{22} = 0, \quad a_{31} = 0, \quad a_{33} = 0.
\end{aligned}$$

Assume that, inside the shell, there is an elastic filler serving as an elastic base with rigidity C. Then the full strain energy takes the form $J = J_1 + J_2 - PJ_3$, where

$$\begin{aligned}
J_1 &= 2\pi\int_0^{2\pi} \Phi_1(\varepsilon_1, \varepsilon_2, \varkappa_1, \varkappa_2)a(R + a\cos\theta)d\theta,\\
J_2 &= 2\pi\int_0^{2\pi} \frac{C}{2}w^2 a(R + a\cos\theta)d\theta,\\
J_3 &= \Delta V.
\end{aligned} \tag{2.86}$$

In the stable equilibrium state, the full energy takes the minimal value. Thus, we arrive at the variational problem

$$J \to \min_{w,u}, \tag{2.87}$$

where the functions w and u satisfy the periodicity condition.

2.5.2. Numerical method. The displacements $w(\theta)$ and $u(\theta)$ will be approximated by interpolation splines. Denote

$$w_i = w(\theta_i), \quad v_i = v(\theta_i), \quad i \in [0..n-1], \quad \theta_i = \frac{2\pi i}{n} \tag{2.88}$$

and introduce the vector $z \in R^{2n}$: $z_1 = w_0$, $z_2 = w_1, \ldots, z_n = w_{n-1}$, $z_{n+1} = u_0, z_{n+2} = u_1, \ldots, z_{2n} = u_{n-1}$. Thus, using finite-dimensional approximation of J_1, J_2, J_3, we obtain the functions $f_1(z)$, $f_2(z)$, $f_3(z)$, $f(z;P) = f_1(z) + f_2(z) - Pf_3(z)$ respectively.

The necessary extremum condition is written as

$$\frac{\partial f\,(z,P)}{\partial z} = 0. \tag{2.89}$$

To solve the stability problem for a shell, it is required to find the minimal value of P such that the solution to the system (2.89) bifurcates. The necessary bifurcation condition is the singularity of the matrix of second order derivatives, i.e.,

$$\det\left[\frac{\partial^2 f\,(z,P)}{\partial z^2}\right] = 0. \tag{2.90}$$

We introduce the matrices

$$Q = \frac{\partial^2 f_1\,(0)}{\partial z^2}, \quad Q_2 = \frac{\partial^2 f_2\,(0)}{\partial z^2}, \quad G = \frac{\partial^2 f_3\,(0)}{\partial z^2}.$$

By (2.90), the system of equations

$$Qz + Q_2 z = \mu G z \tag{2.91}$$

has a nontrivial solution, where $\mu = P$ is a generalized eigenvalue. Note that

$$f_2(z) = \frac{1}{2}\left(Gz, z\right).$$

The problem about a generalized eigenvalue of the system (2.91) can be formulated as the extremum problem

$$\chi(z) = \frac{1}{2}\left(Qz, z\right) + \frac{1}{2}\left(Q_2 z, z\right) \to \min \tag{2.92}$$

with the constraint

$$\xi(z) = \frac{1}{2}\left(Qz, z\right) = 1. \tag{2.93}$$

Assume that the shell can deflect from the filler for $w > 0$, i.e., the reaction force of the filler has the form

$$Cw_- = C\min\{0, w\} = -\frac{C}{2}\left(|w| - w\right), \tag{2.94}$$

whereas the energy connected with the elastic filler is computed by

$$\Psi_1 = \pi \int\limits_0^{2\pi} C(w_-)^2 a(R + a\cos\theta)d\theta. \tag{2.95}$$

We introduce a 2π-periodic function $v(\theta)$ and the functional

$$\Psi_2 = 2\pi \int\limits_0^{2\pi} \frac{C}{2} v^2(\theta) a(R + a\cos\theta)d\theta.$$

We consider the extremum problem

$$\widetilde{J} = J_1 + \Psi_2 - PJ_3 \to \min_{w,u,v} \tag{2.96}$$

with the conditions

$$\begin{aligned} &v(\theta) - w(\theta) \leqslant 0, \\ &v(\theta) \leqslant 0. \end{aligned} \tag{2.97}$$

The problem (2.96), (2.97) is equivalent to minimizing the functional

$$\Psi = J_1 + \Psi_1 - PJ_3. \tag{2.98}$$

Under finite-dimensional approximation of Ψ_1, we have $\widetilde{f}_2(z)$ instead of $f_2(z)$. Therefore, Equation (2.91) fails because the matrix Q_2 does not exist. Note that $\widetilde{f}_2(z)$ is a positive homogeneous function, i.e., $\widetilde{f}_2(\alpha z) = \alpha^2 \widetilde{f}_2(z)$ for any $\alpha > 0$. Therefore, instead of the problem (2.92)–(2.93), we have the problem of minimizing

$$\Phi(z) = \frac{1}{2}(Qz, z) + \widetilde{f}_2(z) \to \min \tag{2.99}$$

with the constraint (2.93). The function $\Phi(z)$ is continuously differentiable, but does not possess continuous second order partial derivatives.

For (2.93)–(2.99) the method of successive approximations (2.11)–(2.13) can be used.

2.5.3. Results of numerical experiments. Based on the linear theory of thin shells, a formula for the critical normal pressure for a torus-like shell is given in [6]:

$$q = \frac{\psi E h}{a(1-\nu^2)}, \tag{2.100}$$

where

$$\begin{aligned} \psi = {} & \frac{4k^2\left(n^2 + \frac{1-\nu^2}{2} n^2 k^2 + (1+\nu)^2 k^2 + (1+\nu)\right)}{(4+k^2)\left(n^4\left(2+k^2\right) + (1+\nu)\, k^2 n^2\right)} \\ & + \frac{2h^2}{3a^2\left(4+k^2\right)} \frac{\left(n^2 - 1 + \frac{n^2k^2}{2}\right)\left(n^2\left(1+\frac{k^2}{2}\right) + k^2\right)}{n^2\left(2+k^2\right) + (1+\nu)\, k^2} + \frac{h^2k^2}{6a^2\left(4+k^2\right)}, \end{aligned}$$

$k = a/R$, $n = 1, 2, 3\ldots$, and n is chosen from the minimum condition for q in (2.100).

Table 5 contains the values of $\widetilde{q} = \frac{h^3}{12(1-\nu^2)} P$ and the parameter q computed by formula (2.100). The numerical results agree with the theoretical results.

TABLE 5. Comparison of results with a given critical pressure.

h	0.346	0.346	0.346	0.489	0.489	0.489
R	20	15	15	10	15	20
a	5	5	4	2.5	5	5
$\widetilde{q}$	0.012	0.017	0.021	0.036	0.014	0.022
q	0.012	0.012	0.019	0.037	0.017	0.017

TABLE 6. The values of the critical pressure for a torus-like shell with an elastic filler inside the shell.

h	0.346	0.346	0.346	0.346	0.489	0.489
R	20	20	15	15	20	20
a	5	5	5	5	5	5
C	1	3	1	3	1	3
q^*	0.017	0.026	0.022	0.030	0.034	0.058
q_*	0.015	0.021	0.021	0.026	0.029	0.041

In Table 6, C is the filler rigidity, q^* is the value of the critical parameter in the case where the shell and elastic filler are rigidly connected (cf. (2.86)), and q_* is the value of the critical parameter in the case where the shell can deflect from the filler (cf. (2.95)).

Thus, a contact between shell and filler should be taken into account if we are interested in the study of stability. Note that stability problems for elastic systems with one-sided constraints were considered in [5, 4, 10].

References

1. E. I. Mikhajlovskii and V. N. Tarasov, "The convergence of the method of generalized reaction in contact problems with a free boundary" [in Russian], *Prikl. Mat. Mekh.* **57**, No. 1, 128–136 (1993); English transl.: *J. Appl. Math. Mech.* **57**, No. 1, 147–157 (1993).

2. I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North Holland, Amsterdam (1976).

3. V. F. Dem'yanov and A. M. Rubinov, *Approximation Methods for Solving Extremum Problems* [in Russian], Leningr. Univ. Press, Leningr. (1968).

4. V. N. Tarasov, *Optimization Methods in the Study of Constructive-Nonlinear Problems of Mechanics of Elastic Systems* [in Russian], *Komi Sci. Center Ural Branch RAS*, Syktyvkar (2013).

5. V. N. Tarasov, "On stability of elastic systems under one-sided constraints on displacements" [in Russian], *Trudy IMM, Ekaterinburg* **11**, No. 1, 177–188 (2005).

6. A. S. Volmir, *Stability of Defomrable Systems* [in Russian], Nauka, Moscow (1967).

7. A. V. Pogorelov, *Differential Geometry* [in Russian], Nauka, Moscow (1974).

8. A. V. Pogorelov, *Geometric Theory of Stability of Shells* [in Russian], Nauka, Moscow (1966).

9. G. M. Fikhtengol'ts, *Course of Differential and Integral Calculus. Vol. 3* [in Russian], Nauka, Moscow (1970).

10. V. Yu. Andryukova and V. N. Tarasov, "Stability of elastic systems with nonretaining connections" [in Russian], *Izv. KOMI Nauchn. Tsentr* **3**, 12–19 (2013).

Submitted on July 11, 2014