DOI 10.1007/s10958-015-2246-3
Journal of Mathematical Sciences, Vol. 205, No. 2, February, 2015

TEMPERATURES OF PHASE TRANSITIONS AND QUASICONVEX HULL OF ENERGY FUNCTIONALS FOR A TWO-PHASE ELASTIC MEDIUM WITH ANISOTROPIC RESIDUAL STRAIN TENSOR

V. G. Osmolovskii
St. Petersburg State University
28, Universitetskii pr., Petrodvorets, St. Petersburg 198504, Russia
vicos@VO8167.spb.edu UDC 517.9

For a two-phase elastic medium with anisotropic residual strain tensors we compute the phase transition temperatures $t_\pm$. We find an explicit expression for the quasiconvex hull of strain energy densities and obtain all solutions to the relaxed variational problem and limit points of equilibrium states as the surface tension coefficient tends to zero. We show that there are no equilibrium states for the initial energy functional if $t \in (t_-, t_+)$. Bibliography: 18 *titles.*

1 Introduction

In quadratic approximation, the strain energy density in an elastic medium occupying a bounded domain $\Omega \subset R^m$, $m \geqslant 2$, is defined by (cf. [1])

$$F(\nabla u) = a_{ijkl}(e(\nabla u) - \zeta)_{ij}(e(\nabla u) - \zeta)_{kl}, \tag{1.1}$$

where the displacement field $u = u(x)$, $x \in \Omega$, is an m-dimensional vector-valued function, $(\nabla u)_{ij} = u^i_{x_j}$, $e(\nabla u)$ is the strain tensor with components $e_{ij}(\nabla u) = 1/2(u^i_{x_j} + u^j_{x_i})$, and the symmetric matrix ζ is interpreted as the residual strain. The elasticity modulus tensor a_{ijkl} satisfies the symmetry and positive definiteness conditions

$$\begin{gathered} a_{ijkl} = a_{jikl} = a_{klij} = a_{ijlk}, \quad a_{ijkl}\xi_{ij}\xi_{kl} \geqslant \nu \xi_{ij}\xi_{ij} \\ \text{for all symmetric matrices } \xi \text{ and some } \nu > 0. \end{gathered} \tag{1.2}$$

We assume summation with respect to repeated indices from 1 to m.

There are elastic media where the crystal structure can be modified under the action of temperature change and internal stresses [2]. If only two different crystal structures (two phases labeled by the subscripts + and −) are possible, then for each of them the strain energy density is defined by

$$F^\pm(\nabla u) = a^\pm_{ijkl}(e(\nabla u) - \zeta^\pm)_{ij}(e(\nabla u) - \zeta^\pm)_{kl}. \tag{1.3}$$

Translated from *Problemy Matematicheskogo Analiza* **77**, December 2014, pp. 119-128.

In (1.3), $a^{\pm}_{ijkl}$ and $\zeta^{\pm}_{ij}$ possess the same properties as in the one-phase case (1.1).

The strain energy functional for a two-phase elastic medium is given by

$$I_0[u,\chi,t] = \int\limits_{\Omega} \{\chi(F^+(\nabla u)+t)+(1-\chi)F^-(\nabla u)\}\,dx. \tag{1.4}$$

The parameter t, interpreted as the temperature, is responsible for the rise of one "potential hole" $F^+(\nabla u)$ over the other $F^-(\nabla u)$. The characteristic function $\chi(x)$ describes the location of the phase labeled by + in the domain Ω.

For the domain of the functional (1.4) we take the set of admissible displacement fields $\mathbb{X}$ and the set of admissible phase distributions $\mathbb{Z}'$ by setting

$$\begin{aligned}&\mathbb{X} = \overset{\circ}{W}{}^1_2(\Omega, R^m),\\ &\mathbb{Z}' \text{ is the set of all measurable characteristic functions.}\end{aligned} \tag{1.5}$$

The choice (1.5) of the set $\mathbb{X}$ means that we deal with only displacement fields that vanish, in a certain sense, on $\partial\Omega$ and are continuous through the phase interface boundary.

By an *equilibrium state* of a two-phase elastic medium with a temperature t we mean the solution to the variational problem

$$I_0[\widehat{u}_t,\widehat{\chi}_t,t] = \inf_{u\in\mathbb{X},\chi\in\mathbb{Z}'} I_0[u,\chi,t], \quad \widehat{u}_t\in\mathbb{X}, \quad \widehat{\chi}_t\in\mathbb{Z}'. \tag{1.6}$$

An equilibrium state is said to be *one-phase* if $\widehat{\chi}_t\equiv 0$ or $\widehat{\chi}_t\equiv 1$ and *two-phase* in the opposite case. It is obvious that $\widehat{u}_t\equiv 0$ in the one-phase equilibrium state. We emphasize that the unknowns of the problem (1.6) is an equilibrium displacement field $\widehat{u}_t$ and an equilibrium phase distribution $\widehat{\chi}_t$. Therefore, the problem (1.6) is related to the class of problems with a free surface. We set

$$\begin{aligned}&F^{\min}(M,t) = \min\{F^+(M)+t, F^-(M)\},\\ &M \text{ is a matrix in } R^m.\end{aligned} \tag{1.7}$$

Minimizing (1.6) with respect to $\chi\in\mathbb{Z}'$, we obtain the equivalent problem for the equilibrium displacement field

$$\begin{aligned}&I_0^{\min}[\widehat{u}_t,t] = \inf_{u\in\mathbb{X}} I_0^{\min}[u,t], \quad \widehat{u}_t\in\mathbb{X},\\ &I_0^{\min}[u,t] = \int\limits_{\Omega} F^{\min}(\nabla u,t)\,dx,\end{aligned} \tag{1.8}$$

from which we can restore $\widehat{\chi}_t$ by the rule

$$\begin{aligned}&\widehat{\chi}_t(x) = \begin{cases} 1, & \Phi(x,t)<0,\\ 0, & \Phi(x,t)>0,\\ \text{arbitrary characteristic} & \\ \text{function of } x\in\Omega, & \Phi(x,t)=0,\end{cases}\\ &\Phi(x,t) = F^+(\nabla\widehat{u}_t(x)) - F^-(\nabla\widehat{u}_t(x)) + t.\end{aligned} \tag{1.9}$$

Since for $F^+(M)\not\equiv F^-(M)$ the function (1.7) is not necessarily convex with respect to the first variable for all temperatures, the problem (1.6) is related to the class of nonconvex problems in Calculus of variations. In our situation (free surface and nonconvexity), the problem (1.6) is not necessarily solvable for any temperatures and residual strain tensors (cf. [3]).

Regarding the solvability of the problem (1.6), we mention the results of [4, 5] concerning the existence of phase transition temperatures $t_- \leqslant t_+$, independent of Ω and such that

- for $t < t_-$ a unique solution to the problem (1.6) is the one-phase equilibrium state $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 1$,
- for $t > t_+$ a unique solution to the problem (1.6) is the one-phase equilibrium state $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 0$,
- for $t \in (t_-, t_+) \neq \varnothing$ the problem (1.6) has no one-phase equilibrium states,
- $t_- \leqslant t^* \leqslant t_+$, $t^* = -(a^+_{ijkl}\zeta^+_{ij}\zeta^+_{kl} - a^-_{ijkl}\zeta^-_{ij}\zeta^-_{kl})$, where equalities hold simultaneously.

In [4, 5], two-sided estimates of $t_\pm$ were obtained, which allows us to derive sufficient conditions for the coincidence ($t_- = t_+$) or difference ($t_- < t_+$) of the temperatures. By properties of temperatures, if for some $t \in (t_-, t_+)$ the problem (1.6) is solvable, then the solution is two-phase. From the physical point of view, it is reasonable to expect the existence of phase transition temperatures $t_\pm$ for the functional (1.4).

Since the solvability of the problem (1.6) is not guaranteed for $t \in (t_-, t_+) \neq \varnothing$, we need to modify the problem by introducing the notion of a weak solution. We discuss two approaches. The first approach is based on considering a relaxed problem instead of (1.8), and the second approach takes into account the surface energy of the phase interface boundary.

We construct the quasiconvex hull $\mathscr{F}(M,t)$ of (1.7) by (cf. [6]–[8])

$$|\Omega|\mathscr{F}(M,t) = \inf_{u\in\mathbb{X}} \int\limits_\Omega F^{\min}(M + \nabla u, t)\, dx \tag{1.10}$$

and associate the variational problem

$$\mathfrak{J}[\check{u}_t, t] = \inf_{u\in\mathbb{X}} \mathfrak{J}[u,t], \quad \check{u}_t \in \mathbb{X}, \quad \mathfrak{J}[u,t] = \int\limits_\Omega \mathscr{F}(\nabla u, t)\, dx. \tag{1.11}$$

The problem (1.11) is the relaxed problem for (1.8). It is known that

- $\mathscr{F}$ is independent of Ω,
- the problem (1.11) is always solvable and each solution to the problem (1.8) is a solution to the problem (1.11); moreover, $\inf_{u\in\mathbb{X}} I_0^{\min}[u,t] = \min_{u\in\mathbb{X}} \mathfrak{J}[u,t]$ for all t,
- every weakly converging in $\mathbb{X}$ minimizing sequence of the functional $I_0^{\min}[.,t]$ converges weakly in $\mathbb{X}$ to some solution to the problem (1.11),
- every solution to the problem (1.11) is a weak limit in $\mathbb{X}$ of some minimizing sequence of the functional $I_0^{\min}[.,t]$.

Owing to the above properties of the problem (1.11), we can take the solution to the problem (1.11) for a weak solution to the problem (1.8). By the relations

$$|\Omega|\mathscr{F}(0,t) = \inf_{u\in\mathbb{X}} I_0^{\min}[u,t] = \min_{u\in\mathbb{X}} \mathfrak{J}[u,t] = \min_{u\in\mathbb{X}} \int\limits_\Omega \mathscr{F}(\nabla u, t)\, dx,$$

the function $\check{u}_t \equiv 0$ is a solution to the problem (1.11) for all t.

We restrict the set of admissible phase distributions $\mathbb{Z}'$ by setting

$$\mathbb{Z} = \mathbb{Z}' \cap BV(\Omega), \tag{1.12}$$

where $BV(\Omega)$ is the space of functions of bounded variation [9, 10]. For $\chi \in \mathbb{Z}$ we define the area of the phase interface boundary:

$$S[\chi] = \sup_{h \in C_0^1(\Omega, R^m), |h(x)| \leqslant 1} \int_\Omega \chi \operatorname{div} h \, dx \equiv \int_\Omega |D\chi| \tag{1.13}$$

(for $\chi \in \mathbb{Z}$ the right-hand side of (1.13) is finite). We assume that the surface energy of the phase interface boundary is proportional to its area. The positive proportionality coefficient σ is called the *surface tension coefficient*. Taking into account the surface energy, we replace the functional (1.4) with

$$I[u, \chi, t, \sigma] = I_0[u, \chi, t] + \sigma S[\chi], \quad u \in \mathbb{X}, \ \chi \in \mathbb{Z}, \ t \in R^1, \ \sigma > 0. \tag{1.14}$$

By an *equilibrium state* of the energy functional of a two-phase elastic medium (1.14) we mean the solution to the problem

$$I[\widehat{u}_{t,\sigma}, \widehat{\chi}_{t,\sigma}, t, \sigma] = \inf_{u \in \mathbb{X}, \chi \in \mathbb{Z}} I[u, \chi, t, \sigma], \quad \widehat{u}_{t,\sigma} \in \mathbb{X}, \quad \widehat{\chi}_{t,\sigma} \in \mathbb{Z}. \tag{1.15}$$

As is known [11, 12], if Ω is a Lipschitz domain, then the problem (1.15) is solvable and any sequence $\widehat{u}_{t,\sigma_n}$ as $\sigma_n \to 0$ is minimizing for the functional $I_0^{\min}[., t]$.

Since the functional (1.14) is coercive, from any sequence $\widehat{u}_{t,\sigma_n}$, $\sigma_n \to 0$, we can extract a weakly converging subsequence in $\mathbb{X}$. It is natural to call its weak limit a *weak solution* to the problem (1.8).

A question arises about connections between solutions to the problem (1.8) (consequently, the problem (1.6)) and the problems (1.11) and (1.15).

Explicit formulas for $t_\pm$ and the quasiconvex hull $\mathscr{F}(M, t)$ are found in [12]–[15]; moreover the set of all solutions to the problems (1.6), (1.8), (1.11) and the set of limit points of the family $\widehat{u}_{t,\sigma}$ in the weak topology of the space $\mathbb{X}$ as $\sigma \to 0$ are described in the case of the isotropic elasticity modulus tensors $a_{ijkl}^\pm$ and the residual strain tensors $\zeta_{ij}^\pm$

$$a_{ijkl}^\pm = \frac{a_\pm}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + b_\pm \delta_{ij}\delta_{kl}, \quad a_\pm > 0, \quad b_\pm \geqslant 0, \tag{1.16}$$

$$\zeta_{ij}^\pm = c_\pm \delta_{ij} \tag{1.17}$$

under the additional condition $a_\pm = a$.

The goal of this paper is to realize the same program for the coefficients (1.16) under the additional assumption $b_\pm = 0$ (and, sometimes, $a_\pm = a$), for the anisotropic residual strain tensors

$$\zeta_{ij}^\pm = c_\pm (\lambda \otimes \lambda)_{ij} = c_\pm \lambda_i \lambda_j, \quad \lambda \in R^m, \quad |\lambda| = 1, \quad c_\pm \in R^1. \tag{1.18}$$

Under the above assumptions, the formula for the strain energy density takes the form

$$F^\pm(\nabla u) = a_\pm \operatorname{tr}(e(\nabla u) - c_\pm \lambda \otimes \lambda)^2. \tag{1.19}$$

2 The Main Results

We formulate the main results. The square brackets mean the jump $[\beta] = \beta_+ - \beta_-$.

Theorem 1. *For the energy densities* (1.19) *the phase transition temperatures are given by*

$$t_+ = t^* + \frac{[ac]^2}{a_+}, \quad t_- = t^* - \frac{[ac]^2}{a_-}, \quad t^* = -[ac^2]. \tag{2.1}$$

By (2.1), for $a_\pm = 1$, $[c] \neq \varnothing$ the phase transition temperatures are different.

Theorem 2. *For the energy densities* (1.19), *under the additional condition* $a_\pm = 1$, $c_+ \neq c_-$, *the quasiconvex hull is given by*

$$\mathscr{F}(M,t) = e_{ij}(M)e_{ij}(M) - z^2(M) + H_c^{\min}(z(M),t), \tag{2.2}$$

where $z(M) = (e(M)\lambda, \lambda)$ *and* $H_c^{\min}(.,t)$ *for every* t *is the convex hull of* $H^{\min}(.,t)$,

$$\begin{aligned} &H^{\min}(z,t) = \min\{H^+(z,t), H^-(z)\}, \quad z \in R^1, \\ &H^+(z,t) = (z-c_+)^2 + t, \quad H^-(z) = (z-c_-)^2. \end{aligned} \tag{2.3}$$

Theorem 3. *For the energy densities* (1.19) *and every* t *the function* $\check{u}_t = 0$ *is a unique solution to the problem* (1.11).

Theorem 4. *For the energy densities* (1.19) *the problem* (1.6) *is not necessarily solvable for any* $t \in (t_-, t_+)$.

Theorem 5. *For the energy densities* (1.19) *any solution* $\widehat{u}_{t,\sigma}$, $\widehat{\chi}_{t,\sigma}$ *to the problem* (1.15) *satisfies the relation*

$$\widehat{u}_{t,\sigma} \rightharpoonup 0 \quad \text{in } \mathbb{X} \text{ as } \sigma \to 0. \tag{2.4}$$

In Theorems 1–4, it is assumed that the domain $\Omega \subset R^m$, $m \geqslant 2$, is bounded, whereas the domain Ω is Lipschitz in Theorem 5.

Thus, for the densities (1.19) the problem (1.6) (and, consequently, the problem (1.8)) is not solvable for any $t \in (t_-, t_+)$, whereas its weak solution vanishes in both approaches (the relaxed problem and the method of vanishing surface tension).

3 Proof of Theorem 1

(1) *One-dimensional model case.* In the one-dimensional case, $\Omega = (0,l)$, $l > 0$, $\mathbb{X} = \overset{\circ}{W}{}_2^1(0,l)$, $\mathbb{Z}'$ is the set of measurable characteristic functions on $(0,l)$, $F^\pm(M) = a_\pm(M - c_\pm)^2$, $M \in R^1$. Since for all $M, \lambda \in R^1$

$$F^\pm(M) = a_\pm(M - \Big(c_\pm - \frac{\lambda}{a_\pm})\Big)^2 + 2c_\pm\lambda - \frac{\lambda^2}{a_\pm} - 2M\lambda,$$

for any $\chi \in \mathbb{Z}'$ we have

$$\chi F^+(M) + (1-\chi)F^-(M) = (a_+\chi + a_-(1-\chi))\Big(M - \chi\Big(c_+ - \frac{\lambda}{a_+}\Big) - (1-\chi)\Big(c_- - \frac{\lambda}{a_-}\Big)\Big)^2$$

$$+ 2\lambda(\chi c_+ + (1-\chi)c_-) - \lambda^2\Big(\frac{\chi}{a_+} + \frac{1-\chi}{a_-}\Big) - 2M\lambda.$$

It is easy to verify this equality by comparing with the above equality for $\chi(x) = 1$ and $\chi(x) = 0$. We fix λ by the condition

$$\int_0^l \Big(\chi\Big(c_+ - \frac{\lambda}{a_+}\Big) + (1-\chi)\Big(c_- - \frac{\lambda}{a_-}\Big)\Big)\,dx = 0.$$

After some transformations, for all $u \in \mathbb{X}$, $\chi \in \mathbb{Z}'$ we find

$$\begin{aligned} I_0[u,\chi,t] &= \int_0^l \{\chi(F^+(u') + t) + (1-\chi)F^-(u')\}\,dx \\ &= \int_0^l (a_+\chi + a_-(1-\chi))(u' - \alpha(Q)(\chi - Q))^2\,dx + lG(Q,t), \end{aligned} \tag{3.1}$$

$$Q = \frac{1}{l}\int_0^l \chi\,dx, \quad \alpha(Q) = \frac{[ac]}{a_-Q + a_+(1-Q)},$$

$$G(Q,t) = tQ + a_+c_+^2Q + a_-c_-^2(1-Q) - [ac]\alpha(Q)Q(1-Q).$$

By (3.1), the set of all solutions to the problem (1.6) in the one-dimensional case has the form

$$\begin{aligned} &\widehat{\chi}_t(x) \in \mathbb{Z}' : \quad \frac{1}{l}\int_0^l \widehat{\chi}_t(x)\,dx = \widehat{Q}(t), \quad x \in (0,l), \\ &\widehat{u}_t(x) = \alpha(\widehat{Q}(t))\int_0^x (\widehat{\chi}_t(y) - \widehat{Q}(t))\,dy, \quad x \in (0,l), \end{aligned} \tag{3.2}$$

where $\widehat{Q}(t)$ is found from

$$G(\widehat{Q}(t),t) = \min_{Q\in[0,1]} G(Q,t), \quad \widehat{Q}(t) \in [0,1]. \tag{3.3}$$

Analyzing the problem (3.3), we obtain the following result (we use the notation (2.1)):

$$\text{for } t_- < t_+\text{:} \quad \widehat{Q}(t) = \begin{cases} 1, & t \leqslant t_-, \\ 0, & t \geqslant t_+, \\ h(t), & [a] = 0,\ t \in (t_-,t_+), \\ \dfrac{a_+ + a_-}{2[a]} + \dfrac{1}{2} - \dfrac{1}{[a]g^{1/2}(t)}, & [a] \neq 0,\ t \in (t_-,t_+), \end{cases} \tag{3.4}$$

$$h(t) = \frac{t_+ - t}{t_+ - t_-}, \quad g(t) = \frac{h(t)}{a_-^2} + \frac{1-h(t)}{a_+^2},$$

$$\text{for } t_- = t_+ (= t^*)\text{:} \quad \widehat{Q}(t) = \begin{cases} 1, & t < t^*, \\ 0, & t > t^*, \end{cases} \quad \widehat{Q}(t^*) \in [0,1]. \tag{3.5}$$

From the equalities (3.4), (3.5) it follows that $t_\pm$ defined in (2.1) are the phase transition temperatures for the energy densities (1.19) in the one-dimensional case.

Here, we used some methods of [16].

(2) *Lower estimate for the energy functional.* Since the phase transition temperatures are independent of Ω, we consider the cylinder $\Omega = B \times (0, l)$, where B is the unit ball in R^{m-1}. The cylinder axis and the coordinate x_m-axis are directed along the vector λ. We denote by x' the remaining coordinates $x_1, \dots, x_{m-1}$. Since

$$\operatorname{tr}(e(M) - c_\pm \lambda \otimes \lambda)^2 = (e_{mm}(M) - c_\pm)^2 + \Sigma_{i,j=1,\dots,m, i+j<2m} e_{ij}(M) e_{ij}(M),$$

we have

$$\begin{aligned}
&I_0[u, \chi, t] = J_1[u, \chi, t] + J_2[u, \chi],\\
&J_1[u, \chi, t] = \int_\Omega \{\chi(a_+(e_{mm}(\nabla u) - c_+)^2 + t) + (1-\chi)a_-(e_{mm}(\nabla u) - c_-)^2\}\, dx,\\
&J_2[u, \chi] = \int_\Omega (a_+\chi + a_-(1-\chi))\Sigma_{i,j=1,\dots,m, i+j<2m} e_{ij}(\nabla u) e_{ij}(\nabla u)\, dx,
\end{aligned} \tag{3.6}$$

where $u \in \mathbb{X}$, $\chi \in \mathbb{Z}'$. Consequently,

$$I_0[u, \chi, t] \geqslant J_1[u, \chi, t] = \int_B \left(\int_0^l \{\chi(a_+(u^m_{x_m} - c_+)^2 + t) + (1-\chi)a_-(u^m_{x_m} - c_-)^2\}\, dx_m \right) dx', \tag{3.7}$$

where $\chi = \chi(x', x_m)$, $u^m = u^m(x', x_m)$.

For almost all $x' \in B$ the function $\chi(x', .)$ is measurable and characteristic on $(0, l)$, whereas the function $u^m(x', .)$ belongs to the space $\overset{\circ}{W}{}^1_2(0, l)$. Since the internal integral on the right-hand side of (3.7) coincides with the one-dimensional energy functional, it is bounded from below by $lG(\widehat{Q}(t), t)$. Therefore,

$$I_0[u, t, \chi] \geqslant |\Omega| G(\widehat{Q}(t), t) \tag{3.8}$$

for all $u \in \mathbb{X}$, $\chi \in \mathbb{Z}'$, $t \in R^1$. Thus,

$$i(t, \Omega) = \inf_{u \in \mathbb{X}, \chi \in \mathbb{Z}'} I_0[u, \chi, t] \geqslant |\Omega| G(\widehat{Q}(t), t). \tag{3.9}$$

(3) *Computation of* $i(t, \Omega)$. In fact, we have equality in (3.9). Indeed, let us consider the pair of functions

$$\widetilde{u}(x) = \widehat{u}_t(x_m)\varphi_r(x')e_m, \quad \widetilde{\chi}(x) = \widehat{\chi}_t(x_m), \tag{3.10}$$

where e_m is the unit vector directed along the x_m-axis and the cut-off function $\varphi_r(x')$ possesses the properties

$$\begin{aligned}
&\varphi_r \in C_0^\infty(B), \quad 0 \leqslant \varphi_r(x') \leqslant 1,\\
&\varphi_r(x') = 1, \quad |x'| \leqslant r \in [1/2, 1), \quad |\nabla \varphi_r(x')| \leqslant C(1-r)^{-1},
\end{aligned}$$

whereas the pair $\widehat{u}_t(x_m)$, $\widehat{\chi}_t(x_m)$ is an arbitrary equilibrium state for the one-dimensional problem in (3.2). We note that $\widetilde{u} \in \mathbb{X}$, $\widetilde{\chi} \in \mathbb{Z}'$.

For the pair (3.10) and a fixed t we have

$$\begin{aligned} &|J_1[\widetilde{u},\widetilde{\chi},t]-|\Omega|G(\widehat{Q}(t),t)|\leqslant L(1-r)(1+\|\widehat{u}_t'\|^2_{L_\infty(0,l)}),\\ &|J_2[\widetilde{u},\widetilde{\chi}]|\leqslant\frac{L}{(1-r)}\|\widehat{u}_t\|^2_{C[0,l]},\quad 0\leqslant L\neq L(r,\widehat{u}_t,\widehat{\chi}_t). \end{aligned}\tag{3.11}$$

Consequently,

$$i(t,\Omega)\leqslant I_0[\widetilde{u},\widetilde{\chi},t]\leqslant|\Omega|G(\widehat{Q}(t),t)+L\Big(\frac{\|\widehat{u}_t\|^2_{C[0,l]}}{(1-r)}+(1-r)(1+\|\widehat{u}_t'\|^2_{L_\infty(0,l)})\Big).\tag{3.12}$$

We set $r=r_n=1-1/n$, $n=2,3,\ldots$. In this case, the second factor of the last term on the right-hand side of (3.12) takes the form

$$n\|\widehat{u}_t\|^2+n^{-1}(1+\|\widehat{u}_t'\|^2_{L_\infty(0,l))}).$$

If we can construct the sequence $\widehat{u}_t^{(n)}$, $\widehat{\chi}_t^{(n)}$ of solutions to the one-dimensional problem such that

$$n\|\widehat{u}_t^{(n)}\|^2_{C[0,l]}+n^{-1}(1+\|(\widehat{u}_t^{(n)})'\|^2_{L_\infty(0,l)})\to 0,\quad n\to\infty,$$

then, passing to the limit in (3.12) and using (3.9), we conclude that

$$i(t,\Omega)=|\Omega|G(\widehat{Q}(t),t).\tag{3.13}$$

For any pair $\widehat{u}_t$, $\widehat{\chi}_t$ in (3.2)

$$\|\widehat{u}_t'\|_{L_\infty(0,l)}\leqslant\alpha(\widehat{Q}(t)).\tag{3.14}$$

We divide $[0,l]$ into n equal intervals l_k, $k=1,\ldots,n$. For $\widehat{\chi}_t^{(n)}$ we take a measurable characteristic function on $(0,l)$ such that

$$\frac{1}{|l_k|}\int\limits_{l_k}\widehat{\chi}_t^{(n)}(y)\,dy=\widehat{Q}(t),\quad k=1,\ldots,n.$$

This function satisfies the condition (3.2) and the constructed function $\widehat{u}_t^{(n)}$ vanishes at the endpoints of every interval l_k of length ln^{-1}. Taking into account (3.14), we conclude that

$$\|\widehat{u}_t^{(n)}\|_{C[0,l]}\leqslant\alpha(\widehat{Q}(t))\frac{l}{n}.$$

(4) *Computation of* $i_{\min}(t,\Omega)$. We set

$$\begin{aligned} &i^\pm(t,\Omega)=\inf_{u\in\mathbb{X}}I_0[u,\chi^\pm,t],\quad\chi^+\equiv 1,\quad\chi^-\equiv 0,\\ &i_{\min}(t,\Omega)=\min\{i^+(t,\Omega),i^-(t,\Omega)\}. \end{aligned}\tag{3.15}$$

Since the infimum in (3.15) is attained only for $u=0$, the function $i_{\min}(t,\Omega)$ can be explicitly found, which implies that the function $i_{\min}(t,(0,l))$ in the one-dimensional problem and the function $i_{\min}(t,B\times(0,l))$ in the multi-dimensional problem are connected by

$$|B|i_{\min}(t,(0,l))=i_{\min}(t,B\times(0,l)).\tag{3.16}$$

By (3.1), (3.2), and (3.13), a similar connection holds between $i(t,(0,l))$ and $i(t, B\times(0,l))$:

$$|B| i(t,(0,l)) = i(t, B\times(0,l)). \tag{3.17}$$

(5) *Proof of Theorem 1 (continued).* As was established in [4], the interval (t_-, t_+) coincides with the interval, where the difference $i_{min}(t,\Omega) - i(t,\Omega)$ is positive. From (3.16) and (3.17) it follows that this interval is the same for the one-dimensional problem on the interval $(0,l)$ and the multi-dimensional problem in the cylinder $B\times(0,l)$. Therefore, the phase transition temperatures also coincide. It remains to recall that for a fixed m the phase transition temperatures are independent of the domain $\Omega \subset R^m$.

4 Proof of Theorem 2

(1) *Computation of $\mathscr{F}(0,t)$.* Since $|\Omega|\mathscr{F}(0,t) = i(t,\Omega)$, from (3.13) with $a_\pm = 1$, $c_+ \neq c_-$ we find

$$\mathscr{F}(0,t) = t\widehat{Q}(t) + c_+^2\widehat{Q}(t) + c_-^2(1-\widehat{Q}(t)) - [c]^2\widehat{Q}(t)(1-\widehat{Q}(t)),$$
$$\widehat{Q}(t) = \begin{cases} 1, & t \leqslant t_-. \\ \dfrac{t_+ - t}{t_+ - t_-}, & t\in(t_-,t_+), \quad t_\pm = t^* \pm [c]^2, \quad t^* = -[c^2]. \\ 0, & t\geqslant t_+, \end{cases} \tag{4.1}$$

(2) *Formula for $\mathscr{F}(M,t)$.* Using the obvious relations

$$\begin{aligned} &\operatorname{tr}(e(\nabla u) - c_\pm\lambda\otimes\lambda + e(M))^2 \\ &= \operatorname{tr}(e(\nabla u) - c_\pm\lambda\otimes\lambda)^2 + 2\operatorname{tr} e(\nabla u)e(M) + \operatorname{tr} e^2(M) - 2c_\pm z, \\ &z = z(M) = \operatorname{tr} e(M)\lambda\otimes\lambda = (e(M)\lambda,\lambda) \end{aligned} \tag{4.2}$$

(the last parentheses denote the inner product in R^m), we find

$$\int\limits_\Omega \{\chi(\operatorname{tr}(e(\nabla u) - c_+\lambda\otimes\lambda + e(M))^2 + t) + (1-\chi)\operatorname{tr}(e(\nabla u) - c_-\lambda\otimes\lambda + e(M))^2\}\,dx$$
$$= I_0[u,\chi,t(z)] + |\Omega|(\operatorname{tr} e^2(M) - 2c_- z), \quad t(z) = t - 2[c]z. \tag{4.3}$$

Consequently,

$$\begin{aligned} &\mathscr{F}(M,t) = \mathscr{F}(0,t(z)) + \operatorname{tr} e^2(M) - 2c_- z = \operatorname{tr} e^2(M) - z^2 + H(z,t), \\ &H(z,t) = z^2 + t(z)\widehat{Q}(t(z)) + c_+^2\widehat{Q}(t(z)) \\ &\qquad + c_-^2(1-\widehat{Q}(t(z)) - [c]^2\widehat{Q}(t(z))(1-\widehat{Q}(t(z))) - 2c_- z. \end{aligned} \tag{4.4}$$

(3) *Function $H(z,t)$.* We set

$$z_-(t) = \frac{t-t_-}{2[c]}, \quad z_+(t) = \frac{t-t_+}{2[c]}. \tag{4.5}$$

It is obvious that $z_+(t) - z_-(t) = -[c]$. A direct analysis of (4.4) leads to the conclusion that $H(.,t)$ is continuously differentiable with respect to $z\in R^1$,

$$H(z,t)=\begin{cases} H^+(z,t), & z \geqslant z_-(t), \\ H^-(z), & z \leqslant z_+(t), \quad [c]>0, \\ H^+(z,t), & z \leqslant z_-(t), \\ H^-(z), & z \geqslant z_+(t), \quad [c]<0, \end{cases} \tag{4.6}$$

$H(\cdot,t)$ is linear on the remaining part.

Since $H(.,t)$ is continuously differentiable and $H^+(.,t)$, $H^-(.)$ are convex, from (4.6) we see that the derivative $H_z(.,t)$ is monotone increasing, which implies the convexity of $H(.,t)$. It is obvious that we cannot place between the graphs of $H^{\min}(.,t)$ and $H(.,t)$ the graph of some other convex function. Hence $H(.,t)=H_c^{\min}(.,t)$.

5 Proof of Theorem 3

As was established in Section 3, for a cylindrical domain $\Omega = B\times(0,l)$

$$|\Omega|^{-1}\inf_{u\in\mathbb{X},\chi\in\mathbb{Z}'} I_0[u,\chi,t]=G(\widehat{Q}(t),t). \tag{5.1}$$

Since ((1.7), (1.8), (1.10)) the left-hand side of (5.1) coincides with the function $\mathscr{F}(M,t)$, $M=0$, independent of Ω, the equality (5.1) holds for all $\Omega\subset R^m$.

We choose coordinates with the x_m-axis directed along the vector λ and denote by x' the remaining coordinates. The intersection of the line $\{x'\in R^{m-1}$ is fixed, $x_m\in R^1$ is arbitrary$\}$ with Ω is open in this line and, consequently, it either is empty or consists of at most countably many open intervals l_j (depending on x'), $j=1,2,\ldots$. For each pair $u\in\mathbb{X}$, $\chi\in\mathbb{Z}'$ and almost all x' the restriction of u on l_j belongs to $\overset{\circ}{W}{}^1_2(l_j)$ and χ is a measurable characteristic function of x_m on these intervals. By (3.1),

$$\begin{gathered} \text{for a.e. } x' \text{ and all } l_j \\ \int\limits_{l_j}\{\chi((u^m_{x_m}-c_+)^2+t)+(1-\chi)(u^m_{x_m}-c_-)^2\}\,dx_m \geqslant |l_j|G(\widehat{Q}(t),t), \\ \chi=\chi(x',x_m), \quad u=u(x',x_m). \end{gathered} \tag{5.2}$$

Using (3.6) and (5.2), we find

$$J_1[u,\chi,t]\geqslant|\Omega|G(\widehat{Q}(t),t), \quad J_2[u,\chi]\geqslant 0, \quad \forall\, u\in\mathbb{X},\ \chi\in\mathbb{Z}'. \tag{5.3}$$

Let $\check{u}_t$ be a solution to the problem (1.11) for the energy densities (1.19). Then there exists a minimizing sequence u_n of the functional $I_0^{\min}[u,t]$ such that $u_n \rightharpoonup \check{u}_t$ in the space $\mathbb{X}$. Using the rule (1.9), we restore for u_n a function χ_n such that the pair u_n, χ_n is minimizing for the functional $I_0[u,\chi,t]$. From (5.1) and (5.3) it follows that

$$J_2[u_n,\chi_n]\to 0, \quad n\to\infty. \tag{5.4}$$

By (5.4), $e_{ij}(\nabla\check{u}_t)=0$, $i,j=1,\ldots,m-1$, and $e_{mi}(\nabla\check{u}_t)=0$, $i=1,\ldots,m-1$. Hence $\check{u}^i_t(x)=0$, $i=1,\ldots,m-1$, and $\check{u}^m_t(x)=v(x_m)$ with some scalar function v. Consequently, $\check{u}_t(x)=\lambda v(x_m)$. Such a function can belong to the space $\mathbb{X}$ only if $v=0$.

6 Proof of Theorem 4

As was mentioned in Section 1, for one-phase equilibrium states $\widehat{u}_t$, $\widehat{\chi}_t$ the equilibrium displacement field vanishes. Let us prove the converse assertion for the strain energy densities (1.3). Suppose that $t_- < t_+$ and $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t$ is a solution to the problem (1.6). Then this equilibrium state is one-phase. Since for $\widehat{u}_t \equiv 0$

$$I_0[\widehat{u}_t, \widehat{\chi}_t, t] = (t - t^*) \int\limits_{\Omega} \widehat{\chi}_t \, dx + |\Omega| a^-_{ijkl} \zeta^-_{ij} \zeta^-_{kl}, \quad t \neq t^*,$$

only one-phase equilibrium states are possible. We note that for $\widehat{u}_t \equiv 0$ (cf. (3.9) and (3.15))

$$i(t, \Omega) = I_0[\widehat{u}_t, \widehat{\chi}_t, t] = \inf_{\chi \in \mathbb{Z}'} I_0[\widehat{u}_t, \chi, t] = i_{\min}(t, \Omega).$$

Since $t^* \in (t_-, t_+)$ for $t_- < t_+$, we have $i(t^*, \Omega) < i_{\min}(t^*, \Omega)$. Consequently, $t \neq t^*$ for the equilibrium state $\widehat{u}_t$, $\widehat{\chi}_t$ with $\widehat{u}_t \equiv 0$, which means that the state is one-phase.

Let for $t \in (t_-, t_+) \neq \varnothing$ the problem (1.6) have a solution $\widehat{u}_t$, $\widehat{\chi}_t$. By properties of the phase transition temperatures, this equilibrium state is two-phase. Since $\widehat{u}_t$ is a solution to the problem (1.8) and, consequently, the problem (1.11), from Theorem 3 it follows that for the densities (1.19) this function vanishes. It remains to note that for $t_- < t_+$ there are no two-phase equilibrium states with zero equilibrium displacement field.

Here, we used the observation from [17] that the uniqueness of a solution to the relaxed problem can imply that the initial problem has no solutions.

7 Proof of Theorem 5

We assume that for fixed t and some sequence $\sigma_n > 0$, $\sigma_n \to 0$, there are solutions $\widehat{u}_{t,\sigma_n}$, $\widehat{\chi}_{t,\sigma_n}$ to the problem (1.15) such that $|l(\widehat{u}_{t,\sigma_n})| \geqslant \epsilon$ for some linear bounded functional l in $\mathbb{X}$ and a positive number ϵ. Passing to $\widehat{u}_{t,\sigma_{n'}}$, we can reach the weak convergence in $\mathbb{X}$ to some function $u \in \mathbb{X}$. Since u is a solution to the relaxed problem, from Theorem 3 it follows that $u = 0$ contradicts the assumption.

Remark. For details of results in the above-cited works of the author we refer to [18].

Acknowledgments

This work was supported by the Russian Foundation for Basic Research (grant No. 14-01-00534) and the program Scientific Schools (grant No. NSh-1771.2014.1).

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Submitted on June 30, 2014