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ON A POWER OF OPTIMAL TEST FOR ASYMPTOTIC DISTINCTION OF STATISTICAL HYPOTHESES FOR DISTRIBUTIONS WITH HEAVY TAILS*

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In the paper, the asymptotic behavior of the power function of the most powerful test in the problem of testing a simple hypothesis against a simple alternative from a homogeneous sample of independent observations is studied under the assumption that the likelihood ratio has a heavy-tailed distribution belonging to the domain of attraction of a stable law.

1. Introduction

Consider the sequence of series $\{X_{n,j}\}$, $n \geqslant 1$, $j \geqslant 1$, of independent and identically distributed for each n random variables, defined on the same measurable space $(\Omega, \mathcal{A})$. Consider the sequence of pairs of probability measures $\{\mathsf{P}_n^{(0)}, \mathsf{P}_n^{(1)}\}_{n\geqslant 1}$, defined on $\mathcal{A}$. The density of $X_{n,1}$ with respect to probability measure $\mathsf{P}_n^{(i)}$ we denote by $p_n^{(i)}(x)$, $i = 0, 1$.

The main subject of study in this paper is an asymptotic behavior of power $\beta_{n,k}$ of the most powerful test in the problem of testing a simple hypothesis $H_n^{(0)} = \mathsf{P}_n^{(0)}$ against a simple alternative $H_n^{(1)} = \mathsf{P}_n^{(1)}$ by homogeneous sample $X_{n,1}, \ldots, X_{n,k}$ of independent real observations ($k \geqslant 1$ is a natural number, in the particular case $k = n$).

In this problem log-likelihood ratio $\Lambda_{n,k}$ equals

$$\Lambda_{n,k} = \log \prod_{j=1}^{k} \frac{p_n^{(1)}(X_{n,j})}{p_n^{(0)}(X_{n,j})} = \sum_{j=1}^{k} \left[\ell_n^{(1)}(X_{n,j}) - \ell_n^{(0)}(X_{n,j}) \right],$$

where $\ell_n^{(i)}(x) = \log p_n^{(i)}(x)$, $i = 0, 1$. It is known that for given significance level a the most powerful test, constructed with the use of fundamental Neumann–Pearson's lemma, rejects $H_n^{(0)}$ if

$$\Lambda_{n,k} \ > \ c_{n,k},$$

where $c_{n,k}$ is determined from the condition

$$\mathsf{P}_n^{(0)} \left(\Lambda_{n,k} > c_{n,k} \right) \ = \ a$$

(for simplicity we assume that all distributions are continuous). If the random variable $L_n = \ell_n^{(1)}(X_{n,1}) - \ell_n^{(0)}(X_{n,1})$ has variance with respect to measures $\mathsf{P}_n^{(0)}$ and $\mathsf{P}_n^{(1)}$, then with the use of central limit theorem it is easy to conclude, that the power $\beta_{n,k}$ of this test,

$$\beta_{n,k} \ = \ \mathsf{P}_n^{(1)} \left(\Lambda_{n,k} > c_{n,k} \right),$$

for each n satisfies the relation

$$\lim_{k \to \infty} \beta_{n,k} \ = \ 1$$

(see, e.g., [8]). Moreover, if $\{k_n\}$ is a sequence of natural numbers, such that $k_n \to \infty$ as $n \to \infty$, then a non-trivial, different from 1, limit of β_{n,k_n} (as $n \to \infty$) exists only when alternatives $H_n^{(0)}$ and $H_n^{(1)}$ are chosen in such a way that

$$\mu_n^{(1)} - \mu_n^{(0)} \ \sim \ k_n^{-1/2},$$

where

$$\mu_n^{(i)} \ = \ \mathsf{E}_n^{(i)} \left[\ell_n^{(1)}(X_{n,1}) - \ell_n^{(0)}(X_{n,1}) \right], \quad i = 0, 1, \tag{1}$$

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and symbol $\mathsf{E}_n^{(i)}$ denotes mathematical expectation with respect to the measure $\mathsf{P}_n^{(i)}$, see, e.g., [8]. In other words, under condition of existence of log-likelihood ratio variances, non-trivial asymptotic results hold only in the situation, when one considers contigual hypotheses, which approach each other in the above sense as $n \to \infty$ at the rate $O\left(k_n^{-1/2}\right)$.

In recent years we see a significantly increased interest in the problems connected with so-called heavy-tailed distributions, attributing considerable probabilities to the large deviations of corresponding random variables, see, e.g., [11, 12]. This interest is apparently due to the need in studying large risks associated with catastrophic events. At the same time above-mentioned case, when there exist variances of log-likelihood ratio, does not exhaust all the possible cases. Let us consider the possible behavior of log-likelihood ratio distribution tails in more detail.

Let X be a real random variable on the measurable space $(\Omega, \mathcal{A})$. Suppose that two measures P_0 and P_1 are defined on $\mathcal{A}$. Mathematical expectations with respect to these measures are denoted by E_0 and E_1 respectively. Suppose that distributions of random variable X,

$$\mathsf{P}_X^0(B) = \mathsf{P}_0(\{\omega : X(\omega) \in B\}) \text{ and } \mathsf{P}_X^1(B) = \mathsf{P}_1(\{\omega : X(\omega) \in B\}), \quad B \in \mathcal{B},$$

where $\mathcal{B}$ is Borel σ-algebra, are absolutely continuous with respect to Lebesgue measure. Corresponding densities we denote by $p_0(x)$ and $p_1(x)$. Let us study the behavior of tails of random variable

$$L = \log \frac{p_1(X)}{p_0(X)}$$

both with respect to measure P_0, and measure P_1.

Proposition 1. *For any $x > 0$*

$$\mathsf{P}_0(L > x) \leqslant e^{-x}.$$

Proof. By the Tchebyshev inequality

$$\mathsf{P}_0(L > x) = \mathsf{P}_0\left(\frac{p_1(X)}{p_0(X)} > e^x\right) \leqslant e^{-x}\mathsf{E}_0\frac{p_1(X)}{p_0(X)} = e^{-x}\int\limits_{-\infty}^{\infty}\frac{p_1(y)}{p_0(y)}p_0(y)dy = e^{-x}.$$

Remark 1. Proposition 1 means that log-likelihood ratio distribution under null hypothesis always has light tails, guarantying the existence of its variance under null hypothesis.

Proposition 2. *For any arbitrarily small $\varepsilon > 0$ there exists a random variable X, $C > 0$ and densities $p_0(x)$ and $p_1(x)$ such that as $x \to \infty$*

$$\mathsf{P}_1(L > x) \sim \frac{C}{x^{\varepsilon}}.$$

Proof. For simplicity let us assume that

$$\mathsf{P}_0(X \geqslant 1) = \mathsf{P}_1(X \geqslant 1) = 1,$$

and put

$$p_1(x) \equiv \frac{C_0}{x^2}, \quad p_0(x) \equiv C_1(\varepsilon)\exp\{-x^{1/\varepsilon}\}$$

(here C_0 and $C_1(\varepsilon)$ are positive normalization constants). Then denoting $C' = C'(\varepsilon) = C_1(\varepsilon)/C_0$, for $x \geqslant 1$ we have

$$\mathsf{P}_1(L > x) = \mathsf{P}_1\left(\log\frac{e^{X^{1/\varepsilon}}}{C'X^2} > x\right) = \mathsf{P}_1\left(\frac{e^{X^{1/\varepsilon}}}{C'X^2} > e^x\right) = \mathsf{P}_1\left(e^{X^{1/\varepsilon}} > e^x C' X^2\right) =$$

$$= \mathsf{P}_1\left(X^{1/\varepsilon} > x + 2\log X + \log C'\right) = \mathsf{P}_1\left(X^{1/\varepsilon} - 2\log X - \log C' > x\right).$$

Since for any $\alpha > 0$

$$x^{-\alpha}\log x \longrightarrow 0$$

as $x \to \infty$, then for any $\varepsilon > 0$ and any arbitrarily small $\delta > 0$ there exists $x' = x'(\varepsilon, \delta)$, such that for all ω, satisfying $X(\omega) \geqslant x'$, the following inequalities hold:

$$(1-\delta)X^{1/\varepsilon} \leqslant X^{1/\varepsilon} - 2\log X - \log C' = X^{1/\varepsilon}\left(1 - 2X^{-1/\varepsilon}\log X - \log C' X^{-1/\varepsilon}\right) \leqslant (1+\delta)X^{1/\varepsilon}.$$

Thus for large enough x (for $x \geqslant (x' + 2\log x' + \log C')^{\varepsilon}$) the following inequalities hold:

$$\mathsf{P}_1\left((1-\delta)X^{1/\varepsilon} > x\right) \leqslant \mathsf{P}_1\left(X^{1/\varepsilon} > x + 2\log X + \log C'\right) \leqslant$$

$$\leqslant \mathsf{P}_1\left((1+\delta)X^{1/\varepsilon} > x\right).$$

Hence, for those large enough x

$$\mathsf{P}_1(L > x) \leqslant \mathsf{P}_1\left((1+\delta)X^{1/\varepsilon} > x\right) = C_0 \int\limits_{(1+\delta)^{-\varepsilon}x^{\varepsilon}}^{\infty} \frac{dz}{z^2} = \frac{C_0(1+\delta)^{\varepsilon}}{x^{\varepsilon}},$$

$$\mathsf{P}_1(L > x) \geqslant \mathsf{P}_1\left((1-\delta)X^{1/\varepsilon} > x\right) = C_0 \int\limits_{(1-\delta)^{-\varepsilon}x^{\varepsilon}}^{\infty} \frac{dz}{z^2} = \frac{C_0(1-\delta)^{\varepsilon}}{x^{\varepsilon}}.$$

Now it is clear that for large enough x

$$(1-\delta)^{\varepsilon} \leqslant \frac{x^{\varepsilon}}{C_0}\mathsf{P}_1(L > x) \leqslant (1+\delta)^{\varepsilon}.$$

The possibility to choose δ arbitrarily small as $x \to \infty$ concludes the proof.

Remark 2. Proposition 2 means that log-likelihood ratio distribution under alternative hypothesis may have a tail that is at least not lighter than the tail of the stable law with arbitrarily small positive characteristic index.

Remark 3. In the proof of Proposition 2 it is established that log-likelihood ratio distribution under alternative hypothesis may have a tail that decays as a power law. This means that log-likelihood ratio distribution under alternative hypothesis may belongs to the domain of attraction of a stable law with arbitrarily small positive characteristic index.

Remark 4. Condition $\mathsf{P}_0(X \geqslant 1) = \mathsf{P}_1(X \geqslant 1) = 1$ is used only to simplify computations. It is neither restrictive nor fundamental.

Example 1. Consider the problem of testing hypothesis

$$H_0: \text{ the density of random variable } X \text{ is equal to } \ p_0(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

(random variable X has a normal distribution) against the alternative

$$H_1: \text{ the density of random variable } X \text{ is equal to } p_1(x) = \frac{1}{\pi(1+x^2)}$$

(random variable X has Cauchy distribution). Using the proof scheme of Proposition 2, it can be shown that as $x \to \infty$

$$\mathsf{P}_1(L > x) \sim Cx^{-1/2},$$

and it follows (see [8]), that in this problem log-likelihood ratio distribution under alternative hypothesis belongs to the domain of attraction of a stable law with characteristic index $\alpha = 1/2$ (Levy distribution), which does not have moments of order $\geqslant 1/2$, including mathematical expectation.

Example 2. Similar result is obtained in the problem of testing the above hypothesis H_0 from Example 1 about normality of random variable X against the alternative $H_{1,\theta}$: the density of random variable X is equal to

$$p_{1,\theta}(x) = \frac{1-\theta}{\sqrt{2\pi}}e^{-x^2/2} + \frac{\theta}{\pi(1+x^2)},$$

where $\theta \in (0,1)$ is a small parameter (the problem of possibility of observations, contaminating normal sample).

Example 3. Consider the problem of testing hypothesis

$$H_0: \text{ the density of random variable } X \text{ is equal to } p_0(x) = \frac{2}{3}x^{-1/3}e^{-x^{2/3}}, \quad x > 0,$$

(random variable X has Weibull distribution with form parameter equal to $2/3$) against the alternative

$$H_1: \text{ the density of random variable } X \text{ is equal to } p_1(x) = \frac{2}{\pi(1+x^2)}, \quad x > 0,$$

(random variable X has Pareto distribution with form parameter equal to 2). Then as $x \to \infty$

$$\mathsf{P}_1(L > x) \sim C x^{-3/2},$$

and it follows (see [8]), that in this problem log-likelihood ratio distribution under alternative hypothesis belongs to the domain of attraction of a stable law with characteristic index $\alpha = 3/2$, which has mathematical expectation, but does not have moments of order $\geqslant 3/2$, including variance.

Example 4. Similar result is obtained in the problem of testing the above hypothesis H_0 from Example 3, that random variable X has Weibull distribution with parameter $2/3$, against the alternative $H_{1,q}$: the density of random variable X is equal to

$$p_{1,q}(x) = \frac{2(1-q)}{3} x^{-1/3} e^{-x^{2/3}} + \frac{2q}{\pi(1+x^2)}, \quad x > 0,$$

where $q \in (0,1)$ is a small parameter (the problem of possibility of "long-lived" observations, contaminating sample, with lifetime, having the above Weibull distribution).

Let us now return to the above problem of asymptotic distinction of two simple hypotheses. As we saw, distribution of random variable $L_n = \ell_n^{(1)}(X_{n,1}) - \ell_n^{(0)}(X_{n,1})$ under hypothesis $H_n^{(0)}$ always has variance, and it follows, that it belongs to the domain of attraction of a normal law, i.e. there exist numbers $a_{n,k}^{(0)} \in \mathbf{R}$ and $b_{n,k}^{(0)} > 0$ such that for each n

$$\mathcal{L}_n^{(0)}\left(\frac{\Lambda_{n,k} - a_{n,k}^{(0)}}{b_{n,k}^{(0)}}\right) \Longrightarrow \Phi, \quad k \to \infty, \tag{2}$$

where Φ is a standard normal distribution function, and symbols $\Longrightarrow$ and $\mathcal{L}_n^{(i)}(\cdot)$ here and below denote respectively weak convergence and distribution of random variable in brackets with respect to measure $\mathsf{P}_n^{(i)}$, $i = 0, 1$.

Below we will focus on the situation when distribution of random variable $L_n = \ell_n^{(1)}(X_{n,1}) - \ell_n^{(0)}(X_{n,1})$ under alternative $H_n^{(1)}$ has heavy tails. By "heavy-tailedness" of distribution we mean the following. We consider the special case of series scheme and assume that under hypothesis $H_n^{(1)}$ this distribution belongs to the domain of attraction of a stable law G_α with characteristic index $\alpha \in (0,2)$ (the same for all n). It means that there exist numbers $a_{n,k}^{(1)} \in \mathbf{R}$ and $b_{n,k}^{(1)} > 0$ such that for each n

$$\mathcal{L}_n^{(1)}\left(\frac{\Lambda_{n,k} - a_{n,k}^{(1)}}{b_{n,k}^{(1)}}\right) \Longrightarrow G_\alpha, \quad k \to \infty. \tag{3}$$

Our goal is to study conditions of nontrivial limits existence for power β_{n,k_n} in situation, when log-likelihood ratio distribution under alternative hypothesis has heavy tails (in the above sense).

2. The case $\alpha > 1$

Assumption $\alpha \in (1,2)$ in the section title allows us without loss of generality to assume that mathematical expectations $\mu_n^{(1)}$ exist, see (1). Existence of mathematical expectation $\mu_n^{(0)}$ follows from proposition 1, which allows us to take

$$a_{n,k}^{(0)} = k\mu_n^{(0)} \text{ and } b_{n,k}^{(0)} = \sigma_n^{(0)}\sqrt{k},$$

where

$$(\sigma_n^{(0)})^2 = \mathsf{D}_n^{(0)}\left[\ell_n^{(1)}(X_{n,1}) - \ell_n^{(0)}(X_{n,1})\right].$$

Condition $\alpha \in (1,2)$ allows us in turn to take

$$a_{n,k}^{(1)} = k\mu_n^{(1)} \text{ and } b_{n,k}^{(1)} = B_n(k) k^{1/\alpha},$$

where $B_n(k)$ is slowly changing function as $k \to \infty$, see, e.g., [13]. And from (2) we have

$$\mathcal{L}_n^{(0)}\left(\frac{\Lambda_{n,k} - k\mu_n^{(0)}}{\sigma_n^{(0)}\sqrt{k}}\right) \Longrightarrow \Phi, \quad k \to \infty,$$

and hence,

$$c_{n,k} \;=\; u(1-a)\sigma_n^{(0)}\sqrt{k} + k\mu_n^{(0)} + o(1), \;\; k \to \infty, \tag{4}$$

where $u(1-a)$ is $(1-a)$-order quantile of standard normal distribution. From (3) we also obtain

$$\mathcal{L}_n^{(1)}\left(\frac{\Lambda_{n,k} - k\mu_n^{(1)}}{B_n(k)k^{1/\alpha}}\right) \Longrightarrow G_\alpha, \quad k \to \infty,$$

and hence, with the use of (4) we have

$$\beta_{n,k} \;=\; \mathsf{P}_n^{(1)}\left(\Lambda_{n,k} > c_{n,k}\right) \;=\; 1 - G_\alpha\left(\frac{c_{n,k} - k\mu_n^{(1)}}{B_n(k)k^{1/\alpha}}\right) + o(1) \;=$$

$$=\; 1 - G_\alpha\Bigg(u(1-a)\frac{\sigma_n^{(0)}}{B_n(k)}k^{1/2-1/\alpha} + \frac{k^{1-1/\alpha}}{B_n(k)}(\mu_n^{(0)} - \mu_n^{(1)})\Bigg) + o(1), k \to \infty. \tag{5}$$

Using Jensen inequality, we conclude that

$$\mu_n^{(0)} \;<\; 0, \quad \mu_n^{(1)} \;>\; 0.$$

Let $k = k_n \to \infty$ and $\sigma_n^{(0)}$ be a slowly changing function as $n \to \infty$. Since $\alpha < 2$,

$$\frac{\sigma_n^{(0)}}{B_n(k_n)}k_n^{1/2-1/\alpha} \;\longrightarrow\; 0, \;\; n \to \infty,$$

and the asymptotics of distribution function G_α's argument in the right-hand side of (5) is determined by the behavior of the expression $k_n^{1-1/\alpha}(\mu_n^{(0)} - \mu_n^{(1)})/B_n(k_n)$. This expression has a finite limit only if

$$\mu_n^{(1)} - \mu_n^{(0)} \;=\; O\Bigg(B_n(k_n)k_n^{1/\alpha-1}\Bigg), \quad n \to \infty.$$

In summary we conclude that with regard to analyzed model of "heavy-tailedness" in order to obtain non-trivial results on asymptotic behavior of the power of asymptotically optimal tests for distinction simple hypotheses, one should consider contigual alternatives, which approach each other (in the sense of approaching $\mu_n^{(1)}$ to $\mu_n^{(0)}$) at the rate $O\left(B_n(k_n)k_n^{1/\alpha-1}\right)$, i.e. the choice of contiguity parameter, ensuring non-triviality of the problem, depends on parameter α, defining the asymptotics of a more "heavy-tailed" alternative. In particular, if $\mu_n^{(0)} \equiv \mu^{(0)}$, $\sigma_n^{(0)} \equiv \sigma^{(0)}$ and $k_n = n$, then non-trivial power limit is guaranteed only by the alternatives, for which

$$\mu_n^{(1)} \;=\; \mu^{(0)} + O\Bigg(B(n)n^{1/\alpha-1}\Bigg),$$

where $B(n)$ is some slowly changing function.

3. The case $\alpha \leqslant 1$

In the view of the above-formulated idea about what heavy-tailed distribution is, let us assume that $\alpha \leqslant 1$, and

$$\mathsf{P}_n^{(1)}(\ell_n^{(1)}(X_{n,1}) - \ell_n^{(0)}(X_1) \geqslant x) \;=\; \frac{\delta + o(1)}{x^\alpha}h(x),$$

as $x \to \infty$, where $\delta \in (0,\infty)$ and $h(x)$ are slowly changing functions (see, e.g., [6]). Since $\alpha \leqslant 1$, the value of $\mu_n^{(1)}$ is not defined, but there is a convergence of normalized log-likelihood ratio without centering (which is not needed in this situation):

$$\mathcal{L}_n^{(1)}\left(\frac{\Lambda_{n,k}}{B_n(k)k^{1/\alpha}}\right) \Longrightarrow G_\alpha, \quad k \to \infty,$$

see, e.g., [7]. But for hypothesis $H_n^{(0)}$ we still have

$$\mathcal{L}_n^{(0)}\left(\frac{\Lambda_{n,k}-k\mu_n^{(0)}}{\sigma_n^{(0)}\sqrt{k}}\right)\Longrightarrow\Phi,\quad k\to\infty,$$

and hence,

$$c_{n,k} \;=\; u(1-a)\sigma_n^{(0)}\sqrt{k}+k\mu_n^{(0)}+o(1)m,\;\; k\to\infty.$$

Thus

$$\beta_{n,k} \;=\; \mathsf{P}_n^{(1)}\left(\Lambda_{n,k}>c_{n,k}\right) \;=\; 1-G_{\alpha_1}\left(\frac{c_{n,k}-k\mu_n^{(1)}}{B_n(k)k^{1/\alpha}}\right)+o(1) \;=$$

$$=\; 1-G_\alpha\left(u(1-a)\frac{\sigma_n^{(0)}}{B_n(k)}k^{1/2-1/\alpha}+\frac{k^{1-1/\alpha}}{B_n(k)}\mu_n^{(0)}\right)+o(1),\;\; k\to\infty. \qquad (6)$$

Let $k=k_n\to\infty$ as $n\to\infty$. Since $\alpha<1$,

$$\frac{\sigma_n^{(0)}}{B_n(k_n)}k_n^{1/2-1/\alpha} \;\longrightarrow\; 0,\;\; n\to\infty,$$

and the asymptotics of distribution function G_α's argument in the right-hand side of (6) is determined by the behavior of the expression $k_n^{1-1/\alpha}\mu_n^{(0)}/B_n(k_n)$. This expression has a finite limit, only if

$$|\mu_n^{(0)}| \;=\; O\left(B_n(k_n)k_n^{1/\alpha-1}\right),\qquad n\to\infty,$$

i.e. if $|\mu_n^{(0)}|$ grows not faster than $O\left(B_n(k_n)k_n^{1/\alpha-1}\right)$. Thus, when $\alpha\leqslant 1$:

1) if

$$\lim_{n\to\infty}\frac{|\mu_n^{(0)}|}{B_n(k_n)k_n^{1/\alpha-1}}=\infty,$$

then $\beta_{n,k_n}\to 1,$ as $n\to\infty$;

2) if

$$\lim_{n\to\infty}\frac{\mu_n^{(0)}}{B_n(k_n)k_n^{1/\alpha-1}}=A\in(-\infty,0],$$

then $\beta_{n,k_n}\to\; 1-G_\alpha(A)$ as $n\to\infty$; particularly, if $k_n=n$ and $\mu_n^{(0)}=\mu^{(0)}$, then $\beta_{n,k_n}\to 1-G_\alpha(0)$ as $n\to\infty$.

In summary we conclude that with regard to analyzed model of “heavy-tailedness” in order to obtain non-trivial results on asymptotic behavior of the power of asymptotically optimal tests for distinction simple hypotheses, when $k_n=n$ and $G_\alpha(0)=0$, one should consider such alternatives, for which the presence of heavy tail is compensated by unlimited growth of mathematical expectation $|\mu_n^{(0)}|$ of log-likelihood ratio at the rate no less than $B(n)n^{1/\alpha-1}$, where $B(n)$ is a slowly changing function. This rate depends on parameter α, defining the “heaviness” of log-likelihood ratio's tail under alternative.

In particular, it follows, that since stable distributions with index $\alpha<1$ are single-sided (concentrated on nonnegative or nonpositive semiaxis, i.e. either $G_\alpha(0)=0$ or $G_\alpha(0)=1$), condition $\mu_n^{(0)}=\mu^{(0)}=\text{const}$ necessitates the presence of trivial power limits only (unity or zero) no matter how heavy are the tails of alternative with $\alpha<1$.

4. On the rate of convergence of optimal test power

In this section we give estimates of convergence rate of optimal test power to its limit values as $n\to\infty$.

4.1. Asymptotically normal case

Suppose that random variables $\{\ell_n^{(1)}(X_{n,j})-\ell_n^{(0)}(X_{n,j})\}_{j\geqslant 1}$ are such that their growing sums are asymptotically normal. Denote

$$(\sigma_n^{(i)})^2 = \mathsf{E}_n^{(i)}\left[(\ell_n^{(1)}(X_{n,1}) - \ell_n^{(0)}(X_{n,1})) - (\mu_n^{(1)} - \mu_n^{(0)})\right]^2,$$

$$\left(m_n^{(i)}\right)^3 = \mathsf{E}_n^{(i)}\left|\left(\ell_n^{(1)}(X_{n,1}) - \ell_n^{(0)}(X_{n,1})\right) - (\mu_n^{(1)} - \mu_n^{(0)})\right|^3, \quad i=0,1,$$

and assume that $(m_n^{(i)})^3 < \infty$. Then by Berry–Esseen inequality, when

$$k \geqslant \frac{\left(\sigma_n^{(0)}\right)^2 u^2(1-a)}{\left(\sigma_n^{(1)}\right)^2\left(\mu_n^{(1)} - \mu_n^{(0)}\right)^2},$$

where $u(\gamma)$ is, as before, γ-quantile of standard normal distribution, we have

$$\left|\beta_{n,k} - \Phi\left(\sqrt{k}(\mu_n^{(1)} - \mu_n^{(0)}) - u(1-a)\frac{\sigma_n^{(0)}}{\sigma_n^{(1)}}\right)\right| \leqslant A_0\frac{\left(m_n^{(1)}\right)^3}{\left(\sigma_n^{(1)}\right)^3\sqrt{k}}.$$

Hence,

$$\sqrt{k}(1-\beta_{n,k}) \leqslant A_0\frac{\left(m_n^{(1)}\right)^3}{\left(\sigma_n^{(1)}\right)^3} + \frac{\exp\left\{-\frac{1}{2}\left[u(1-a)\sigma_n^{(0)}/\sigma_n^{(1)} - \sqrt{k}(\mu_n^{(1)} - \mu_n^{(0)})\right]^2\right\}}{\sqrt{2\pi}\left[\mu_n^{(1)} - \mu_n^{(0)} - \frac{1}{\sqrt{k}}u(1-a)\sigma_n^{(0)}/\sigma_n^{(1)}\right]},$$

where A_0 is an absolute constant, $A_0 \leqslant 0.7655$.

Particularly, if $k=n$, $\mu_n^{(i)} \equiv \mu_i$ and $\left(m_n^{(i)}\right)^3/\left(\sigma_n^{(i)}\right)^3 < M < \infty$, $i=0,1$, then

$$1-\beta_{n,n} = O\left(n^{-1/2}\right).$$

4.2. The case of heavy tails

Now suppose that the alternative is such that (2) holds with $B_n(k) \equiv b_n$ (in this case we say that general distribution function $F_n(x)$ (under alternative $H_n^{(1)}$) of random variables $\{\ell_n^{(1)}(X_{n,j}) - \ell_n^{(0)}(X_{n,j})\}_{j\geqslant 1}$ belongs to the domain of *normal* attraction of stable law G_α).

We need the estimate for the value

$$\Delta_{n,k} = \sup_x\left|\mathsf{P}_n^{(1)}\left(\frac{\Lambda_{n,k} - k\mu_n^{(1)}}{b_n k^{1/\alpha}} < x\right) - G_\alpha(x)\right|.$$

Let us use the estimates from [10]. To do so, denote characteristic functions, corresponding to distributions $F_n(x)$ and $G_\alpha(x)$, by $f_n(s)$ and $g_\alpha(s)$ respectively, $s\in\mathbf{R}$. Let $r>0$ and $\nu_{n,r} = \nu_{n,r}(F_n - G_\alpha)$ is such a functional of difference between distribution functions $F_n(x)$ and $G_\alpha(x)$, that for some $M_n>0$, $\varepsilon_n>0$ and all $s\in(-\varepsilon_n, \varepsilon_n)$ the following inequality holds:

$$|f_n(s) - g_\alpha(s)| \leqslant M_n|s|^r\nu_{n,r}. \tag{7}$$

Let us note without additional notations, that in different papers (se, e.g., [1–5,9] and others) $\nu_{n,r}$ was chosen as various difference pseudomoments.

In [10] there is a statement (Theorem 3.11), from which the following proposition is derived.

Theorem 1. *Let* (7) *hold, and* $\alpha < r \leqslant 1+\alpha$. *There exists a constant* $Q_0(r,\alpha)$, *depending only on* α_1 *and* r, *such that*

$$\Delta_{n,k} \leqslant Q_0(r,\alpha)\frac{\max\left\{\nu_{n,r}\,\nu_{n,r}^{1/(r+1)}\right\}}{k^{(r-\alpha)/\alpha}}$$

for all n, $k\geqslant 1$.

Let us consider different possible options for the location of parameter α.

4.2.1. The case $\alpha > 1$

From (4), using Theorem 1, we derive the following proposition.

Theorem 2. *Let $\alpha > 1$ and (7) hold. Besides, let $\alpha < r \leqslant 1+\alpha$. There exists a constant $Q_0(r,\alpha)$, depending only on α and r, such that*

$$\left|\beta_{n,k} - \left[1 - G_\alpha\left(u(1-a)\frac{\sigma_n^{(0)}}{b_n}k^{1/2-1/\alpha} - \frac{k^{1-1/\alpha}}{b_n}(\mu_n^{(1)} - \mu_n^{(0)})\right)\right]\right| \leqslant$$

$$\leqslant Q_0(r,\alpha)\frac{\max\left\{\nu_{n,r}\,\nu_{n,r}^{1/(r+1)}\right\}}{k^{(r-\alpha)/\alpha_1}}$$

for all n, $k \geqslant 1$.

Particularly, consider the case when $k = n$, $p_n^{(i)}(x) \equiv p^{(i)}(x)$, $i = 0,1$. Then $\mu_n^{(i)} \equiv \mu^{(i)}$, $\sigma_n^{(0)} \equiv \sigma^{(0)}$, $b_n = b$, $\nu_{n,r} = \nu_r$. Using the well-known property of stable law tails:

$$G_\alpha(-x) + 1 - G_\alpha(x) \sim \frac{C(\alpha)}{x^\alpha}$$

as $x \to \infty$, where $C(\alpha) > 0$, we derive from Theorem 2

$$1 - \beta_{n,n} \leqslant G_\alpha\left(u(1-a)\frac{\sigma^{(0)}}{b}n^{1/2-1/\alpha} - \frac{n^{1-1/\alpha}}{b}(\mu^{(1)} - \mu^{(0)})\right) +$$

$$+Q_0(r,\alpha)\frac{\max\left\{\nu_r\,\nu_r^{1/(r+1)}\right\}}{n^{(r-\alpha)/\alpha}} = O\left(n^{1-\alpha}\right) + O\left(n^{(\alpha-r)/\alpha}\right).$$

Therefore:

1) if $1 < \alpha < r \leqslant \min\{\alpha^2, \alpha+1\}$, then $1 - \beta_{n,n} = O\left(n^{(\alpha-r)/\alpha}\right)$;

2) if $\alpha^2 < r < \alpha + 1$, then $1 - \beta_{n,n} = O\left(n^{1-\alpha}\right)$.

4.2.2. The case $\alpha \leqslant 1$

In this case

$$1 - \beta_{n,k} \leqslant G_\alpha\left(u(1-a)\frac{\sigma_n^{(0)}}{B_n(k)}k^{1/2-1/\alpha} + \frac{k^{1-1/\alpha}}{B_n(k)}\mu_n^{(0)}\right) + Q_0(r,\alpha)\frac{\max\left\{\nu_{n,r}\,\nu_{n,r}^{1/(r+1)}\right\}}{k^{(r-\alpha)/\alpha}}.$$

Particularly, if $G_\alpha(0) = 0$, and $k = n$, $\sigma_n^{(0)} = \sigma^{(0)}$, and $\mu_n^{(0)} = \mu^{(0)}$, then

$$1 - \beta_{n,n} = O\left(n^{(\alpha-r)/\alpha}\right),$$

since for $\alpha \leqslant 1$ we have $1/2 - 1/\alpha < 1 - 1/\alpha$, and hence, starting from some n, the argument of function G_α in the expression for $1 - \beta_{n,n}$ becomes negative, because, as is established above, $\mu^{(0)} < 0$.

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