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VARIETY OF INTEGRABLE CASES IN DYNAMICS OF LOW- AND MULTI-DIMENSIONAL RIGID BODIES IN NONCONSERVATIVE FORCE FIELDS

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Abstract. This paper is a survey of integrable cases in dynamics of two-, three-, and four-dimensional rigid bodies under the action of a nonconservative force field. We review both new results and results obtained earlier. Problems examined are described by dynamical systems with so-called variable dissipation with zero mean.

CONTENTS

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To the bright memory of Professor Vladimir Aleksandrovich Kondratiev

One thing I have learned in a long life: that all our science, measured against reality, is primitive and childlike—and yet it is the most precious thing we have.

Albert Einstein,
letter to Hans Muehsam, July 9, 1951.
Einstein Archives 38-408

Introduction

We study nonconservative systems for which the usual methods of the study, e.g., Hamiltonian systems, are inapplicable. Thus, for such systems, we must "directly" integrate the main equation of dynamics. We generalize previously known cases and obtain new cases of the complete integrability in transcendental functions of the equation of dynamics of a four-dimensional rigid body in a nonconservative force field.

Of course, in the general case, the construction of a theory of integration of nonconservative systems (even of low dimension) is a quite difficult task. In a number of cases, where the systems considered have additional symmetries, we succeed in finding first integrals through finite combinations of elementary functions (see [95–97]).

We obtain a series of complete integrable nonconservative dynamical systems with nontrivial symmetries. Moreover, in almost all cases, all first integrals are expressed through finite combinations of elementary functions; these first integrals are transcendental functions of their variables. In this case, the transcendence is understood in the sense of complex analysis, when the analytic continuation of a function into the complex plane has essentially singular points. This fact is caused by the existence of attracting and repelling limit sets in the system (for example, attracting and repelling focuses).

We detect new integrable cases of the motion of a rigid body, including the classical problem of the motion of a multi-dimensional spherical pendulum in a flowing medium.

Chapter 1 is devoted to general aspects of the integrability of dynamical systems with variable dissipation. First, we propose a descriptive characteristic of such systems. The term "variable dissipation" refers to the possibility of alternation of its sign rather than to the value of the dissipation coefficient (therefore, it is more reasonable to use the term "sign-alternating").

Later, we define systems with variable dissipation with zero (nonzero) mean based on the divergence of the vector field of the system, which characterizes the change of the phase volume in the phase space of the system considered (see [21, 22, 26, 27, 30, 104, 113, 172]).

We introduce a class of autonomous dynamical systems with one periodic phase coordinate possessing certain symmetries that are typical for pendulum-type systems. We show that this class of systems can be naturally embedded in the class of systems with variable dissipation with zero mean, i.e., on the average for the period with respect to the periodic coordinate, the dissipation in the system is equal to zero, although in various domains of the phase space, either energy pumping or dissipation can occur, but they balance to each other in a certain sense. We present some examples of pendulum-type systems on lower-dimension manifolds from dynamics of a rigid body in a nonconservative field.

Then we study certain general conditions of the integrability in elementary functions for systems on the two-dimensional plane and the tangent bundles of a one-dimensional sphere (i.e., the two-dimensional cylinder) and a two-dimensional sphere (a four-dimensional manifold). Therefore, we propose an interesting example of a three-dimensional phase portrait of a pendulum-like system which describes the motion of a spherical pendulum in a flowing medium (see [143–145]).

For multi-parametric third-order systems, we present sufficient conditions of the existence of first integrals that are expressed through finite combinations of elementary functions.

We deal with three properties that seem, at first glance, to be independent:

(1) a class of systems with symmetries specified above;
(2) the fact that this class consists of systems with variable dissipation with zero mean (with respect to the existing periodic variable), which allows us to consider them as "almost" conservative systems;
(3) in certain (although lower-dimensional) cases, these systems have a complete set of first integrals, which, in general, are transcendental (in the sense of complex analysis).

In Chaps. 2 and 3, we systematize the obtained results on the study of dynamical equations of motion for symmetrical two-dimensional ($2D$-) rigid body in a nonconservative force field. The form of these equations is taken from the dynamics of realistic rigid bodies that interact with a resisting medium by the laws of jet flow when the motion is influenced by a nonconservative tracing force. Under the action of this force, the following two cases are possible. In the first case, the velocity of some characteristic point of the body remains constant, which means that the system possesses a nonintegrable servo constraint (Chap. 2). In the second case, the body is subjected to a nonconservative tracing force such that throughout the motion the center of mass of the body moves rectilinearly and uniformly; this means that there exists a nonconservative couple of forces in the system (Chap. 3); see [1, 64, 70, 72, 119–121, 157, 164–167, 180–182, 184, 191, 194, 198, 212, 231, 237, 242, 258, 291, 346, 351–354, 374, 390, 414]).

Moreover, in Chap. 2, for an analytical nonintegrable constraint we find an additional transcendental first integral expressed through a finite combination of elementary functions, and in Chap. 3, we find an additional transcendental first integral for an analytical first integral (the square of the velocity of the center of mass).

New obtained results are systematized and are given in invariant form. Moreover, an additional dependence of the moment of nonconservative forces on the angular velocity is introduced. The given dependence can also be generalized to higher-dimensional cases.

In Chaps. 4 and 5, we systematize results on the study of dynamical equations of motion for symmetric three-dimensional ($3D$-) rigid bodies in nonconservative force fields. The form of these equations is also taken from the dynamics of realistic rigid bodies interacting with resisting media by laws of jet flow when the motion is influenced by a nonconservative tracing force. Under the action of this force, the following two cases are possible. In the first case, the velocity of some characteristic point of the body remains constant, which means that the system possesses a nonintegrable servo constraint (Chap. 4). In the second case, the body is subjected to a nonconservative tracing force such that throughout the motion the center of mass of the body moves rectilinearly and uniformly; this means that there exists a nonconservative couple of forces in the system (Chap. 5); see [1, 29, 64, 70, 72, 92–94, 99, 102, 119–122, 146, 147, 157–159, 164–167, 170, 180–182, 184, 191, 194, 198, 212, 231, 237, 242, 252, 258, 259, 277, 343, 346, 351–354, 374, 390, 394, 398–401, 409–411, 414–416, 418–420, 424–426]).

Moreover, in Chap. 4, in addition to analytical invariant relations (a nonintegrable constraint and an integral expressing the vanishing of one of the components of the angular velocity), we find three additional transcendental first integrals expressed through finite combination of elementary functions. Similarly, in Chap. 5, we find three additional transcendental first integrals in addition to analytical first integrals (the square of the velocity of the center of mass and the integral expressing the vanishing of one of the components of the angular velocity).

New results are also systematized and are presented given in an invariant form. An additional dependence of the moment of the nonconservative force on the angular velocity is introduced. This dependence can also be generalized to higher-dimensional cases.

In Chap. 6, we present general aspects of the dynamics of a free multi-dimensional rigid body, i.e., the notion of the tensor of the angular velocity, dynamical equations of motion on the direct product $\mathbf{R}^n \times \mathrm{so}(n)$, the Euler and Rivals formulas in the multi-dimensional case, etc.

We also consider the tensor of inertia of a four-dimensional (4D) rigid body. In this work, we study two possible cases in which there exist *two* relations between the principal moments of inertia:

(i) there are *three* equal principal moments of inertia ($I_2 = I_3 = I_4$);
(ii) there are *two pairs* of equal principal moments of inertia ($I_1 = I_2$ and $I_3 = I_4$).

In Chaps. 6 and 7, we systematize results on the study of equations of motion of a four-dimensional (4D) rigid body in a nonconservative force field for the case (i). The form of these equations is taken from the dynamics of realistic rigid bodies of lesser dimension that interact with a resisting medium by laws of jet flow when the body is influenced by a nonconservative tracing force. Under the action of this force, the following two cases are possible. In the first case, the velocity of some characteristic point of the body remains constant, which means that the system possesses a nonintegrable servo constraint (see Chap. 6). In the second case, the body is subjected to a nonconservative tracing force such that throughout the motion the center of mass of the body moves rectilinearly and uniformly; this means that there exists a nonconservative couple of forces in the system (see Chap. 7); see also [1, 29, 36–39, 45, 47, 64, 70, 72, 119–121, 157, 164–167, 180–184, 191, 194, 198, 212, 221–223, 229, 231, 233, 237, 242, 247, 248, 251, 257, 258, 278, 280, 291, 346, 351–354, 374, 390, 402, 403, 414].

Moreover, in Chap. 6, in addition to the four existing analytic invariant relations (a nonintegrable connection and three integrals that show the vanishing of the components of the tensor of angular velocity), we obtain four additional transcendental first integrals expressed as finite combinations of elementary functions. In Chap. 7, we find additional transcendental first integrals in addition to the four known analytic first integrals (the square of the velocity of the center of mass and the three integrals that show the vanishing of the components of the tensor of angular velocity).

The results relate to the case where all interaction of the medium with the body part is concentrated on a part of the surface of the body, which has the form of a three-dimensional disk, and the action

Table 1. Classification of integrable cases presented in this paper

Dimension of a Rigid Body	Constraint Conditions	
	$v \equiv \text{const}$ ($\beta_2 \equiv \text{const}$)	$\mathbf{V}_C \equiv \mathbf{const}$
$\mathbf{E}^2$	$h = 0$ ⊕ $h \neq 0$ ⊕	$h = 0$ ⊕ $h \neq 0$ ⊕
$\mathbf{E}^3$ $(I_2 = I_3)$	$h = 0$ ⊕ $h \neq 0$ ⊕	$h = 0$ ⊕ $h \neq 0$ ⊕
$\mathbf{E}^4$ $(I_2 = I_3 = I_4)$	$h = 0$ ⊕ $h \neq 0$ ⊕	$h = 0$ ⊕ $h \neq 0$ ⊕
$\mathbf{E}^4$ $(I_1 = I_2,\ I_3 = I_4)$	$h = 0$ ⊕ $h \neq 0$ ⊕	$h = 0$ ⊖ $h \neq 0$ ⊖

of the force is concentrated in a direction perpendicular to this disk. These results are systematized and are presented in invariant form. Moreover, we introduce an extra dependence of the moment of the nonconservative force on the angular velocity. This dependence can be further extended to cases of motion in spaces of higher dimension.

In Chap. 8, we systematize results on the study of the equations of motion of a symmetric four-dimensional rigid body in a nonconservative force field for the case (ii). The form of these equations is taken from the lower-dimensional dynamics of realistic rigid bodies that interact with a resisting medium by laws of jet flow when the body is influenced by a nonconservative tracing force. Under the action of this force, the magnitude of the velocity of a certain typical point of the rigid body and a certain phase variable remain constant; this means that the system possesses nonintegrable servo-constraints (see also [1, 64, 70, 72, 119–121, 139, 157, 164–167, 182, 184, 191, 198, 237, 242, 294, 310, 318, 321, 326, 328, 329, 335, 341, 346, 351, 352, 390]).

Moreover, in Chap. 8, in addition to the four existing analytical invariant relations (two nonintegrable constraints and two integrals expressing the vanishing of certain components of the tensor of angular velocity), we find two additional transcendental and three analytical first integrals expressed through finite combinations of elementary functions.

The results relate to the case where all interaction of the medium with the body part is concentrated on a part of the surface of the body that has the form of a two-dimensional disk, and the action of the force is concentrated in a direction perpendicular to this disk. These results are systematized and are presented in invariant form. Moreover, we introduce an extra dependence of the moment of the nonconservative force on the angular velocity. This dependence can be further extended to cases of the motion in spaces of higher dimension.

Thus, in Chap. 2–8, many integrable cases in lower- and higher-dimensional dynamics of a rigid body in a nonconservative force field are discussed. All these cases are listed in Table 1.

The notation $h = 0$ (or $h \neq 0$) means that the force field, respectively depends (does not depend) on the components of the angular velocity tensor.

The sign ⊕ means that the corresponding case is discussed in the present survey; the two occurrences of the sign ⊖ in the lower right corner of the table mean that these two cases are not discussed here (indeed, Chap. 8 is devoted to the case $I_1 = I_2$, $I_3 = I_4$).

Nevertheless, for a symmetric n-dimensional rigid body with $I_2 = \cdots = I_n$, some results have already been obtained; they are not included in the present survey.

Many of the results of this paper were presented earlier in a regular basis at various scientific seminars, including the seminar "Actual problems of geometry and mechanics" named after Prof. V. V. Trofimov [78] under the supervision of D. V. Georgievsky and M. V. Shamolin (see also [1, 2, 67, 68, 70–73, 75, 79–84, 86–88, 230, 231, 261–264, 266, 269, 270, 272, 274, 275, 295, 296, 303, 304, 306, 311, 340, 344]).

Chapter 1

INTEGRABILITY IN ELEMENTARY FUNCTIONS OF CERTAIN CLASSES OF NONCONSERVATIVE SYSTEMS

We study nonconservative systems for which the usual methods of study, e.g., Hamiltonian systems, are inapplicable. Thus, for such systems, we must "directly" integrate the main equation of dynamics. We recall known facts in a more universal form and also present some new cases of the complete integrability in transcendental functions in the dynamics of 2D-, 3D-, and 4D-rigid bodies in a nonconservative field.

The results of the present paper have been developed from previous studies, including an applied problem from the dynamics of a rigid body [1, 64, 70, 72, 119–121, 157, 164–167, 182, 184, 191, 198, 237, 242, 346, 351, 352, 390], for which complete lists of transcendental first integrals that can be expressed through finite combinations of elementary functions were obtained. Later, this allowed one to perform a complete analysis of all phase trajectories and to specify those properties that are preserved for systems of a more general form. The complete integrability of such systems is related to hidden symmetries.

As is known, the notion of integrability is, generally speaking, quite vague. We must always take into account in what sense this notion is understood (what criterion allows one to judge whether trajectories of the dynamical system considered are simple in one or another sense), in what functional class first integrals are sought, etc. (see [23, 24, 31, 90, 103, 105, 109, 110, 123, 131]).

In this paper, we consider first integrals that belong to the functional class consisting of transcendental elementary functions. Here the term "transcendental" is meant in the sense of complex analysis, i.e., a transcendental function is a function that possesses essential singularities after analytic continuation in the complex plane (see also [132, 148, 153, 265]).

1. Preliminaries

The construction of a theory of integration of nonconservative systems (even lower-dimensional) is a difficult problem. However, in some cases where systems studied possess additional symmetries, one can find first integrals in the form of finite combinations of elementary functions [265, 271, 276, 279, 281, 287–290, 297–302, 314, 332, 334, 336–339].

The present paper is a development of the plane problem on the motion of a rigid body in a resisting medium in which the domain of the contact between the body and the medium is a planar part of the exterior surface of the body. The force field in this problem is constructed by accounting for the action of the medium on the body in the quasi-stationary jet or separated flow. It turns out that the study of such motions can be reduced to systems either with dissipation of energy [(purely) dissipative systems or systems in dissipative force fields] or to systems with energy pumping (so-called systems with antidissipation or systems with accelerating forces). Note that similar problems appeared earlier in applied aerodynamics (see also [265, 271, 276, 279, 281, 287–290, 297–302, 314, 332, 334, 336–339]).

Problems considered earlier stimulated the development of qualitative tools that substantially supplement the qualitative theory of nonconservative systems with dissipation of either sign (see also [265]).

Nonlinear effects in problems of the plane and spatial dynamics of a rigid body were examined by qualitative methods. We justify the necessity of the introduction of the notions of relative roughness and relative nonroughness of different orders (see also [184, 188, 190, 191, 197, 208, 227, 265, 282]).

In the present work, the following results are obtained.

(1) We develop methods of qualitative analysis of dissipative and antidissipative systems, which allows us to obtain bifurcation conditions for the appearance of stable and unstable self-oscillations and conditions of the absence of singular trajectories. We succeed in the study of plane topographical Poincaré systems and comparison systems and generalize them to higher dimensions. We obtain sufficient Poisson-stability conditions (everywhere density near itself) of some classes of nonclosed trajectories of dynamical systems (see [181, 194, 198, 199, 206, 228, 327, 351, 361, 362, 370, 371, 375, 389, 397];

(2) in 2D- and 3D-dynamics of a rigid body, we obtain complete lists of first integrals of dissipative and antidissipative systems that are transcendental (in the sense of the classification of their singularities) functions, which, in some cases, can be expressed through elementary functions. We introduce the notions of relative roughness and relative nonroughness of different orders for integrated systems (see [184, 188, 190, 191, 197, 208, 227, 265, 282]);

(3) we obtain multi-parameter families of topologically nonequivalent phase portraits that appear in purely dissipative systems (i.e., systems with variable dissipation with nonzero (positive) mean). Almost all portraits of such families are (absolutely) rough (see [265];

(4) we detect new qualitative analogies between the motion of a free body in a resisting medium and the motion of a fixed body in a flowing medium (see [265]).

2. Dynamical Systems with Variable Dissipation as a Class of Systems Admitting Complete Integration

2.1. Descriptive characteristics of dynamical systems with variable dissipation. At initial modeling of the action of a medium on a rigid body, we use experimental information on the properties of jet flow, and the necessity of the study of the class of dynamical systems that possess the property of (relative) roughness (relative structural stability) naturally appears. Therefore, it is natural to introduce these notions for such systems. Herewith, many of the systems considered are rough in the sense of Andronov and Pontryagin (see [15–19]).

After some transformations (for example, in the 2D-dynamics), the dynamical part of the general system of the equations of plane-parallel motion can be reduced to a pendulum system of second order containing a linear nonconservative (sign-alternating dissipative) force with a coefficient, which can change sign for different values of the periodic phase coordinate of the system. Thus, in this case, we speak of systems with so-called variable dissipation, where the term "variable" refers not only to the value of the dissipation coefficient but to its sign (and so the term "sign-alternating" is more adequate).

In the average over a period (with respect to the periodic coordinate), dissipation can be positive ("purely" dissipative systems), negative (systems with accelerating forces), or zero (but it does not vanish identically). In the last case, we speak of systems with variable dissipation with zero mean (these systems can be associated with "almost" conservative systems).

As was noted above, we obtain important mechanical analogies appearing in the comparison of qualitative properties of a free body and the equilibrium of a pendulum in the flow of a medium. Such analogies have a deep sense since they allow one to transfer properties of the nonlinear dynamical system for a pendulum to the dynamical system for a free body. Both systems belong to the class of so-called pendulum dynamical systems with variable dissipation with zero mean.

Under additional conditions, the equivalence described above can be spread to the case of the spatial motion, which allows one to speak of a general character of symmetries of systems with variable dissipation with zero mean in plane-parallel and spatial motions (for planar and spatial versions of a pendulum in a flow of a medium, see also [265]).

In the sequel, we present some classes of nonlinear systems of the second, third, and higher orders that are integrable in the class of transcendental (in the sense of the theory of functions of complex variables) elementary functions, for example, five-parameter dynamical systems including the majority of systems examined earlier in the dynamics of a low-dimensional (2D and 3D) rigid body interacting with a medium:

$$\begin{aligned} \dot\alpha &= a\sin\alpha + b\omega + \gamma_1\sin^5\alpha + \gamma_2\omega\sin^4\alpha + \gamma_3\omega^2\sin^3\alpha + \gamma_4\omega^3\sin^2\alpha + \gamma_5\omega^4\sin\alpha, \\ \dot\omega &= c\sin\alpha\cos\alpha + d\omega\cos\alpha + \gamma_1\omega\sin^4\alpha\cos\alpha + \gamma_2\omega^2\sin^3\alpha\cos\alpha + \gamma_3\omega^3\sin^2\alpha\cos\alpha + \\ &\quad + \gamma_4\omega^4\sin\alpha\cos\alpha + \gamma_5\omega^5\cos\alpha. \end{aligned}$$

In this connection, we have introduced the notions of relative structural stability (relative roughness) and relative structural instability (relative nonroughness) of various degrees. These properties were proved for systems that arise, e.g., in [265].

Purely dissipative dynamical systems [and also (purely) antidissipative], which, in our case, can belong to the class of systems with variable dissipation with nonzero mean, are, as a rule, structurally stable [(absolutely) rough], whereas systems with variable dissipation with zero mean (which usually possess additional symmetries) are either structurally unstable (nonrough) or only relatively structurally stable (relatively rough). However, the proof of the last assertion in the general case is a difficult problem.

For example, the dynamical system of the form

$$\begin{aligned} \dot\alpha &= \Omega + \beta\sin\alpha, \\ \dot\Omega &= -\beta\sin\alpha\cos\alpha \end{aligned} \tag{1.1}$$

is relatively structurally stable (relatively rough) and is topologically equivalent to the system describing a fixed pendulum in a flowing medium (see [265]).

Below we present its first integral, which is a transcendental (in the sense of the theory of functions of a complex variable, as a function whose analytical continuation in the complex plane has essential singularities) function of phase variables that can be expressed through a finite combination of elementary functions (see [265]). The phase cylinder $\mathbb{R}^2\{\alpha, \Omega\}$ of quasi-velocities of the system considered has an interesting topological structure of the splitting into trajectories (for more detail, see [265]).

Although the dynamical system considered is not conservative, in the rotational domain (and only in it) of its phase plane $\mathbb{R}^2\{\alpha, \Omega\}$, it admits the preservation of the invariant measure with variable density. This property characterizes this system as a system with variable dissipation with zero mean (see [67, 68, 243, 260, 262, 265, 282–286, 291, 295, 309, 324, 333, 340, 384, 391, 404–408, 412, 413, 421, 437, 438]).

2.2. Definition of a system with variable dissipation with zero mean. We study systems of ordinary differential equations that have a periodic phase coordinate. Such systems possess symmetries under which their average phase volume with respect to the periodic coordinate is preserved. For example, the following pendulum system, with a smooth and periodic (of period T) with respect to α

right-hand side $\mathbf{V}(\alpha, \omega)$ of the form

$$\begin{aligned} \dot{\alpha} &= -\omega + f(\alpha), \;\; f(\alpha + T) = f(\alpha), \\ \dot{\omega} &= g(\alpha), \;\; g(\alpha + T) = g(\alpha), \end{aligned} \tag{1.2}$$

preserved its phase area on the phase cylinder within the period T:

$$\int\limits_0^T \operatorname{div} \mathbf{V}(\alpha, \omega) d\alpha = \int\limits_0^T \left(\frac{\partial}{\partial \alpha}(-\omega + f(\alpha)) + \frac{\partial}{\partial \omega} g(\alpha) \right) d\alpha = \int\limits_0^T f'(\alpha) d\alpha = 0. \tag{1.3}$$

This system is equivalent to the equation of a pendulum

$$\ddot{\alpha} - f'(\alpha)\dot{\alpha} + g(\alpha) = 0, \tag{1.4}$$

in which the integral of the coefficient $f'(\alpha)$ of the dissipative term $\dot{\alpha}$ over the period is equal to zero. We see that this system has symmetries under which it becomes a system with *variable dissipation with zero mean* in the sense of the following definition (see [265]).

Definition 1.1. Consider a smooth autonomous system of order $(n+1)$ in the normal form defined on the cylinder $\mathbb{R}^n\{x\} \times \mathbf{S}^1\{\alpha \mod 2\pi\}$, where α is a periodic coordinate of period $T > 0$. The divergence of the right-hand side $\mathbf{V}(x, \alpha)$ (which, in general, is a function of all phase variables and does not vanish identically) of this system is denoted by $\operatorname{div} \mathbf{V}(x, \alpha)$. This system is called a system with variable dissipation with zero (respectively, nonzero) mean if the function

$$\int\limits_0^T \operatorname{div} \mathbf{V}(x, \alpha) d\alpha \tag{1.5}$$

vanishes (respectively, does not vanish) identically. In some cases (for example, when at some points of the circle $\mathbf{S}^1\{\alpha \mod 2\pi\}$ singularities appear), this integral is meant in the sense of the principal value.

We note that it is quite difficult to give a general definition to a system with variable dissipation with zero (nonzero) mean. The definition presented above is based on the notion of the divergence (as is known, the divergence of the right-hand side of a system in the normal form characterizes the change of the phase volume in the phase space of the given system).

3. Systems with Symmetries and Variable Dissipation with Zero Mean

Consider a system of the following form (the dot means the derivative with respect to time):

$$\begin{aligned} \dot{\alpha} &= f_\alpha(\omega, \sin\alpha, \cos\alpha), \\ \dot{\omega_k} &= f_k(\omega, \sin\alpha, \cos\alpha), \quad k = 1, \dots, n, \end{aligned} \tag{1.6}$$

defined on the set

$$\mathbf{S}^1\{\alpha \mod 2\pi\} \setminus K \times \mathbb{R}^n\{\omega\}, \quad \omega = (\omega_1, \dots, \omega_n), \tag{1.7}$$

where sufficiently smooth functions $f_\lambda(u_1, u_2, u_3)$, $\lambda = \alpha, 1, \dots, n$, of three variables u_1, u_2, u_3 are such that

$$\begin{aligned} f_\lambda(-u_1, -u_2, u_3) &= -f_\lambda(u_1, u_2, u_3), \\ f_\alpha(u_1, u_2, -u_3) &= f_\alpha(u_1, u_2, u_3), \\ f_k(u_1, u_2, -u_3) &= -f_k(u_1, u_2, u_3); \end{aligned} \tag{1.8}$$

moreover, the functions $f_k(u_1, u_2, u_3)$ are defined for $u_3 = 0$ for any $k = 1, \dots, n$.

The set K is either empty or consists of a finite number of points of the circle $\mathbf{S}^1\{\alpha \mod 2\pi\}$.

The last two variables u_2, u_3 in the functions $f_\lambda(u_1, u_2, u_3)$ depend on the same parameter α, but we assume that these variables belong to different groups for the following reasons: first, they cannot be uniquely expressed one through another on their whole domain and, second, u_2 is an odd function of α, whereas u_3 is even, which affects the symmetries of system (1.6).

To system (1.6), we put in correspondence the following nonautonomous system:

$$\frac{d\omega_k}{d\alpha} = \frac{f_k(\omega, \sin\alpha, \cos\alpha)}{f_\alpha(\omega, \sin\alpha, \cos\alpha)}, \quad k = 1, \ldots, n. \tag{1.9}$$

By the substitution $\tau = \sin\alpha$, it can be reduced to the form

$$\begin{gathered} \frac{d\omega_k}{d\tau} = \frac{f_k(\omega, \tau, \varphi_k(\tau))}{f_\alpha(\omega, \tau, \varphi_\alpha(\tau))}, \quad k = 1, \ldots, n, \\ \varphi_\lambda(-\tau) = \varphi_\lambda(\tau), \quad \lambda = \alpha, 1, \ldots, n. \end{gathered} \tag{1.10}$$

The last system, in particular, can have an algebraic right-hand side (i.e., it can be the ratio of two polynomials), which simplifies the search for its first integrals in explicit form.

The following theorem states that the class of systems (1.6) is a subclass of the class of dynamical systems with variable dissipation with zero mean. Note that, in general, the converse is invalid.

Theorem 1.1. *Systems of the form* (1.6) *are dynamical systems with variable dissipation with zero mean.*

Proof. The proof of this theorem is based on some symmetries (1.8) of system (1.6) listed above and the periodicity of the right-hand side of the system with respect to α.

Indeed, the divergence of the vector field of system (1.6) equals

$$\frac{\partial f_\alpha(\omega, \sin\alpha, \cos\alpha)}{\partial u_2}\cos\alpha - \frac{\partial f_\alpha(\omega, \sin\alpha, \cos\alpha)}{\partial u_3}\sin\alpha + \sum_{k=1}^{n} \frac{\partial f_k(\omega, \sin\alpha, \cos\alpha)}{\partial u_1}. \tag{1.11}$$

The following integral of the first two terms in (1.11) vanishes:

$$\int_0^{2\pi} \left\{ \frac{\partial f_\alpha(\omega, \sin\alpha, \cos\alpha)}{\partial u_2} d\sin\alpha + \frac{\partial f_\alpha(\omega, \sin\alpha, \cos\alpha)}{\partial u_3} d\cos\alpha \right\}$$
$$= \int_0^{2\pi} \frac{\partial f_\alpha(\omega, \sin\alpha, \cos\alpha)}{\partial \alpha} d\alpha = h_\alpha(\omega) \equiv 0, \tag{1.12}$$

since the function $f_\alpha(\omega, \sin\alpha, \cos\alpha)$ is periodic with respect to α.

Further, by the third equation in (1.8), for any $k = 1, \ldots, n$ we have

$$\frac{\partial f_k(\omega, \sin\alpha, \cos\alpha)}{\partial u_1} = \cos\alpha \cdot \frac{\partial g_k(\omega, \sin\alpha)}{\partial u_1}, \tag{1.13}$$

where the function $g_k(u_1, u_2)$ is sufficiently smooth for any $k = 1, \ldots, n$.

Then the integral over the period 2π of the right-hand side of Eq. (1.13) equals

$$\int_0^{2\pi} \frac{\partial g_k(\omega, \sin\alpha)}{\partial u_1} d\sin\alpha = h_k(\omega) \equiv 0 \tag{1.14}$$

for any $k = 1, \ldots, n$. From Eqs. (1.12) and (1.14) we obtain Theorem 1.1. □

The converse assertion is invalid: there exist dynamical systems on the two-dimensional cylinder that are systems with variable dissipation with zero mean, but they do not possess symmetries listed above.

In this paper, we basically consider the case where the functions $f_\lambda(\omega, \tau, \varphi_k(\tau))$ $(\lambda = \alpha, 1, \dots, n)$ are polynomials of ω and τ.

Example 1.1. Below we consider, in particular, pendulum-type systems on the two-dimensional cylinder $\mathbf{S}^1\{\alpha \mod 2\pi\} \times \mathbf{R}^1\{\omega\}$ with parameter $b > 0$ from the rigid body dynamics (see [265]):

$$\begin{aligned} \dot\alpha &= -\omega + b\sin\alpha, \\ \dot\omega &= \sin\alpha\cos\alpha, \end{aligned} \tag{1.15}$$

and

$$\begin{aligned} \dot\alpha &= -\omega + b\sin\alpha\cos^2\alpha + b\omega^2\sin\alpha, \\ \dot\omega &= \sin\alpha\cos\alpha - b\omega\sin^2\alpha\cos\alpha + b\omega^3\cos\alpha, \end{aligned} \tag{1.16}$$

with the variables (ω, τ). To these systems, we can put in correspondence the following equations with algebraic right-hand sides:

$$\frac{d\omega}{d\tau} = \frac{\tau}{-\omega + b\tau} \tag{1.17}$$

and

$$\frac{d\omega}{d\tau} = \frac{\tau + b\omega[\omega^2 - \tau^2]}{-\omega + b\tau + b\tau[\omega^2 - \tau^2]}, \tag{1.18}$$

which have the form (1.10). Moreover, one can easily verify that these systems are dynamical systems with variable dissipation with zero mean. Indeed, the divergences of their right-hand sides are equal to

$$b\cos\alpha, \quad b\cos\alpha[4\omega^2 + \cos^2\alpha - 3\sin^2\alpha],$$

respectively. It is easy to see that they belong to the class of systems (1.6).

Moreover, each of these systems has a first integral that is a transcendental (in the sense of complex analysis) function expressed through a finite combination of elementary functions (see Chaps. 2 and 3 below).

We present another important example of a higher-order system that possesses the properties listed above.

Example 1.2. Consider the following system with a parameter b, which is defined in the three-dimensional domain

$$\mathbf{S}^1\{\alpha \mod 2\pi\} \setminus \{\alpha = 0, \alpha = \pi\} \times \mathbb{R}^2\{z_1, z_2\} \tag{1.19}$$

(this system is separated from a system on the tangent bundle $T_*\mathbf{S}^2$ of the two-dimensional sphere $\mathbf{S}^2$):

$$\begin{aligned} \dot\alpha &= -z_2 + b\sin\alpha, \\ \dot z_2 &= \sin\alpha\cos\alpha - z_1^2\frac{\cos\alpha}{\sin\alpha}, \\ \dot z_1 &= z_1 z_2\frac{\cos\alpha}{\sin\alpha}. \end{aligned} \tag{1.20}$$

This system describes the motion of a rigid body in a resistive medium (see Chaps. 4 and 5). We put in correspondence to this system the following nonautonomous system with algebraic right-hand side $(\tau = \sin\alpha)$:

$$\begin{aligned} \frac{dz_2}{d\tau} &= \frac{\tau - z_1^2/\tau}{-z_2 + b\tau}, \\ \frac{dz_1}{d\tau} &= \frac{z_1 z_2/\tau}{-z_2 + b\tau}. \end{aligned} \tag{1.21}$$

We see that system (1.20) is a system with variable dissipation with zero mean. To obtain the full correspondence with the definition, we introduce the new phase variable

$$z_1^* = \ln |z_1|. \tag{1.22}$$

The divergence of the right-hand side of system (1.20) in the Cartesian coordinates α, z_1^*, z_2 is equal to $b\cos\alpha$. Taking into account (1.19), we have (in the sense of principal value)

$$\lim_{\varepsilon\to 0} \int\limits_{\varepsilon}^{\pi-\varepsilon} b\cos\alpha + \lim_{\varepsilon\to 0} \int\limits_{\pi+\varepsilon}^{2\pi-\varepsilon} b\cos\alpha = 0. \tag{1.23}$$

Moreover, this system possesses two first integrals (i.e., a complete set) that are transcendental functions, which can be expressed through a finite combination of elementary functions (see Chaps. 4 and 5). This becomes possible after putting in correspondence to it a system (nonautonomous, generally speaking) of equations with an algebraic (polynomial) right-hand side (1.21).

Systems (1.15), (1.16), and (1.20) belong to the class of systems (1.6), possess a variable dissipation with zero mean, and have a complete set of transcendental first integrals that can be expressed through a finite combination of elementary functions.

So, to find first integrals of the systems considered, it is convenient to reduce systems of the form (1.6) to systems with polynomial right-hand sides (1.10), which allows one to perform integration in elementary functions of the initial system. Thus, we find sufficient conditions for the integrability in elementary functions of systems with polynomial right-hand sides and examine systems of the most general form.

4. Systems on the Plane and Two-Dimensional Cylinder

Earlier, the author proved a series of assertions regarding many-parameter systems of ordinary differential equations with algebraic right-hand side (see, e.g., [265]). We recall some of them.

Proposition 1.1. *A seven-parameter family of systems of equations on the plane* $\mathbb{R}^2\{x, y\}$

$$\begin{aligned} \dot x &= a_1 x + b_1 y + \beta_1 x^3 + \beta_2 x^2 y + \beta_3 x y^2, \\ \dot y &= c_1 x + d_1 y + \beta_1 x^2 y + \beta_2 x y^2 + \beta_3 y^3, \end{aligned} \tag{1.24}$$

possesses a first integral (*in general, transcendental*), *which can be expressed through elementary functions.*

Corollary. *For any parameters* a_1, b_1, c_1, d_1, β_1, β_2, *and* β_3, *the system*

$$\begin{aligned} \dot\alpha &= a_1 \sin\alpha + b_1\omega + \beta_1 \sin^3\alpha + \beta_2\omega\sin^2\alpha + \beta_3\omega^2\sin\alpha, \\ \dot\omega &= c_1\sin\alpha\cos\alpha + d_1\omega\cos\alpha + \beta_1\omega\sin^2\alpha\cos\alpha + \beta_2\omega^2\sin\alpha\cos\alpha + \beta_3\omega^3\cos\alpha \end{aligned} \tag{1.25}$$

on the two-dimensional cylinder $\{(\alpha,\omega) \in \mathbb{R}^2 : \alpha \mod 2\pi\}$ *possesses a first integral* (*in general, transcendental*), *which can be expressed through elementary functions.*

In particular, systems (1.15) and (1.16) can be obtained from this system if

$$a_1 = b, \quad b_1 = -1, \quad c_1 = 1, \quad d_1 = \beta_1 = \beta_2 = \beta_3 = 0$$

and

$$a_1 = b, \quad b_1 = -1, \quad c_1 = 1, \quad d_1 = -b, \quad \beta_1 = -b, \quad \beta_2 = 0, \quad \beta_3 = b,$$

respectively.

The above reasons can be easily generalized. We consider the possibility of the complete integration (in elementary functions) of systems of a more general form: the nonlinearity is characterized by an arbitrary homogeneous form of odd degree $2n-1$.

In this case, we have the following assertion, which is more general than Proposition 1.1.

Proposition 1.2. *The* $(2n+3)$*-parameter family of systems of equations*

$$\begin{aligned} \dot{x} &= a_1 x + b_1 y + \delta_1 x^{2n-1} + \delta_2 x^{2n-2} y + \cdots + \delta_{2n-2} x^2 y^{2n-3} + \delta_{2n-1} x y^{2n-2}, \\ \dot{y} &= c_1 x + d_1 y + \delta_1 x^{2n-2} y + \delta_2 x^{2n-3} y^2 + \cdots + \delta_{2n-2} x y^{2n-2} + \delta_{2n-1} y^{2n-1} \end{aligned} \tag{1.26}$$

on the plane $\mathbb{R}^2\{x,y\}$ *possesses a first integral* (*in general, transcendental*), *which can be expressed through elementary functions.*

Indeed, the family of Eqs. (1.26) depends on $2n-1+4$ independent parameters since the total nonlinearity of an odd degree is characterized by $4n$ parameters subject to $2n+1$ conditions (the other 4 parameters are contained in the linear part).

Corollary. *For any parameters* a_1, b_1, c_1, d_1, *and* $\delta_1,\ldots,\delta_{2n-1}$, *the systems*

$$\begin{aligned} \dot{\alpha} &= a\sin\alpha + b\omega + \delta_1 \sin^{2n-1}\alpha + \delta_2\omega\sin^{2n-2}\alpha + \cdots + \delta_{2n-1}\omega^{2n-2}\sin\alpha, \\ \dot{\omega} &= c\sin\alpha\cos\alpha + d\omega\cos\alpha + \delta_1\omega\sin^{2n-2}\alpha\cos\alpha + \delta_2\omega^2\sin^{2n-3}\alpha\cos\alpha + \cdots + \delta_{2n-1}\omega^{2n-1}\cos\alpha \end{aligned} \tag{1.27}$$

on the two-dimensional cylinder $\{(\alpha,\omega)\in\mathbb{R}^2 : \alpha \mod 2\pi\}$ *possesses a transcendental first integral, which can be expressed through elementary functions.*

Systems (1.15), (1.16), and (1.20) are sufficiently rough (see [265]), but if we break symmetries (1.8) introduced for systems of the general form (1.6) (for example, by introducing additional terms in their right-hand sides), then the number of topologically distinct phase portraits can substantially change.

In [265], we obtained a multi-parametric family of phase portraits of a system with variable dissipation with nonzero mean (whose typical portraits are (absolutely) rough), which is a perturbation of a dynamical system with variable dissipation with zero mean of the form (1.16). This family (as families obtained earlier, see [265]) contains an infinite number of topologically nonequivalent phase portraits on a two-dimensional phase cylinder.

5. Systems on the Tangent Bundle of the Two-Dimensional Sphere

On the tangent bundle $T_*\mathbf{S}^2$ of the two-dimensional sphere $\mathbf{S}^2\{\theta,\psi\}$, we consider the following dynamical system:

$$\begin{aligned} \ddot{\theta} + b\dot{\theta}\cos\theta + \sin\theta\cos\theta - \dot{\psi}^2\frac{\sin\theta}{\cos\theta} &= 0, \\ \ddot{\psi} + b\dot{\psi}\cos\theta + \dot{\theta}\dot{\psi}\left[\frac{1+\cos^2\theta}{\sin\theta\cos\theta}\right] &= 0. \end{aligned} \tag{1.28}$$

This system describes a spherical pendulum in flowing medium (see [265]). Herewith, the system possesses the conservative moment

$$\sin\theta\cos\theta \tag{1.29}$$

and the force moment, which linearly depends of the velocity with a variable coefficient:

$$b\begin{pmatrix}\dot{\theta}\\ \dot{\psi}\end{pmatrix}\cos\theta. \tag{1.30}$$

The other coefficients in the equations are the connection coefficients, namely,

$$\Gamma^{\theta}_{\psi\psi} = -\frac{\sin\theta}{\cos\theta}, \quad \Gamma^{\psi}_{\theta\psi} = \frac{1+\cos^2\theta}{\sin\theta\cos\theta}. \tag{1.31}$$

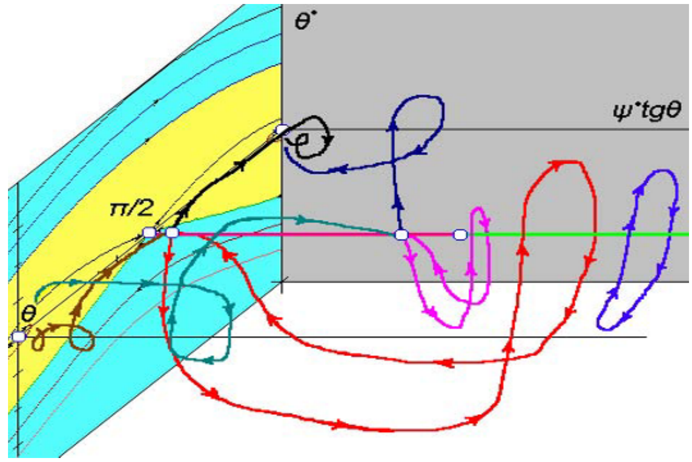

Fig. 1. Relatively rough phase portrait in a three-dimensional domain

In fact, system (1.28) has order 3 since the variable ψ is cyclic and the system contains only the variable $\dot{\psi}$.

Proposition 1.3. *The equation*

$$\dot{\psi} = 0 \tag{1.32}$$

defines a family of integral planes for system (1.28)*.*

Moreover, Eq. (1.32) reduces system (1.28) to the equation that describes a cylindrical pendulum in a flowing medium (see [265]).

Proposition 1.4. *System* (1.28) *is equivalent to the following system*:

$$\begin{aligned} \dot{\theta} &= -z_2 + b\sin\theta, \\ \dot{z_2} &= \sin\theta\cos\theta - z_1^2\frac{\cos\theta}{\sin\theta}, \\ \dot{z_1} &= z_1 z_2\frac{\cos\theta}{\sin\theta}, \\ \dot{\psi} &= z_1\frac{\cos\theta}{\sin\theta} \end{aligned} \tag{1.33}$$

on the tangent bundle $T_*\mathbf{S}^2\{z_1, z_2, \theta, \psi\}$ *of the two-dimensional sphere* $\mathbf{S}^2\{\theta, \psi\}$*.*

Moreover, the first three equations of system (1.33) form a closed system of the third order and coincide with system (1.20) (if we set $\alpha = \theta$). The fourth equation of system (1.33) has been separated due to the cyclicity of the variable ψ.

The construction of the phase portrait of system (1.28) is shown in Fig. 1 (see [265]).

Example 1.3. We examine a system of the form (1.20), which can be reduced to (1.21), and the following system, which appears in the spatial (3D) dynamics of a rigid body interacting with a medium (see Chaps. 4 and 5):

$$\begin{aligned} \dot{\alpha} &= -z_2 + b\left(z_1^2 + z_2^2\right)\sin\alpha + b\sin\alpha\cos^2\alpha, \\ \dot{z_2} &= \sin\alpha\cos\alpha + bz_2\left(z_1^2 + z_2^2\right)\cos\alpha - bz_2\sin^2\alpha\cos\alpha - z_1^2\frac{\cos\alpha}{\sin\alpha}, \\ \dot{z_1} &= bz_1\left(z_1^2 + z_2^2\right)\cos\alpha - bz_1\sin^2\alpha\cos\alpha + z_1 z_2\frac{\cos\alpha}{\sin\alpha}, \end{aligned} \tag{1.34}$$

which corresponds to the following system with algebraic right-hand side:

$$\begin{aligned}\frac{dz_2}{d\tau} &= \frac{\tau + bz_2\left(z_1^2+z_2^2\right) - bz_2\tau^2 - z_1^2/\tau}{-z_2 + b\tau\left(z_1^2+z_2^2\right) + b\tau\left(1-\tau^2\right)},\\ \frac{dz_1}{d\tau} &= \frac{bz_1\left(z_1^2+z_2^2\right) - bz_1\tau^2 + z_1z_2/\tau}{-z_2 + b\tau\left(z_1^2+z_2^2\right) + b\tau(1-\tau^2)}.\end{aligned} \tag{1.35}$$

So, we consider two systems: the initial system (1.34) and the corresponding algebraic system (1.35).

Similarly, we can pass to the homogeneous coordinates u_k, $k=1,2$, by the formulas

$$z_k = u_k\tau. \tag{1.36}$$

By this change of variables, system (1.21) (see above) can be transformed to the form

$$\begin{aligned}\tau\frac{du_2}{d\tau} + u_2 &= \frac{\tau - u_1^2\tau}{-u_2\tau + b\tau},\\ \tau\frac{du_1}{d\tau} + u_1 &= \frac{u_1u_2\tau}{-u_2\tau + b\tau},\end{aligned} \tag{1.37}$$

which, in turn, corresponds to the equation

$$\frac{du_2}{du_1} = \frac{1 - bu_2 + u_2^2 - u_1^2}{2u_1u_2 - bu_1}. \tag{1.38}$$

Since the identity

$$d\left(\frac{1-\beta u_2 + u_2^2}{u_1}\right) + du_1 = 0 \tag{1.39}$$

is integrable, this equation can be integrated in elementary functions and in the coordinates (τ, z_1, z_2) it has the first integral of the form (cf. [265])

$$\frac{z_1^2 + z_2^2 - \beta z_2\tau + \tau^2}{z_1\tau} = \text{const}\,.$$

System (1.34) after reduction corresponds to the system

$$\begin{aligned}\tau\frac{du_2}{d\tau} + u_2 &= \frac{\tau + bu_2\tau^3\left(u_1^2+u_2^2\right) - bu_2\tau^3 - u_1^2\tau}{-u_2\tau + b\tau^3\left(u_1^2+u_2^2\right) + b\tau\left(1-\tau^2\right)},\\ \tau\frac{du_1}{d\tau} + u_1 &= \frac{bu_1\tau^3\left(u_1^2+u_2^2\right) - bu_1\tau^3 + u_1u_2\tau}{-u_2\tau + b\tau^3\left(u_1^2+u_2^2\right) + b\tau\left(1-\tau^2\right)},\end{aligned} \tag{1.40}$$

which can also be reduced to (1.38).

6. Some Generalizations

The following question arises: Can the system

$$\begin{aligned}\frac{dz}{dx} &= \frac{ax + by + cz + c_1z^2/x + c_2zy/x + c_3y^2/x}{d_1x + ey + fz},\\ \frac{dy}{dx} &= \frac{gx + hy + iz + i_1z^2/x + i_2zy/x + i_3y^2/x}{d_1x + ey + fz},\end{aligned} \tag{1.41}$$

possessing a singularity of the type $1/x$, be integrated in elementary functions? This system is a generalization of systems (1.21) and (1.35) in three-dimensional phase domains.

A series of results concerning this question has already been obtained (see [265]); we here present a brief review of these results.

As above, we introduce the substitutions

$$y = ux, \quad z = vx \tag{1.42}$$

and reduce system (1.41) to the following form:

$$x\frac{dv}{dx} + v = \frac{ax + bux + cvx + c_1v^2x + c_2vux + c_3u^2x}{d_1x + eux + fvx}, \tag{1.43}$$

$$x\frac{du}{dx} + u = \frac{gx + hux + ivx + i_1v^2x + i_2vux + i_3u^2x}{d_1x + eux + fvx}, \tag{1.44}$$

which is equivalent to

$$x\frac{dv}{dx} = \frac{ax + bux + (c - d_1)vx + (c_1 - f)v^2x + (c_2 - e)vux + c_3u^2x}{d_1x + eux + fvx}, \tag{1.45}$$

$$x\frac{du}{dx} = \frac{gx + (h - d_1)ux + ivx + i_1v^2x + (i_2 - f)vux + (i_3 - e)u^2x}{d_1x + eux + fvx}. \tag{1.46}$$

To this system, we put in correspondence the following nonautonomous equation with algebraic right-hand side:

$$\frac{dv}{du} = \frac{a + bu + cv + c_1v^2 + c_2vu + c_3u^2 - v[d_1 + eu + fv]}{g + hu + iv + i_1v^2 + i_2vu + i_3u^2 - u[d_1 + eu + fv]}. \tag{1.47}$$

Integration of this equation is reduced to integration of the equation in complete differentials

$$\begin{aligned}\left[g + hu + iv + i_1v^2 + i_2vu + i_3u^2 - d_1u - eu^2 - fuv\right] dv \\ = \left[a + bu + cv + c_1v^2 + c_2vu + c_3u^2 - d_1v - euv - fv^2\right] du.\end{aligned} \tag{1.48}$$

Generally speaking, we have a 15-parameter family of equations of the form (1.48). To integrate the last identity in elementary functions as a homogeneous equation, it suffices to impose the following six restrictions:

$$g = 0, \quad i = 0, \quad i_1 = 0, \quad e = c_2, \quad h = c, \quad i_2 = 2c_1 - f. \tag{1.49}$$

We introduce nine parameters $\beta_1, \ldots, \beta_9$ and consider them as independent:

$$\beta_1 = a, \quad \beta_2 = b, \quad \beta_3 = c, \quad \beta_4 = c_1, \quad \beta_5 = c_2, \quad \beta_6 = c_3, \quad \beta_7 = d_1, \quad \beta_8 = f, \quad \beta_9 = i_3. \tag{1.50}$$

Thus, Eq. (1.48) under the conditions (1.49) and (1.50) is reduced to the form

$$\frac{dv}{du} = \frac{\beta_1 + \beta_2u + (\beta_3 - \beta_7)v + (\beta_4 - \beta_8)v^2 + \beta_6u^2}{(\beta_3 - \beta_7)u + 2(\beta_4 - \beta_8)vu + (\beta_9 - \beta_5)u^2}, \tag{1.51}$$

whereas system (1.45), (1.46) is reduced to the form

$$x\frac{dv}{dx} = \frac{\beta_1 + \beta_2u + (\beta_3 - \beta_7)v + (\beta_4 - \beta_8)v^2 + \beta_6u^2}{\beta_7 + \beta_5u + \beta_8v}, \tag{1.52}$$

$$x\frac{du}{dx} = \frac{(\beta_3 - \beta_7)u + 2(\beta_4 - \beta_8)vu + (\beta_9 - \beta_5)u^2}{\beta_7 + \beta_5u + \beta_8v}. \tag{1.53}$$

After this, Eq. (1.51) can be integrated by a finite combination of elementary functions.

Indeed, integrating identity (1.48), we obtain

$$d\left[\frac{(\beta_3 - \beta_7)v}{u}\right] + d\left[\frac{(\beta_4 - \beta_8)v^2}{u}\right] + d[(\beta_9 - \beta_5)v] + d\left[\frac{\beta_1}{u}\right] - d[\beta_2 \ln|u|] - d[\beta_6u] = 0, \tag{1.54}$$

which implies the following invariant relation:

$$\frac{(\beta_3 - \beta_7)v}{u} + \frac{(\beta_4 - \beta_8)v^2}{u} + (\beta_9 - \beta_5)v + \frac{\beta_1}{u} - \beta_2 \ln|u| - \beta_6u = C_1 = \text{const}, \tag{1.55}$$

and then in the coordinates (x,y,z) the first integral

$$\frac{(\beta_4-\beta_8)z^2-\beta_6y^2+(\beta_3-\beta_7)zx+(\beta_9-\beta_5)zy+\beta_1x^2}{yx}-\beta_2\ln\left|\frac{y}{x}\right|=\text{const}\,. \tag{1.56}$$

Therefore, we can confirm the integrability in elementary functions of the following, generally speaking nonconservative, system of third order depending on 9 parameters:

$$\begin{aligned}\frac{dz}{dx}&=\frac{\beta_1x+\beta_2y+\beta_3z+\beta_4z^2/x+\beta_5zy/x+\beta_6y^2/x}{\beta_7x+\beta_5y+\beta_8z},\\ \frac{dy}{dx}&=\frac{\beta_3y+(2\beta_4-\beta_8)zy/x+\beta_9y^2/x}{\beta_7x+\beta_5y+\beta_8z}.\end{aligned} \tag{1.57}$$

Corollary. *On the set*

$$\mathbf{S}^1\{\alpha \mod 2\pi\}\setminus\{\alpha=0,\ \alpha=\pi\}\times\mathbb{R}^2\{z_1,z_2\}, \tag{1.58}$$

the third-order system

$$\begin{aligned}\dot\alpha&=\beta_7\sin\alpha+\beta_5z_1+\beta_8z_2,\\ \dot z_2&=\beta_1\sin\alpha\cos\alpha+\beta_2z_1\cos\alpha+\beta_3z_2\cos\alpha+\beta_4z_2^2\frac{\cos\alpha}{\sin\alpha}+\beta_5z_1z_2\frac{\cos\alpha}{\sin\alpha}+\beta_6z_1^2\frac{\cos\alpha}{\sin\alpha},\\ \dot z_1&=\beta_3z_1\cos\alpha+(2\beta_4-\beta_8)z_1z_2\frac{\cos\alpha}{\sin\alpha}+\beta_9z_1^2\frac{\cos\alpha}{\sin\alpha},\end{aligned} \tag{1.59}$$

depending on 9 parameters, possesses, generally speaking, a transcendental first integral, which can be expressed through elementary functions:

$$\frac{(\beta_4-\beta_8)z_2^2-\beta_6z_1^2+(\beta_3-\beta_7)z_2\sin\alpha+(\beta_9-\beta_5)z_2z_1+\beta_1\sin^2\alpha^2}{z_1\sin\alpha}-\beta_2\ln\left|\frac{z_1}{\sin\alpha}\right|=\text{const}\,. \tag{1.60}$$

In particular, system (1.59) *for* $\beta_1=1$, $\beta_2=\beta_3=\beta_4=\beta_5=\beta_9=0$, $\beta_6=\beta_8=-1$, *and* $\beta_7=b$ *coincides with system* (1.20)*.*

To find an additional first integral of the nonautonomous system (1.41), we can use the first integral (1.56), which is expressed through a finite combination of elementary functions.

First, we transform relation (1.55) as follows:

$$(\beta_4-\beta_8)v^2+[(\beta_9-\beta_5)u+(\beta_3-\beta_7)]\,v+f_1(u)=0, \tag{1.61}$$

where

$$f_1(u)=\beta_1-\beta_6u^2-\beta_2u\ln|u|-C_1u.$$

Formally, v can be found from the relation

$$v_{1,2}(u)=\frac{1}{2(\beta_4-\beta_8)}\left\{(\beta_5-\beta_9)u+(\beta_7-\beta_3)\pm\sqrt{f_2(u)}\right\}, \tag{1.62}$$

where

$$f_2(u)=A_1+A_2u+A_3u^2+A_4u\ln|u|,$$
$$A_1=(\beta_3-\beta_7)^2-4\beta_1(\beta_4-\beta_8),\quad A_2=2(\beta_9-\beta_5)(\beta_3-\beta_7)+4C_1(\beta_4-\beta_8),$$
$$A_3=(\beta_9-\beta_5)^2+4\beta_6(\beta_4-\beta_8),\quad A_4=4\beta_2(\beta_4-\beta_8).$$

Then the required quadrature for the additional (in general, transcendental) first integral (for example, of system (1.52), (1.53) or (1.45), (1.46)), where Eq. (1.53) is used) becomes

$$\int\frac{dx}{x}=\int\frac{[\beta_7+\beta_5u+\beta_8v_{1,2}(u)]du}{(\beta_3-\beta_7)u+(\beta_9-\beta_5)u^2+2(\beta_4-\beta_8)uv_{1,2}(u)}=\int\frac{\left[B_1+B_2u+B_3\sqrt{f_2(u)}\right]du}{B_4u\sqrt{f_2(u)}}, \tag{1.63}$$

where $B_k=\text{const}$, $k=1,\ldots,4$.

The required quadrature for the search for an additional (in general, transcendental) first integral (for system (1.52), (1.53) or (1.45), (1.46), where Eq. (1.52) is used) becomes

$$\int \frac{dx}{x} = \int \frac{[\beta_7 + \beta_5 u(v) + \beta_8 v]dv}{\beta_1 + \beta_2 u(v) + (\beta_3 - \beta_7)v + (\beta_4 - \beta_8)v^2 + \beta_6 u^2(v)}; \tag{1.64}$$

in this case, the function $u(v)$ must be obtained by solving the implicit equation (1.55) with respect to u (which, in the general case, is not obvious).

Sufficient conditions of the expressability of integrals in (1.64) through finite combinations of elementary functions are stated by the following lemma.

Lemma 1.1. *For* $A_4 = 0$*, i.e., for*

$$\beta_2 = 0 \tag{1.65}$$

or for

$$\beta_4 = \beta_8, \tag{1.66}$$

the indefinite integral in (1.64) *can be expressed through a finite combinations of elementary functions.*

Theorem 1.2. *Under sufficient conditions of Lemma* 1.1 (*in this case, property* (1.65) *holds*), *system* (1.59) *possesses a complete set of first integrals that can be expressed through a finite combination of elementary functions.*

Dynamical systems considered in the present paper are systems with variable dissipation with zero mean with respect to the periodic coordinate. In many cases, such systems possess a complete set of first integrals that can be expressed through elementary functions.

We have presented several cases of the complete integrability in the dynamics of the spatial (3D) motion of a body in a nonconservative field. Herewith, we deal with three properties that, at first glance, seem to be independent:

(1) the class of systems (1.6) with marked symmetries specified above;
(2) this class of systems possesses variable dissipation with zero mean (with respect to the variable α); this allows one to consider them as "almost" conservative systems;
(3) in some (sufficiently low-dimensional) cases, these systems possess a complete set of (generally speaking, transcendental from the standpoint of complex analysis) first integrals.

The method of reduction of initial systems whose right-hand sides contain polynomials of trigonometric functions to systems with polynomial right-hand sides allows one to find (or to prove the absence) of first integrals for systems of a more general form that perhaps do not possess the symmetries mentioned above (see [265]).

Chapter 2

CASES OF INTEGRABILITY CORRESPONDING TO THE MOTION OF A RIGID BODY ON THE TWO-DIMENSIONAL PLANE, I

In this chapter, we systematize some earlier and new results on the study of the equations of motion of dynamically symmetric two-dimensional (2D) rigid bodies in nonconservative force fields. The form of these equations is taken from the dynamics of real rigid bodies interacting with a resisting medium by laws of jet flows where a body is influenced by a nonconservative tracing force; under action of this force, the magnitude of the velocity of some characteristic point of the body remains constant, which means that the system possesses a nonintegrable servo constraint (see [1, 64, 70, 72, 119–121, 157, 164–167, 182, 184, 191, 198, 237, 242, 346, 351, 352, 390]).

Earlier (see [164–167]), the author already proved the complete integrability of the equations of a plane-parallel motion of a body in a resisting medium under the jet flow conditions when the system of dynamical equations possesses a first integral, which is a transcendental (in the sense of the theory of functions of a complex variable, i.e., it has essential singularities) function of quasi-velocities. It was assumed that the interaction of the medium with the body is concentrated on a part of the surface of the body that has the form of a (one-dimensional) plate.

In this chapter, we review both new results and results obtained earlier. We systematize these results and present them in the invariant form. Moreover, we introduce an extra dependence of the moment of the nonconservative force on the angular velocity. This dependence can be further extended to cases of the motion in spaces of higher dimension.

1. General Problem on the Motion under a Tracing Force

Let us consider the plane-parallel motion of a symmetric rigid body with flat front end face (one-dimensional plate) in the field of a resisting force under the assumption of quasi-stationarity [28, 50, 51, 62–66, 98, 112, 119–121, 160–169, 171, 431, 432]. If (v, α) are the polar coordinates of the velocity vector of a certain typical point D of a rigid body (D is the center of the plate), Ω is its angular velocity, and I and m are the characteristics of inertia and mass, then the dynamical part of the equations of motion (including also Chaplygin analytical functions [50, 51], see below) in which the tangent forces of the interaction of the body with the medium are absent, has the form

$$\begin{gathered}\dot v\cos\alpha - \dot\alpha v\sin\alpha - \Omega v\sin\alpha + \sigma\Omega^2 = \frac{F_x}{m},\\ \dot v\sin\alpha + \dot\alpha v\cos\alpha + \Omega v\cos\alpha - \sigma\dot\Omega = 0,\end{gathered} \tag{2.1}$$

$$I\dot\Omega = y_N\left(\alpha, \frac{\Omega}{v}\right)s(\alpha)v^2,$$

where

$$F_x = -S, \quad S = s(\alpha)v^2, \quad \sigma > 0, \quad v > 0. \tag{2.2}$$

The first two equations in (2.1) describe the motion of the center of mass in the two-dimensional Euclidean plane $\mathbf{E}^2$ in the coordinate system Dx_1x_2 attached to the body. Here Dx_1 is the perpendicular to the plate passing through the center of mass C of the symmetric body and Dx_2 is an axis along the plate. The third equation of (2.1) is obtained from the theorem on the change of the angular moment of a rigid body.

Thus, the direct product

$$\mathbf{R}^1 \times \mathbf{S}^1 \times \mathrm{so}(2) \tag{2.3}$$

of the two-dimensional cylinder and the Lie algebra so(2) is the phase space of system (2.1).

If we consider a more general problem on the motion of a body under the action of a certain tracing force $\mathbf{T}$ passing through the center of mass and providing the fulfillment of the equality

$$v \equiv \mathrm{const} \tag{2.4}$$

during the motion (see also [164–167]), then F_x in system (2.1) must be replaced by

$$T - s(\alpha)v^2, \quad \sigma = DC. \tag{2.5}$$

As a result of an appropriate choice of the magnitude T of the tracing force, we can achieve the fulfillment of Eq. (2.4) during the motion. Indeed, if we formally express the value T by virtue of system (2.1), we obtain (for $\cos\alpha \neq 0$):

$$T = T_v(\alpha, \Omega) = m\sigma\Omega^2 + s(\alpha)v^2 \left[1 - \frac{m\sigma}{I} y_N \left(\alpha, \frac{\Omega}{v}\right) \frac{\sin\alpha}{\cos\alpha}\right]. \tag{2.6}$$

Note that we have used condition (2.4).

This procedure can be viewed from two standpoints. First, a transformation of the system has occurred at the presence of the tracing (control) force in the system which provides the corresponding class of motions (2.4). Second, we can consider this procedure as a procedure that allows one to reduce the order of the system. Indeed, system (2.1) generates an independent second-order system of the following form:

$$\begin{gathered} \dot\alpha v \cos\alpha + \Omega v \cos\alpha - \sigma\dot\Omega = 0, \\ I\dot\Omega = y_N \left(\alpha, \frac{\Omega}{v}\right) s(\alpha) v^2, \end{gathered} \tag{2.7}$$

where the parameter v is supplemented by the constant parameters specified above.

We can see from (2.7) that the system cannot be solved uniquely with respect to $\dot\alpha$ on the manifold

$$O = \left\{(\alpha, \Omega) \in \mathbf{R}^2 : \ \alpha = \frac{\pi}{2} + \pi k, \ k \in \mathbf{Z}\right\}. \tag{2.8}$$

Thus, formally speaking, the uniqueness theorem is violated on manifold (2.8).

This implies that system (2.7) outside of the manifold (2.8) (and only outside it) is equivalent to the following system:

$$\begin{aligned} \dot\alpha &= -\Omega + \frac{\sigma v}{I} \frac{y_N\left(\alpha, \dfrac{\Omega}{v}\right) s(\alpha)}{\cos\alpha}, \\ \dot\Omega &= \frac{1}{I} y_N \left(\alpha, \frac{\Omega}{v}\right) s(\alpha) v^2. \end{aligned} \tag{2.9}$$

The uniqueness theorem is violated for system (2.7) on the manifold (2.8) in the following sense: regular phase trajectories of system (2.7) pass through almost all points of the manifold (2.8) and intersect the manifold (2.8) at a right angle, and also there exists a phase trajectory that completely coincides with the specified point at all time instants. However, these trajectories are different since they correspond to different values of the tracing force. Let us prove this.

As was shown above, to fulfill constraint (2.4), one must choose the value of T for $\cos\alpha \neq 0$ in the form (2.6).

Let

$$\lim_{\alpha \to \pi/2} \frac{y_N\left(\alpha, \dfrac{\Omega}{v}\right) s(\alpha)}{\cos\alpha} = L\left(\frac{\Omega}{v}\right). \tag{2.10}$$

Note that $|L| < +\infty$ if and only if

$$\lim_{\alpha \to \pi/2} \left| \frac{\partial}{\partial\alpha} \left(y_N \left(\alpha, \frac{\Omega}{v}\right) s(\alpha) \right) \right| < +\infty. \tag{2.11}$$

For $\alpha = \pi/2$, the necessary magnitude of the tracing force can be found from the equality

$$T = T_v\left(\frac{\pi}{2}, \Omega\right) = m\sigma\Omega^2 - \frac{m\sigma L v^2}{I}, \tag{2.12}$$

where Ω is arbitrary.

On the other hand, if we support the rotation around a certain point W by means of the tracing force, then the tracing force has the form

$$T = T_v\left(\frac{\pi}{2}, \Omega\right) = \frac{mv^2}{R_0}, \tag{2.13}$$

where R_0 is the distance CW.

Generally speaking, Eqs. (2.6) and (2.13) define different values of the tracing force T for almost all points of manifold (2.8), and the proof is complete.

2. Case where the Moment of Nonconservative Forces Is Independent of the Angular Velocity

2.1. Reduced system. Similarly to the choice of Chaplygin analytical functions (see [50, 51]), we take the dynamical functions s and y_N of the following form:

$$s(\alpha) = B\cos\alpha, \quad y_N\left(\alpha, \frac{\Omega}{v}\right) = y_0(\alpha) = A\sin\alpha, \quad A, B > 0, \quad v \neq 0, \tag{2.14}$$

which shows that for the considered system, the moment of the nonconservative forces is independent of the angular velocity (it depends only on the angle α).

Then, owing to the nonintegrable constraint (2.4), outside the manifold (2.8) (and only outside it) the dynamical part of the equations of motion (system (2.9)) has the form of the following analytical system:

$$\begin{aligned} \dot\alpha &= -\Omega + \sigma n_0^2 v\sin\alpha, \\ \dot\Omega &= n_0^2 v^2 \sin\alpha\cos\alpha, \quad n_0^2 = \frac{AB}{I}. \end{aligned} \tag{2.15}$$

Introducing the dimensionless variable, the parameter, and the differentiation as follows:

$$\Omega = n_0 v\omega, \quad b = \sigma n_0, \quad \langle\cdot\rangle = n_0 v\langle'\rangle, \tag{2.16}$$

we reduce system (2.15) to the form

$$\begin{aligned} \alpha' &= -\omega + b\sin\alpha, \\ \omega' &= \sin\alpha\cos\alpha. \end{aligned} \tag{2.17}$$

2.2. Complete list of invariant relations. We put in correspondence to system (2.17) the following nonautonomous first-order equation:

$$\frac{d\omega}{d\alpha} = \frac{\sin\alpha\cos\alpha}{-\omega + b\sin\alpha}. \tag{2.18}$$

Using the substitution $\tau = \sin\alpha$, we rewrite Eq. (2.18) in the algebraic form

$$\frac{d\omega}{d\tau} = \frac{\tau}{-\omega + b\tau}. \tag{2.19}$$

Further, introducing the homogeneous variable by the formula $\omega = u\tau$, we reduce Eq. (2.19) to the following quadrature:

$$\frac{(b-u)du}{1 - bu + u^2} = \frac{d\tau}{\tau}. \tag{2.20}$$

Integration of quadrature (2.20) leads to the following three cases. Simple calculations yield the following first integrals.

I. $b^2 - 4 < 0$:

$$\ln(1 - bu + u^2) - \frac{2b}{\sqrt{4-b^2}}\arctan\frac{2u-b}{\sqrt{4-b^2}} + \ln\tau^2 = \text{const}; \tag{2.21}$$

II. $b^2 - 4 > 0$:

$$\ln|1 - bu + u^2| + \frac{b}{\sqrt{b^2-4}} \ln\left|\frac{2u - b + \sqrt{b^2-4}}{2u - b - \sqrt{b^2-4}}\right| + \ln\tau^2 = \text{const}; \tag{2.22}$$

III. $b^2 - 4 = 0$:

$$\ln|u - 1| + \frac{1}{u-1} + \ln|\tau| = \text{const}. \tag{2.23}$$

In other words, in the variables (α, ω) the found first integrals have the following forms:
I. $b^2 - 4 < 0$:

$$[\sin^2\alpha - b\omega\sin\alpha + \omega^2] \exp\left\{-\frac{2b}{\sqrt{4-b^2}} \arctan\frac{2\omega - b\sin\alpha}{\sqrt{4-b^2}\sin\alpha}\right\} = \text{const}; \tag{2.24}$$

II. $b^2 - 4 > 0$:

$$[\sin^2\alpha - b\omega\sin\alpha + \omega^2]\left|\frac{2\omega - b\sin\alpha + \sqrt{b^2-4}\sin\alpha}{2\omega - b\sin\alpha - \sqrt{b^2-4}\sin\alpha}\right|^{b/\sqrt{b^2-4}} = \text{const}; \tag{2.25}$$

III. $b^2 - 4 = 0$:

$$(\omega - \sin\alpha)\exp\left\{\frac{\sin\alpha}{\omega - \sin\alpha}\right\} = \text{const}. \tag{2.26}$$

Therefore, in the considered case the system of dynamical equations (2.1) has two invariant relations: there exist the analytical nonintegrable constraint (2.4) and the first integral expressed by relations (2.24)–(2.26) (or (2.21)–(2.23)), which is a transcendental function of its phase variables (in the sense of complex analysis) and is expressed as a finite combination of elementary functions.

Theorem 2.1. *Under conditions* (2.4) *and* (2.14), *system* (2.1) *possesses two invariant relations* (*the complete set*), *one of which is a transcendental function* (*in the sense of complex analysis*). *Moreover, both these relations are expressed as a finite combination of elementary functions.*

2.3. Topological analogies. Let us consider the following second-order equation:

$$\ddot\theta + b_*\dot\theta\cos\theta + \sin\theta\cos\theta = 0, \quad b_* > 0, \tag{2.27}$$

describing a fixed pendulum in a flowing medium in the case where the moment of forces is independent of the angular velocity, i.e., a mechanical system in a nonconservative force field (see [120, 162]).

Its phase space is the two-dimensional cylinder

$$\mathbf{S}^1 \times \mathbf{R}^1. \tag{2.28}$$

It is easy to verify that the given equation is equivalent to a dynamical system with variable dissipation with zero mean on the tangent bundle $T\mathbf{S}^1$ (or (2.28)) of a one-dimensional sphere (circle). Moreover, the following theorem holds.

Theorem 2.2. *Under conditions* (2.4) *and* (2.14), *system* (2.1) *is equivalent to Eq.* (2.27).

Indeed, it suffices to take $\alpha = \theta$ and $b = -b_*$.

On more general topological analogies, see also [253, 265, 267, 309, 340, 342].

3. Case where the Moment of Nonconservative Forces Depends on the Angular Velocity

3.1. Dependence of the moment of nonconservative forces on the angular velocity. Chapter 2 is devoted to the dynamics of a two-dimensional rigid body on the plane. In the present section, we examine the case of motion where the moment of forces depends on the angular velocity. We introduce this dependence in more general terms. In addition, this point of view will also allow us to introduce this dependence for three-dimensional and higher-dimensional bodies.

Let $x = (x_{1N}, x_{2N})$ be the coordinates of the point N of application of a nonconservative force (interaction with a medium) to a one-dimensional plate and $Q = (Q_1, Q_2)$ be the components independent on the angular velocity. We consider only the linear dependence of the functions $(x_{1N}, x_{2N}) = (x_N, y_N)$ on the angular velocity since the introduction of this dependence itself is not a priori obvious (see [33, 34, 48, 49, 57–66, 120, 121, 169, 182, 203, 205, 249, 250, 261, 274–276, 440].

Thus, we accept the following dependence:

$$x = Q + R, \tag{2.29}$$

where $R = (R_1, R_2)$ is a vector-valued function containing the angular velocity. Here, the dependence of the function R on the angular velocity is gyroscopic:

$$R = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = -\frac{1}{v}\begin{pmatrix} 0 & -\Omega \\ \Omega & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \tag{2.30}$$

where (h_1, h_2) are positive parameters (cf. [322, 330, 331, 345, 393]).

Since $x_{1N} = x_N \equiv 0$, we have for our problem

$$x_{2N} = y_N = Q_2 - h_1 \frac{\Omega}{v}. \tag{2.31}$$

3.2. Reduced system. Similarly to the choice of Chaplygin analytical functions [50, 51]

$$Q_2 = A \sin\alpha, \quad A > 0, \tag{2.32}$$

we take the dynamical functions s and y_N as follows:

$$s(\alpha) = B\cos\alpha, \quad y_N\left(\alpha, \frac{\Omega}{v}\right) = A\sin\alpha - h\frac{\Omega}{v}, \quad A, B,\ h = h_1 > 0, \quad v \neq 0 \tag{2.33}$$

which shows that an additional dependence of the damping (or accelerating in some domains of the phase space) moment of the nonconservative forces is also present in the system considered (i.e., the moment depends on the angular velocity).

Then, owing to the nonintegrable constraint (2.4), outside the manifold (2.8) (and only outside it), the dynamical part of the equations of motion (system (2.9)) has the following form:

$$\begin{aligned} \dot\alpha &= -\left(1 + \frac{\sigma B h}{I}\right)\Omega + \frac{\sigma A B v}{I}\sin\alpha, \\ \dot\Omega &= \frac{ABv^2}{I}\sin\alpha\cos\alpha - \frac{Bhv}{I}\Omega\cos\alpha. \end{aligned} \tag{2.34}$$

Introducing the dimensionless variable, the parameters, and the differentiation as follows:

$$\Omega = n_0 v\omega, \quad n_0^2 = \frac{AB}{I}, \quad b = \sigma n_0, \quad H_1 = \frac{Bh}{In_0}, \quad \langle\cdot\rangle = n_0 v\langle'\rangle, \tag{2.35}$$

we reduce system (2.34) to the form

$$\begin{aligned} \alpha' &= -(1 + bH_1)\omega + b\sin\alpha, \\ \omega' &= \sin\alpha\cos\alpha - H_1\omega\cos\alpha. \end{aligned} \tag{2.36}$$

3.3. Complete list of invariant relations. We put in correspondence to system (2.36) the following nonautonomous first-order equation:

$$\frac{d\omega}{d\alpha} = \frac{\sin\alpha\cos\alpha - H_1\omega\cos\alpha}{-(1+bH_1)\omega + b\sin\alpha}. \tag{2.37}$$

Using the substitution $\tau = \sin\alpha$, we rewrite Eq. (2.37) in the algebraic form:

$$\frac{d\omega}{d\tau} = \frac{\tau - H_1\omega}{-(1+bH_1)\omega + b\tau}. \tag{2.38}$$

Introducing the homogeneous variable by the formula $\omega = u\tau$, we reduce Eq. (2.38) to the following quadrature:

$$\frac{(b-(1+bH_1)u)du}{1-(b+H_1)u+(1+bH_1)u^2} = \frac{d\tau}{\tau}. \tag{2.39}$$

Integration of quadrature (2.39) leads to the following three cases. Simple calculations yield the following first integrals:

I. $|b - H_1| < 2$:

$$\ln(1-(b+H_1)u+(1+bH_1)u^2) - \frac{2b}{\sqrt{4-(b-H_1)^2}}\arctan\frac{2(1+bH_1)u-(b+H_1)}{\sqrt{4-(b-H_1)^2}} + \ln\tau^2 = \text{const}. \tag{2.40}$$

II. $|b - H_1| > 2$:

$$\frac{1}{1+bH_1}\ln|1-(b+H_1)u+(1+bH_1)u^2| + \ln\tau^2 + \frac{b\sqrt{1+bH_1}}{\sqrt{(b-H_1)^2-4}}\ln\left|\frac{2(1+bH_1)^{3/2}u-(b+H_1)\sqrt{1+bH_1}+\sqrt{(b-H_1)^2-4}}{2(1+bH_1)^{3/2}u-(b+H_1)\sqrt{1+bH_1}-\sqrt{(b-H_1)^2-4}}\right| = \text{const}. \tag{2.41}$$

III. $|b - H_1| = 2$:

$$\ln\left|u - \frac{b+H_1}{2(1+bH_1)}\right| + \frac{b-H_1}{2(1+bH_1)u-(b+H_1)} + \ln|\tau| = \text{const}. \tag{2.42}$$

The obtained first integrals have a rather cumbersome form in the variables (α, ω). However, for the case **III**, we present it in the explicit form:

$$\left(\omega - \frac{b+H_1}{2(1+bH_1)}\sin\alpha\right)\exp\left\{\frac{(b-H_1)\sin\alpha}{2(1+bH_1)\omega-(b+H_1)\sin\alpha}\right\} = \text{const}. \tag{2.43}$$

Therefore, in the considered case, the system of dynamical equations (2.1) has two invariant relations: the analytical nonintegrable constraint (2.4) and the first integral expressed by relations (2.40)–2.42 (or, in particular, (2.43) in the case **III**), which is a transcendental function of its phase variables (in the sense of complex analysis) and is expressed as a finite combination of elementary functions.

Theorem 2.3. *Under conditions* (2.4) *and* (2.33), *system* (2.1) *possesses two invariant relations* (*a complete set*), *one of which is a transcendental function* (*in the sense of complex analysis*). *Both relations are expressed as finite combinations of elementary functions.*

3.4. Topological analogies. We consider the following second-order equation:

$$\ddot{\theta} + (b_* - H_{1*})\dot{\theta}\cos\theta + \sin\theta\cos\theta = 0, \quad b_*,\ H_{1*} > 0. \tag{2.44}$$

It describes a fixed pendulum in a flowing medium in the case where the moment of forces depends on the angular velocity, i.e., the mechanical system in a nonconservative force field (see [265]).

Its phase space is the two-dimensional cylinder

$$\mathbf{S}^1 \times \mathbf{R}^1. \tag{2.45}$$

It is easy to verify that the given equation is equivalent to a dynamical system with variable dissipation with zero mean on the tangent bundle $T\mathbf{S}^1$ (or (2.45)) to the one-dimensional sphere (circle). Moreover, the following theorem holds.

Theorem 2.4. *Under conditions* (2.4) *and* (2.14), *system* (2.1) *is equivalent to Eq.* (2.44).

Indeed, it suffices to set $\alpha = \theta$, $b = -b_*$, and $H_1 = -H_{1*}$.

On more general topological analogies, see also [253, 265, 267, 309, 340, 342].

Chapter 3

CASES OF INTEGRABILITY CORRESPONDING TO THE MOTION OF A RIGID BODY ON THE TWO-DIMENSIONAL PLANE, II

In this chapter, we systematize some earlier and new results on the study of the equations of motion of dynamically symmetric two-dimensional (2D) rigid bodies in nonconservative force fields. The form of these equations is taken from the dynamics of real rigid bodies interacting with a resisting medium by laws of jet flows where a body is influenced by a nonconservative tracing force. Under the action of this force, the center of mass of the body moves rectilinearly and uniformly; this means that there exists a nonconservative couple of forces in the system (see also [1, 64, 70, 72, 119–121, 157, 164–167, 180, 181, 184, 191, 194, 212, 231, 258, 291, 353, 354, 374, 390, 414]).

Earlier (see [164–167]), the author already proved the complete integrability of the equations of a plane-parallel motion of a body in a resisting medium under the jet flow conditions when the system of dynamical equations possesses a first integral, which is a transcendental (in the sense of the theory of functions of a complex variable) function of quasi-velocities having essential singularities. It was assumed that the interaction of the medium with the body is concentrated on a part of the surface of the body that has the form of a (one-dimensional) plate.

In this chapter, we review both new results and results obtained earlier. We systematize these results and present them in the invariant form. Moreover, we introduce an extra dependence of the moment of the nonconservative force on the angular velocity. This dependence can be further extended to cases of the motion in spaces of higher dimension.

1. General Problem on the Motion under a Tracing Force

Let us consider the plane-parallel motion of a symmetric rigid body with flat front end face (one-dimensional plate) in the field of a resisting force under the assumption of quasi-stationarity [28, 50, 51, 62–66, 98, 112, 119–121, 160–169, 171, 431, 432]. If (v, α) are the polar coordinates of the velocity vector of a certain typical point D of a rigid body (D is the center of the plate), Ω is its angular velocity, and I and m are the characteristics of inertia and mass, then the dynamical part of the equations of motion (including also Chaplygin analytical functions [50, 51], see below) in which the tangent forces of the interaction of the body with the medium are absent, has the form

$$\begin{gathered}\dot v\cos\alpha-\dot\alpha v\sin\alpha-\Omega v\sin\alpha+\sigma\Omega^2=\frac{F_x}{m},\\ \dot v\sin\alpha+\dot\alpha v\cos\alpha+\Omega v\cos\alpha-\sigma\dot\Omega=0,\\ I\dot\Omega=y_N\left(\alpha,\frac{\Omega}{v}\right)s(\alpha)v^2,\end{gathered}\tag{3.1}$$

where

$$F_x=-S,\quad S=s(\alpha)v^2,\quad \sigma>0,\quad v>0.\tag{3.2}$$

The first two equations in (3.1) describe the motion of the center of mass on the two-dimensional Euclidean plane $\mathbf{E}^2$ in the coordinate system Dx_1x_2 attached to the body. Here Dx_1 is the perpendicular to the plate passing through the center of mass C of the symmetric body and Dx_2 is an axis along the plate. The third equation of (3.1) is obtained from the theorem on the change of the angular moment of a rigid body.

Thus, the direct product

$$\mathbf{R}^1\times\mathbf{S}^1\times\mathrm{so}(2)\tag{3.3}$$

of the two-dimensional cylinder and the Lie algebra so(2) is the phase space of system (3.1).

If we consider a more general problem on the motion of a body under the action of a certain tracing force $\mathbf{T}$ passing through the center of mass and providing the fulfillment of the equality

$$\mathbf{V}_C\equiv\mathrm{const},\tag{3.4}$$

during the motion ($\mathbf{V}_C$ is the velocity of the center of mass, see also [164–167]), then F_x in system (3.1) must be replaced by zero since the nonconservative couple of the forces acts on the body:

$$T-s(\alpha)v^2\equiv 0,\quad \sigma=DC.\tag{3.5}$$

Obviously, we must choose the value of the tracing force T as follows:

$$T=T_v(\alpha,\Omega)=s(\alpha)v^2,\quad \mathbf{T}\equiv-\mathbf{S}.\tag{3.6}$$

The choice (3.6) of the magnitude of the tracing force T is a particular case of the possibility of separation of an independent second-order subsystem after a certain transformation of the third-order system (3.1).

Indeed, let the following condition hold for T:

$$T=T_v(\alpha,\Omega)=\tau_1\left(\alpha,\frac{\Omega}{v}\right)v^2+\tau_2\left(\alpha,\frac{\Omega}{v}\right)\Omega v+\tau_3\left(\alpha,\frac{\Omega}{v}\right)\Omega^2=T_1\left(\alpha,\frac{\Omega}{v}\right)v^2.\tag{3.7}$$

We can rewrite system (3.1) as follows:

$$\begin{gathered}\dot v+\sigma\Omega^2\cos\alpha-\sigma\sin\alpha\left[\frac{v^2}{I}y_N\left(\alpha,\frac{\Omega}{v}\right)s(\alpha)\right]=\frac{T_1\left(\alpha,\frac{\Omega}{v}\right)v^2-s(\alpha)v^2}{m}\cos\alpha,\\ \dot\alpha v+\Omega v-\sigma\cos\alpha\left[\frac{v^2}{I}y_N\left(\alpha,\frac{\Omega}{v}\right)s(\alpha)\right]-\sigma\Omega^2\sin\alpha=\frac{s(\alpha)v^2-T_1\left(\alpha,\frac{\Omega}{v}\right)v^2}{m}\sin\alpha,\\ \dot\Omega=\frac{v^2}{I}y_N\left(\alpha,\frac{\Omega}{v}\right)s(\alpha).\end{gathered}\tag{3.8}$$

If we introduce the new dimensionless phase variable and the differentiation by the formulas

$$\Omega=n_1v\omega,\quad \langle\cdot\rangle=n_1v\langle'\rangle,\quad n_1>0,\quad n_1=\mathrm{const},\tag{3.9}$$

then system (3.8) is reduced to the following form:

$$v' = v\Psi(\alpha, \omega), \tag{3.10}$$

$$\alpha' = -\omega + \sigma n_1\omega^2 \sin\alpha + \left[\frac{\sigma}{In_1} y_N(\alpha, n_1\omega)\, s(\alpha)\right]\cos\alpha - \frac{T_1(\alpha, n_1\omega) - s(\alpha)}{mn_1}\sin\alpha, \tag{3.11}$$

$$\omega' = \frac{1}{In_1^2} y_N(\alpha, n_1\omega)\, s(\alpha) - \omega\left[\frac{\sigma}{In_1} y_N(\alpha, n_1\omega)\, s(\alpha)\right]\sin\alpha + \sigma n_1\omega^3\cos\alpha - \omega\frac{T_1(\alpha, n_1\omega) - s(\alpha)}{mn_1}\cos\alpha,$$

$$\Psi(\alpha, \omega) = -\sigma n_1\omega^2\cos\alpha + \left[\frac{\sigma}{In_1} y_N(\alpha, n_1\omega)\, s(\alpha)\right]\sin\alpha + \frac{T_1(\alpha, n_1\omega) - s(\alpha)}{mn_1}\cos\alpha.$$

We see that the independent second-order subsystem (3.11) can be substituted into the third-order system (3.10) and can be considered separately on its own two-dimensional phase cylinder.

In particular, if condition (3.6) holds, then the method of separation of an independent second-order subsystem is also applicable.

2. Case where the Moment of Nonconservative Forces Is Independent of the Angular Velocity

2.1. Reduced system. Similarly to the choice of the Chaplygin analytical functions (see [50, 51]), we take the dynamical functions s and y_N as follows:

$$s(\alpha) = B\cos\alpha, \quad y_N\left(\alpha, \frac{\Omega}{v}\right) = y_0(\alpha) = A\sin\alpha, \quad A, B > 0, \quad v \neq 0, \tag{3.12}$$

which shows that for the considered system, the moment of the nonconservative forces is independent of the angular velocity (it depends only on the angle α).

Then, owing to the of conditions (3.4) and (3.12), the transformed dynamical part of the equations of motion (system (3.10), (3.11)) has the following form:

$$v' = v\Psi(\alpha, \omega), \tag{3.13}$$

$$\begin{aligned} \alpha' &= -\omega + b\sin\alpha\cos^2\alpha + b\omega^2\sin\alpha, \\ \omega' &= \sin\alpha\cos\alpha - b\omega\sin^2\alpha\cos\alpha + b\omega^3\cos\alpha, \end{aligned} \tag{3.14}$$

$$\Psi(\alpha, \omega) = -b\omega^2\cos\alpha + b\sin^2\alpha\cos\alpha.$$

Here we choose the dimensionless variable, the parameter b, and the constant n_1 as follows:

$$b = \sigma n_0, \quad n_0^2 = \frac{AB}{I}, \quad n_1 = n_0. \tag{3.15}$$

Therefore, system (3.13), (3.14) can be considered on its own three-dimensional phase cylinder

$$W_1 = \mathbf{R}^1_+\{v\} \times \mathbf{S}^1\{\alpha \mod 2\pi\} \times \mathbf{R}^1\{\omega\}. \tag{3.16}$$

2.2. Complete list of first integrals. The independent second-order system (3.14) was extracted from system (3.13), (3.14).

Note that the magnitude of the velocity of the center of mass is a first integral of system (3.1) by virtue of (3.4), since the function of the phase variables

$$\Psi_0(v, \alpha, \Omega) = v^2 + \sigma^2\Omega^2 - 2\sigma\Omega v \sin\alpha = V_C^2 \tag{3.17}$$

is constant on its phase trajectories.

By virtue of the nondegenerate change of the independent variable (for $v \neq 0$), system (3.13), (3.14) also has an analytical integral since the function of the phase variables

$$\Psi_1(v, \alpha, \omega) = v^2(1 + b^2\omega^2 - 2b\omega \sin\alpha) = V_C^2 \tag{3.18}$$

is constant on its phase trajectories.

Equality (3.18) allows one to find the dependence of the velocity of a certain point of a rigid body (namely, the center of the plate) on other phase variables without the solution of system (3.13), (3.14), since the equality

$$v^2 = \frac{V_C^2}{1 + \sigma^2\omega^2 - 2\sigma\omega \sin\alpha} \tag{3.19}$$

holds for $V_C \neq 0$.

Since the phase space (3.16) of system (3.13), (3.14) is three-dimensional and there exist asymptotic limit sets in the phase space, Eq. (3.18) defines a unique analytical (even continuous) first integral of system (3.13), (3.14) in the whole phase space (cf. [23, 24, 43, 53, 103, 105, 137, 152–154, 156, 213, 427, 438]).

We consider in detail the problem of the existence of the second (additional) first integral of system (3.13), (3.14). Its phase space is stratified into surfaces

$$\{(v, \alpha, \omega) \in W_1 : V_C = \text{const}\}, \tag{3.20}$$

on which the dynamics is defined by means of the first integral of system (3.14).

We associate the separated second-order system (3.14) with the following nonautonomous differential equation:

$$\frac{d\omega}{d\tau} = \frac{\tau + b\omega[\omega^2 - \tau^2]}{-\omega + b\tau + b\tau[\omega^2 - \tau^2]}, \quad \tau = \sin\alpha. \tag{3.21}$$

Introduce the following notation (cf. [181]):

$$C_1 = 2 - b, \quad C_2 = b > 0, \quad C_3 = -2 - b < 0. \tag{3.22}$$

After the change of variables

$$u_1 = \omega - \tau, \quad v_1 = \omega + \tau, \tag{3.23}$$

Eq. (3.21) is transformed to the form

$$du_1\left\{-\left(1 + \frac{b}{2}\right)u_1 + \frac{b}{2}v_1 + bu_1v_1^2\right\} = dv_1\left\{\left(1 - \frac{b}{2}\right)v_1 + \frac{b}{2}u_1 + bu_1^2v_1\right\}. \tag{3.24}$$

Then, using two substitutions

$$u_1 = v_1t_1, \quad v_1^2 = p_1, \tag{3.25}$$

we reduce Eq. (3.24) to the Bernoulli equation

$$2p_1\{C_3t_1 + C_2 + 2C_2t_1p_1\} = \frac{dp_1}{t_1}\{C_1 - C_3t_1^2\}, \tag{3.26}$$

which can be easily transformed to a linear nonhomogeneous equation by the substitution $p_1 = 1/q_1$:

$$q_1' = a_1(t_1)q_1 + a_2(t_1), \tag{3.27}$$

where

$$a_1(t_1) = \frac{2(C_3t_1 + C_2)}{C_3t_1^2 - C_1}, \quad a_2(t_1) = \frac{4C_2t_1}{C_3t_1^2 - C_1}. \tag{3.28}$$

The solution of the uniform part of Eq. (3.27) is found from the equality

$$q_{1\,\mathrm{hom}}(t_1) = k \exp W(t_1), \quad W(t_1) = 2\int \frac{(C_3t_1 + C_2)dt_1}{C_3t_1^2 - C_1}. \tag{3.29}$$

Consider the following three cases for calculation of integral (3.29).

I. $C_1 > 0$ $(b < 2)$:

$$W(t_1) = \ln(-C_3t_1^2 + C_1) - 2\frac{C_2}{\sqrt{-C_1C_3}} \arctan\sqrt{-\frac{C_3}{C_1}}t_1 + \mathrm{const}\,. \tag{3.30}$$

II. $C_1 < 0$ $(b > 2)$:

$$W(t_1) = \ln|-C_3t_1^2 + C_1| + \frac{C_2}{\sqrt{C_1C_3}} \ln\left|\frac{\sqrt{-C_1} + \sqrt{-C_3}t_1}{\sqrt{-C_1} - \sqrt{-C_3}t_1}\right| + \mathrm{const}\,. \tag{3.31}$$

III. $C_1 = 0$ $(b = 2)$:

$$W(t_1) = 2\ln|t_1| + \frac{1}{t_1}. \tag{3.32}$$

Now we have the general solution of the homogeneous equation:

I. $b < 2$:

$$q_{1\,\mathrm{hom}}(t_1) = k(-C_3t_1^2 + C_1)\exp\left\{-\frac{2b}{\sqrt{4-b^2}}\arctan\sqrt{\frac{2+b}{2-b}}t_1\right\} + \mathrm{const}\,. \tag{3.33}$$

II. $b > 2$:

$$q_{1\,\mathrm{hom}}(t_1) = k(-C_3t_1^2 + C_1)\left|\frac{\sqrt{-C_1} + \sqrt{-C_3}t_1}{\sqrt{-C_1} - \sqrt{-C_3}t_1}\right|^{C_2/\sqrt{C_1C_3}} + \mathrm{const}\,. \tag{3.34}$$

III. $b = 2$:

$$q_{1\,\mathrm{hom}}(t_1) = kt_1^2 \exp\left\{\frac{1}{t_1}\right\} + \mathrm{const}\,. \tag{3.35}$$

To find a solution of the nonhomogeneous equations (3.27), (3.28), we must express the value of k as a function of t_1. We obtain:

I. $b < 2$:

$$k(t_1) = -\frac{b}{8}\exp\left\{\frac{2b}{\sqrt{4-b^2}}\left[\frac{2b}{\sqrt{4-b^2}}\sin 2\zeta - 2\cos 2\zeta\right]\right\} + \mathrm{const}, \tag{3.36}$$

where

$$\tan\zeta = \sqrt{\frac{2-b}{2+b}}t_1. \tag{3.37}$$

II. $b > 2$:

$$k(t_1) = \pm|\zeta|^{b/\sqrt{b^2-4}} \mp \frac{b}{b + 2\sqrt{b^2-4}}|\zeta|^{b/\sqrt{b^2-4}+2} + \mathrm{const}, \tag{3.38}$$

where

$$t_1 = \sqrt{\frac{b-2}{b+2}}\left(\frac{1-\zeta}{1+\zeta}\right). \tag{3.39}$$

III. $b = 2$:

$$k(t_1) = -2\frac{t_1+1}{t_1}\exp\left\{-\frac{1}{t_1}\right\}. \tag{3.40}$$

Thus, Eqs. (3.33)–(3.40) allow one to obtain the required first integral of system (3.14) (and an additional first integral of system (3.13), (3.14)), which is a transcendental function of its own phase variables and is expressed as a finite combination of elementary functions.

We present the obtained first integral only in the case **III** because of the complexity of other cases:

$$\exp\left\{\frac{\sin\alpha+\omega}{\sin\alpha-\omega}\right\}\frac{1-4\omega\sin\alpha+4\omega^2}{(\omega-\sin\alpha)^2}=C_1=\text{const}\,. \tag{3.41}$$

Therefore, the system of dynamical equations (3.13), (3.14) has two invariant relations (first integrals) in the considered case: there exists an analytical first integral of the form (3.18) and also a transcendental first integral, which can be obtained by means of Eqs. (3.33)–(3.40).

Theorem 3.1. *System* (3.13), (3.14) *possesses a complete list of first integrals, one of which is an analytical function and the second is a transcendental function of the phase variables expressed as a finite combination of elementary functions.*

It is necessary to repeat an important remark. In fact, the obtained integral is transcendental from the point of view of the theory of elementary functions (i.e., not algebraic). In this case, the transcendence is understood in the sense of the theory of functions of complex variables, when the formal continuation of the function to the complex domain has essential singular points that correspond to attractive and repelling limit sets of the considered dynamical system.

2.3. Topological analogies. We show that there exists another mechanical and topological analogy.

Theorem 3.2. *The first integral of system* (2.1) *under conditions* (2.4) *and* (2.14) *is constant on the phase trajectories of system* (3.13), (3.14).

Proof. We prove the theorem for the case $b=2$. Rewrite the first integral (3.41) in the following form:

$$\exp\left\{\frac{n_0v\sin\alpha+\Omega}{n_0v\sin\alpha-\Omega}\right\}\frac{n_0^2v^2-2bn_0v\Omega\sin\alpha+b^2\Omega^2}{(\Omega-n_0v\sin\alpha)^2}=\text{const}\,. \tag{3.42}$$

We see that the numerator of the second multiplier is proportional to the square of the velocity of the center of mass $\mathbf{V}_C$ of a rigid body with the constant coefficient n_0^2. But, by virtue of (3.17), the given value is constant on trajectories of system (3.13), (3.14). This means that the function

$$\exp\left\{\frac{n_0v\sin\alpha+\Omega}{n_0v\sin\alpha-\Omega}\right\}\frac{V_C^2}{(\Omega-n_0v\sin\alpha)^2}=\text{const} \tag{3.43}$$

is also constant on its trajectories.

Now we raise the left-hand side of Eq. (3.43) to the power $(-1/2)$ and conclude that the following function is also constant on phase trajectories of system (3.13), (3.14):

$$\exp\left\{\frac{\Omega+n_0v\sin\alpha}{2(\Omega-n_0v\sin\alpha)}\right\}(\Omega-n_0v\sin\alpha)=\text{const}\,. \tag{3.44}$$

Dividing Eq. (3.44) by $\sqrt{e}$, we obtain the function

$$\exp\left\{\frac{n_0v\sin\alpha}{\Omega-n_0v\sin\alpha}\right\}(\Omega-n_0v\sin\alpha)=\text{const}, \tag{3.45}$$

which is constant on phase trajectories of system (3.13), (3.14). But the first integral (3.45) is completely similar to the first integral (2.26), as is required. □

Thus, we have the following topological and mechanical analogies in the sense explained above.

(1) Free motion of a rigid body in a nonconservative force field under a tracing force (in the presence of a nonintegrable constraint).

(2) A motion of a fixed physical pendulum in a flowing medium (nonconservative force fields).

(3) A rotation of a rigid body in a nonconservative force about its center of mass, which moves rectilinearly and uniformly.

On more general topological analogues, see also [253, 265, 267, 309, 340, 342].

3. Case where the Moment of Nonconservative Forces Depends on the Angular Velocity

3.1. Dependence on the moment of the angular velocity and the reduced system. We continue to study the dynamics of a two-dimensional rigid body on the plane. This section (similarly to the corresponding section of Chap. 2) is devoted to the study of the case of the motion where the moment of forces depends on the angular velocity. We introduce this dependence in the same way as was done in the previous chapter. We also recall that this point of view will also allow us to introduce this dependence for three-dimensional and higher-dimensional bodies.

Let $x = (x_{1N}, x_{2N})$ be the coordinates of the point N of application of a nonconservative force (interaction with a medium) to a one-dimensional plate, and let $Q = (Q_1, Q_2)$ be the components that are independent of the angular velocity. We consider only the linear dependence of the functions $(x_{1N}, x_{2N}) = (x_N, y_N)$ on the angular velocity since the introduction of this dependence itself is not a priori obvious (see [33, 34, 48, 49, 57–66, 120, 121, 169, 182, 203, 205, 249, 250, 261, 274–276, 440]).

Thus, we accept the following dependence:

$$x = Q + R, \tag{3.46}$$

where $R = (R_1, R_2)$ is a vector-valued function containing the angular velocity. Here, the dependence of the function R on the angular velocity is gyroscopic (see also the previous chapter):

$$R = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = -\frac{1}{v} \begin{pmatrix} 0 & -\Omega \\ \Omega & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \tag{3.47}$$

where (h_1, h_2) are certain positive parameters (cf. [322, 330, 331, 345, 393]).

Since $x_{1N} = x_N \equiv 0$, we have

$$x_{2N} = y_N = Q_2 - h_1 \frac{\Omega}{v}. \tag{3.48}$$

Similarly to the choice of the Chaplygin analytical functions (see [50, 51])

$$Q_2 = A \sin\alpha, \quad A > 0, \tag{3.49}$$

we take the dynamical functions s and y_N as follows:

$$s(\alpha) = B\cos\alpha, \quad y_N\left(\alpha, \frac{\Omega}{v}\right) = A\sin\alpha - h\frac{\Omega}{v}, \quad A, B, h = h_1 > 0, \quad v \neq 0, \tag{3.50}$$

which shows that an additional dependence of the damping (or accelerating in some domains of the phase space) moment of the nonconservative forces is also present in the system considered (i.e., the moment depends on the angular velocity).

Then, owing to the conditions (3.4) and (3.50), the transformed dynamical part of the equations of motion (system (3.10), (3.11)) has the following form:

$$v' = v\Psi(\alpha, \omega), \tag{3.51}$$

$$\begin{aligned} \alpha' &= -\omega + b\sin\alpha\cos^2\alpha + b\omega^2\sin\alpha - bH_1\omega\cos^2\alpha, \\ \omega' &= \sin\alpha\cos\alpha - b\omega\sin^2\alpha\cos\alpha + b\omega^3\cos\alpha + bH_1\omega^2\sin\alpha\cos\alpha - H_1\omega\cos\alpha, \end{aligned} \tag{3.52}$$

$$\Psi(\alpha, \omega) = -b\omega^2\cos\alpha + b\sin^2\alpha\cos\alpha - bH_1\omega\sin\alpha\cos\alpha.$$

Just as was done earlier, we choose a dimensionless variable, the parameters b and H_1, and the constant n_1 as follows:

$$b = \sigma n_0, \quad n_0^2 = \frac{AB}{I}, \quad H_1 = \frac{Bh}{In_0}, \quad n_1 = n_0. \tag{3.53}$$

Therefore, system (3.53), (3.52) can be considered on its three-dimensional phase cylinder

$$W_1 = \mathbf{R}^1_+\{v\} \times \mathbf{S}^1\{\alpha \mod 2\pi\} \times \mathbf{R}^1\{\omega\}. \tag{3.54}$$

3.2. Complete list of first integrals. The independent second-order system (3.52) was extracted from system (3.51), (3.52).

We note that the magnitude of the velocity of the center mass is a first integral of system (3.1) by virtue of (3.4) and, therefore, the function of phase variables (3.17) is constant on its phase trajectories.

By virtue of a nondegenerate change of the independent variable (for $v \neq 0$), system (3.51), (3.52) has also an analytical integral and, therefore, the function of the phase variables

$$\Psi_1(v, \alpha, \omega) = v^2(1 + b^2\omega^2 - 2b\omega \sin\alpha) = V_C^2 \tag{3.55}$$

is constant on its phase trajectories.

Equality (3.55) allows one to find the dependence of the velocity of a certain point of a rigid body (namely, of the center of the plate) on the other phase variables without solution of the system (3.51), (3.52); therefore, Eq. (3.19) holds for $V_C \neq 0$.

Since the phase space (3.54) of system (3.51), (3.52) is three-dimensional and there exist asymptotic limit sets in the phase space, we see that Eq. (3.55) defines a unique analytical (even continuous) first integral of system (3.51), (3.52) in the whole phase space (cf. [23, 24, 43, 53, 103, 105, 137, 152–154, 156, 213, 427, 438]).

We consider in detail the problem of the existence of the second (additional) first integral of system (3.51), (3.52). Its phase space is stratified into surfaces (3.20) on which the dynamics is defined by means of the first integral of system (3.52).

We associate the separated second-order system (3.52) with the following nonautonomous differential equation:

$$\frac{d\omega}{d\tau} = \frac{\tau + b\omega[\omega^2 - \tau^2] + H_1\omega[b\omega\tau - 1]}{-\omega + b\tau + b\tau[\omega^2 - \tau^2] - bH_1\omega(1 - \tau^2)}, \quad \tau = \sin\alpha. \tag{3.56}$$

Then, after the change of variables

$$u_1 = \omega - \tau, \quad v_1 = \omega + \tau, \tag{3.57}$$

Eq. (3.56) takes the form

$$du_1\left\{ - u_1\left(1 + \frac{b}{2} + \frac{bH_1}{2} + \frac{H_1}{2}\right) + v_1\left(\frac{b}{2} - \frac{bH_1}{2} - \frac{H_1}{2}\right)\right\} + du_1\left\{bu_1v_1^2 + \frac{bH_1}{4}v_1(v_1^2 - u_1^2)\right\}$$
$$= dv_1\left\{u_1\left(\frac{b}{2} + \frac{bH_1}{2} - \frac{H_1}{2}\right) + v_1\left(1 - \frac{b}{2} - \frac{bH_1}{2} - \frac{H_1}{2}\right)\right\} + dv_1\left\{bu_1^2v_1 + \frac{bH_1}{4}u_1(v_1^2 - u_1^2)\right\}. \tag{3.58}$$

Later, using two substitutions

$$u_1 = v_1t_1, \quad v_1^2 = p_1, \tag{3.59}$$

we reduce Eq. (3.58) to the Bernoulli equation

$$2p_1\{-A_2t_1 + A_1 + bp_1(t_1 + H_1(1 - t_1^2)/4)\} = \frac{dp_1}{dt_1}\{A_3 + bH_1t_1 + A_2t_1^2\}, \tag{3.60}$$

where

$$A_1 = \frac{b}{2} - \frac{bH_1}{2} - \frac{H_1}{2}, \quad A_2 = 1 + \frac{b}{2} + \frac{bH_1}{2} + \frac{H_1}{2} > 0, \quad A_3 = 1 - \frac{b}{2} + \frac{bH_1}{2} - \frac{H_1}{2}. \tag{3.61}$$

By the substitution $p_1 = 1/q_1$, Eq. (3.60) can be easily transformed to the linear nonuniform equation

$$q_1' = a_1(t_1)q_1 + a_2(t_1), \tag{3.62}$$

where

$$a_1(t_1) = \frac{2(A_2 t_1 - A_1)}{A_2 t_1^2 + bH_1 t_1 + A_3}, \quad a_2(t_1) = \frac{2b(-t_1 + H_1(t_1^2 - 1)/4))}{A_2 t_1^2 + bH_1 t_1 + A_3}. \tag{3.63}$$

The solution of the homogeneous part of Eq. (3.62) is found from the equality

$$q_{1\,\mathrm{hom}}(t_1) = k \exp W(t_1), \quad W(t_1) = 2 \int \frac{(A_2 t_1 - A_1)dt_1}{A_2 t_1^2 + bH_1 t_1 + A_3}. \tag{3.64}$$

We consider the following three cases of calculation of integral (3.64).

I. $|b - H_1| < 2$:

$$W(t_1) = \ln(A_2 t_1^2 + bH_1 t_1 + A_3) - \frac{2(b - bH_1 - H_1)}{\sqrt{4 - (b - H_1)^2}} \arctan\left\{ \frac{2 + b + bH_1 + H_1}{\sqrt{4 - (b - H_1)^2}} t_1 + \frac{bH_1}{\sqrt{4 - (b - H_1)^2}} \right\} + \mathrm{const}\,. \tag{3.65}$$

II. $|b - H_1| > 2$:

$$W(t_1) = \ln|A_2 t_1^2 + bH_1 t_1 + A_3| + \frac{b - bH_1 - H_1}{\sqrt{-4 + (b - H_1)^2}} \ln\left| \frac{\sqrt{-4 + (b - H_1)^2} + (2 + b + bH_1 + H_1)t_1 + bH_1}{\sqrt{-4 + (b - H_1)^2} - (2 + b + bH_1 + H_1)t_1 - bH_1} \right| + \mathrm{const}\,. \tag{3.66}$$

III. $|b - H_1| = 2$:

$$W(t_1) = 2 \ln\left| t_1 + \frac{bH_1}{2A_2} \right| + \frac{bH_1 + 2A_1}{A_2} \frac{2A_2}{2A_2 t_1 + bH_1} + \mathrm{const}\,. \tag{3.67}$$

Now we write the general solution of the homogeneous equation:

I. $|b - H_1| < 2$:

$$q_{1\,\mathrm{hom}}(t_1) = k(A_2 t_1^2 + bH_1 t_1 + A_3) \times \exp\left\{ -\frac{2(b - bH_1 - H_1)}{\sqrt{4 - (b - H_1)^2}} \arctan\left\{ \frac{2 + b + bH_1 + H_1}{\sqrt{4 - (b - H_1)^2}} t_1 + \frac{bH_1}{\sqrt{4 - (b - H_1)^2}} \right\} \right\}. \tag{3.68}$$

II. $|b - H_1| > 2$:

$$q_{1\,\mathrm{hom}}(t_1) = k(A_2 t_1^2 + bH_1 t_1 + A_3) \times \left| \frac{\sqrt{-4 + (b - H_1)^2} + (2 + b + bH_1 + H_1)t_1 + bH_1}{\sqrt{-4 + (b - H_1)^2} - (2 + b + bH_1 + H_1)t_1 - bH_1} \right|^{(b - bH_1 - H_1)/\sqrt{-4 + (b - H_1)^2}}. \tag{3.69}$$

III. $|b - H_1| = 2$:

$$q_{1\,\mathrm{hom}}(t_1) = k\left(t_1 + \frac{bH_1}{2A_2} \right)^2 \exp\left\{ \frac{2(b - H_1)}{(2 + b + bH_1 + H_1)t_1 + bH_1} \right\}. \tag{3.70}$$

To find a solution of the nonhomogeneous equation (3.62), (3.63), we find k as a function of t_1, which is expressed as a finite combination of elementary functions. The obtained first integrals have a rather cumbersome form. However, for the case **III**, we present it in the explicit form.

Thus, the corresponding equations allow one to obtain the required first integral of system (3.52) (and the additional first integral of system (3.51), (3.52)), which is a transcendental function of its phase variables and is expressed as a finite combination of elementary functions.

In the case **III**, the required first integral has the form

$$\exp\left\{ \frac{-2(b - H_1)\sin\alpha}{2(1 + bH_1)\omega - (b + H_1)\sin\alpha} \right\} \frac{1 - 4\omega \sin\alpha + 4\omega^2}{(\omega - 2\sin\alpha/(b + H_1))^2} = C_1 = \mathrm{const}\,. \tag{3.71}$$

Therefore, the system of dynamical equations (3.51), (3.52) has two invariant relations (first integrals) in the considered case: an analytical first integral of the form (3.55) and also a transcendental first integral which can be obtained by using Eqs. (3.65)–(3.71).

Theorem 3.3. *System* (3.51), (3.52) *possesses a complete set of first integrals, one of which is an analytical function and the other is a transcendental function of the phase variables expressed as a finite combination of elementary functions.*

It is necessary to repeat an important remark. In fact, the obtained integral is transcendental from the point of view of the theory of elementary functions (i.e., not algebraic). In this case, the transcendence is understood in the sense of the theory of functions of complex variables, when the formal continuation of the function to the complex domain has essential singular points that correspond to attractive and repelling limit sets of the considered dynamical system.

3.3. Topological analogies. We show that there exists another mechanical and topological analogy.

Theorem 3.4. *Under conditions* (2.4) *and* (2.33), *the first integral of system* (2.1) *is constant on phase trajectories of system* (3.51), (3.52).

Proof. We prove the theorem for the case $|b - H_1| = 2$. Indeed, we rewrite the first integral (3.71) as follows:

$$\exp\left\{\frac{-2n_0 v(b-H_1)\sin\alpha}{2(1+bH_1)\Omega - n_0 v(b+H_1)\sin\alpha}\right\}\frac{n_0^2 v^2 - 4n_0 v\Omega\sin\alpha + 4\Omega^2}{(\Omega - 2n_0 v\sin\alpha/(b+H_1))^2} = \text{const}\,. \quad (3.72)$$

We see that the numerator of the second multiplier is proportional to the square of the velocity of the center of mass $\mathbf{V}_C$ of a rigid body with a constant coefficient. But, by virtue of (3.17), this value is constant on trajectories of system (3.51), (3.52). This means that the function

$$\exp\left\{\frac{-2n_0 v(b-H_1)\sin\alpha}{2(1+bH_1)\Omega - n_0 v(b+H_1)\sin\alpha}\right\}\frac{V_C^2}{(\Omega - 2n_0 v\sin\alpha/(b+H_1))^2} = \text{const} \quad (3.73)$$

is also constant on its trajectories.

Now we raise the left-hand side of Eq. (3.73) to the power $(-1/2)$ and conclude that the following function is also constant on phase trajectories of system (3.51), (3.52):

$$\exp\left\{\frac{n_0 v(b-H_1)\sin\alpha}{2(1+bH_1)\Omega - n_0 v(b+H_1)\sin\alpha}\right\}(\Omega - 2n_0 v\sin\alpha/(b+H_1)) = \text{const}\,. \quad (3.74)$$

Now it is clear that function (3.74) is equivalent to function (2.43) since, in the case **III**, the equality

$$(b+H_1)^2 = 4(1+bH_1) \quad (3.75)$$

holds. Thus, the required analogy is proved. □

Similarly to the previous chapters, we have the following topological and mechanical analogies in the cases (2.33), (3.50).

(1) A motion of a free rigid body in a nonconservative force field under a tracing force (in the presence of a nonintegrable constraint).

(2) A motion of a fixed physical pendulum in a flowing medium (a nonconservative force field).

(3) A rotation of a rigid body in a nonconservative force about its center of mass, which moves rectilinearly and uniformly.

On more general topological analogies, see also [253, 265, 267, 309, 340, 342].

Chapter 4

CASES OF INTEGRABILITY CORRESPONDING TO THE MOTION OF A RIGID BODY IN THE THREE-DIMENSIONAL SPACE, I

In this chapter, we systematize some earlier results and new results on the study of the equations of motion of dynamically symmetric three-dimensional (3D) rigid bodies in nonconservative force fields. The form of these equations is taken from the dynamics of real rigid bodies interacting with a resisting medium by laws of jet flows where a body is influenced by a nonconservative tracing force; under action of this force, the magnitude of the velocity of some characteristic point of the body remains constant, which means that the system possesses a nonintegrable servo constraint (see [1, 64, 70, 72, 119–121, 157, 164–167, 182, 184, 191, 198, 237, 242, 346, 351, 352, 390]).

Earlier (see [164–167]), the author has already proved the complete integrability of the equations of a plane-parallel motion of a body in a resisting medium under the jet flow conditions when the system of dynamical equations possesses a first integral, which is a transcendental (in the sense of the theory of functions of a complex variable, i.e., it has essential singularities) function of quasi-velocities. It was assumed that the interaction of the medium with the body is concentrated on a part of the surface of the body that has the form of a (one-dimensional) plate.

In [193, 196, 202, 204, 208, 209, 218, 232, 241], the planar problem was generalized to the spatial (three-dimensional) case, where the system of dynamical equations has a complete set of transcendental first integrals. It was assumed that the interaction of the medium with the body is concentrated on a part of the body surface that has the form of a planar (two-dimensional) disk.

In this chapter, we review both new results and results obtained earlier. We systematize these results and present them in the invariant form. Moreover, we introduce an extra dependence of the moment of the nonconservative force on the angular velocity. This dependence can be further extended to cases of the motion in spaces of higher dimension.

1. General Problem on the Motion under the Tracing Force

Let us consider the plane-parallel motion of a symmetric rigid body with flat front end face (two-dimensional disk) in the field of a resisting force under the assumption of quasi-stationarity (see [28, 50, 51, 62–66, 98, 112, 119–121, 160–169, 171, 431, 432]). If (v, α, β_1) are the spherical coordinates of the velocity vector of a certain typical point D of a rigid body (D is the center of the disk lying on the symmetry axis of the body), $\{\Omega_1, \Omega_2, \Omega_3\}$ are the projections of its angular velocity Ω to the coordinate axes of the coordinate system $Dx_1x_2x_3$ attached to the body, where the symmetry axis CD coincides with the axis Dx_1 (C is the center of mass), and the axes Dx_2 and Dx_3 lie in the hyperplane containing the disk, I_1, I_2, $I_3 = I_2$, and m are the characteristics of inertia and mass, then the dynamical part of the equations of motion (including the case of Chaplygin analytical functions [50, 51], see below), where the tangent forces of the interaction of a medium with the body are absent, has the following form:

$$\dot v\cos\alpha - \dot\alpha v\sin\alpha + \Omega_2 v\sin\alpha\sin\beta_1 - \Omega_3 v\sin\alpha\cos\beta_1 + \sigma(\Omega_2^2+\Omega_3^2) = \frac{F_x}{m}, \tag{4.1a}$$

$$\begin{gathered}\dot v\sin\alpha\cos\beta_1 + \dot\alpha v\cos\alpha\cos\beta_1 - \dot\beta_1 v\sin\alpha\sin\beta_1 + \Omega_3 v\cos\alpha \\ -\Omega_1 v\sin\alpha\sin\beta_1 - \sigma\Omega_1\Omega_2 - \sigma\dot\Omega_3 = 0,\end{gathered} \tag{4.1b}$$

$$\dot{v}\sin\alpha\sin\beta_1 + \dot{\alpha}v\cos\alpha\sin\beta_1 + \dot{\beta}_1 v\sin\alpha\cos\beta_1 + \Omega_1 v\sin\alpha\cos\beta_1 \\ -\Omega_2 v\cos\alpha - \sigma\Omega_1\Omega_3 + \sigma\dot{\Omega}_2 = 0, \tag{4.1c}$$

$$I_1\dot{\Omega}_1 = 0, \tag{4.1d}$$

$$I_2\dot{\Omega}_2 + (I_1 - I_2)\Omega_1\Omega_3 = -z_N\left(\alpha, \beta_1, \frac{\Omega}{v}\right)s(\alpha)v^2, \tag{4.1e}$$

$$I_2\dot{\Omega}_3 + (I_2 - I_1)\Omega_1\Omega_2 = y_N\left(\alpha, \beta_1, \frac{\Omega}{v}\right)s(\alpha)v^2, \tag{4.1f}$$

where

$$F_x = -S, \quad S = s(\alpha)v^2, \quad \sigma > 0, \quad v > 0. \tag{4.2}$$

The first two equations of (4.1) describe the motion of the center of mass in the three-dimensional Euclidean space $\mathbf{E}^3$ in the projections on the system of coordinates $Dx_1x_2x_3$. The second three equations of (4.1) are obtained from the theorem on the angular momentum of rigid body in the König axes.

Thus, the direct product of the three-dimensional manifold by the Lie algebra so(3)

$$\mathbf{R}^1 \times \mathbf{S}^2 \times \mathrm{so}(3) \tag{4.3}$$

is the phase space of the sixth-order system (4.1).

We note that system (4.1), by virtue of its dynamical symmetry

$$I_2 = I_3, \tag{4.4}$$

possesses the cyclic first integral

$$\Omega_1 \equiv \Omega_1^0 = \mathrm{const}\,. \tag{4.5}$$

Therefore, we consider the dynamics of the system on the zero level:

$$\Omega_1^0 = 0. \tag{4.6}$$

If we consider a more general problem on the motion of a rigid body under the action of a tracing force $\mathbf{T}$ passing through the center of mass and providing the fulfillment of the condition

$$v \equiv \mathrm{const} \tag{4.7}$$

during the motion (see also [164–167]), then F_x in system (4.1) must be replaced by

$$T - s(\alpha)v^2, \quad \sigma = DC. \tag{4.8}$$

As a result of an appropriate choice of the magnitude T of the tracing force, we can achieve the fulfillment of Eq. (4.7) during the motion. Indeed, if we formally express the value T by virtue of system (4.1), we obtain (for $\cos\alpha \neq 0$):

$$T = T_v(\alpha, \beta_1, \Omega) = m\sigma(\Omega_2^2 + \Omega_3^2) \\ + s(\alpha)v^2\left[1 - \frac{m\sigma}{I_2}\frac{\sin\alpha}{\cos\alpha}\left[z_N\left(\alpha,\ \beta_1,\ \frac{\Omega}{v}\right)\sin\beta_1 + y_N\left(\alpha,\ \beta_1,\ \frac{\Omega}{v}\right)\cos\beta_1\right]\right]. \tag{4.9}$$

Note that we have used conditions (4.5)–(4.7) to obtain Eq. (4.9).

This procedure can be viewed from two standpoints. First, a transformation of the system has occurred for the presence of a tracing (control) force in the system which provides the corresponding class of motions (4.7). Second, we can consider this procedure as a procedure that allows one to reduce the order of the system. Indeed, system (4.1) generates an independent fourth-order system of the following form:

$$\begin{aligned}
&\dot\alpha v\cos\alpha\cos\beta_1 - \dot\beta_1 v\sin\alpha\sin\beta_1 + \Omega_3 v\cos\alpha - \sigma\dot\Omega_3 = 0,\\
&\dot\alpha v\cos\alpha\sin\beta_1 + \dot\beta_1 v\sin\alpha\cos\beta_1 - \Omega_2 v\cos\alpha + \sigma\dot\Omega_2 = 0,\\
&I_2\dot\Omega_2 = -z_N\left(\alpha,\ \beta_1,\ \frac{\Omega}{v}\right)s(\alpha)v^2,\\
&I_2\dot\Omega_3 = y_N\left(\alpha,\ \beta_1,\ \frac{\Omega}{v}\right)s(\alpha)v^2,
\end{aligned}\tag{4.10}$$

where the parameter v is added to the constant parameters specified above.

System (4.10) is equivalent to

$$\begin{aligned}
&\dot\alpha v\cos\alpha + v\cos\alpha\left[\Omega_3\cos\beta_1 - \Omega_2\sin\beta_1\right] + \sigma\left[-\dot\Omega_3\cos\beta_1 + \dot\Omega_2\sin\beta_1\right] = 0,\\
&\dot\beta_1 v\sin\alpha - v\cos\alpha\left[\Omega_2\cos\beta_1 + \Omega_3\sin\beta_1\right] + \sigma\left[\dot\Omega_2\cos\beta_1 + \dot\Omega_3\sin\beta_1\right] = 0,\\
&\dot\Omega_2 = -\frac{v^2}{I_2}z_N\left(\alpha,\ \beta_1,\ \frac{\Omega}{v}\right)s(\alpha),\\
&\dot\Omega_3 = \frac{v^2}{I_2}y_N\left(\alpha,\ \beta_1,\ \frac{\Omega}{v}\right)s(\alpha).
\end{aligned}\tag{4.11}$$

We introduce new quasi-velocities in the system:

$$\begin{aligned}
z_1 &= \Omega_2\cos\beta_1 + \Omega_3\sin\beta_1,\\
z_2 &= -\Omega_2\sin\beta_1 + \Omega_3\cos\beta_1.
\end{aligned}\tag{4.12}$$

As is seen from (4.11), on the manifold

$$O = \left\{(\alpha,\beta_1,\Omega_2,\Omega_3)\in\mathbf{R}^4:\ \alpha = \frac{\pi}{2}k,\ k\in\mathbf{Z}\right\},\tag{4.13}$$

it is impossible to solve the system uniquely with respect to $\dot\alpha$ and $\dot\beta_1$. Thus, the formal violation of the uniqueness theorem occurs on manifold (4.13). Moreover, the indefiniteness occurs for even k because of the degeneration of the spherical coordinates (v,α,β_1), and an obvious violation of the uniqueness theorem for odd k occurs since the first equation of (4.11) is degenerate for this case.

It follows that system (4.11) outside the manifold (4.13) (and only outside it) is equivalent to the system

$$\begin{aligned}
\dot\alpha &= -z_2 + \frac{\sigma v}{I_2}\frac{s(\alpha)}{\cos\alpha}\left[z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1 + y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1\right],\\
\dot z_2 &= \frac{v^2}{I_2}s(\alpha)\left[z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1 + y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1\right]\\
&\quad - z_1^2\frac{\cos\alpha}{\sin\alpha} - \frac{\sigma v}{I_2}\frac{s(\alpha)}{\sin\alpha}z_1\left[z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1 - y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1\right],\\
\dot z_1 &= z_1z_2\frac{\cos\alpha}{\sin\alpha} + \left[-\frac{v^2}{I_2}s(\alpha) + \frac{\sigma v}{I_2}\frac{s(\alpha)}{\sin\alpha}z_2\right]\\
&\quad\times\left[z_N\left(\alpha,\ \beta_1,\ \frac{\Omega}{v}\right)\cos\beta_1 - y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1\right],\\
\dot\beta_1 &= z_1\frac{\cos\alpha}{\sin\alpha} + \frac{\sigma v}{I_2}\frac{s(\alpha)}{\sin\alpha}\left[z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1 - y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1\right].
\end{aligned}\tag{4.14}$$

In the sequel, the dependence on the variables $(\alpha, \beta_1, \Omega/v)$ must be treated as the composite dependence on $(\alpha, \beta_1, z_1/v, z_2/v)$ by virtue of (4.12).

The uniqueness theorem is violated for system (4.11) for odd k on manifold (4.13) in the following sense: for almost every point of manifold (4.13), there exists a regular phase trajectory of system (4.14) passing through this point and intersecting manifold (4.13) at a right angle, and also there exists a phase trajectory that completely coincides with the specified point at all time instants. However, these trajectories are distinct since different values of the tracing force correspond to them.

Indeed, as was shown above, to satisfy constraint (4.7), it is necessary to choose a value T for $\cos\alpha \neq 0$ in the form (4.9).

Let

$$\lim_{\alpha\to\pi/2} \frac{\left[z_N\left(\alpha, \beta_1, \dfrac{\Omega}{v}\right)\sin\beta_1 + y_N\left(\alpha, \beta_1, \dfrac{\Omega}{v}\right)\cos\beta_1\right] s(\alpha)}{\cos\alpha} = L\left(\beta_1, \frac{\Omega}{v}\right). \tag{4.15}$$

Note that $|L| < +\infty$ if and only if

$$\lim_{\alpha\to\pi/2} \left|\frac{\partial}{\partial\alpha}\left(\left[z_N\left(\alpha, \beta_1, \frac{\Omega}{v}\right)\sin\beta_1 + y_N\left(\alpha, \beta_1, \frac{\Omega}{v}\right)\cos\beta_1\right] s(\alpha)\right)\right| < +\infty. \tag{4.16}$$

The required value of the tracing force for $\alpha = \pi/2$ can be found from the equality

$$T = T_v\left(\frac{\pi}{2}, \beta_1, \Omega\right) = m\sigma(\Omega_2^2 + \Omega_3^2) - \frac{m\sigma L v^2}{I_2}, \tag{4.17}$$

where the values of Ω_2 and Ω_3 are arbitrary.

On the other hand, if we support the rotation about a certain point W by means of the tracing force, we must choose the tracing force in the form

$$T = T_v\left(\frac{\pi}{2}, \beta_1, \Omega\right) = \frac{mv^2}{R_0}, \tag{4.18}$$

where R_0 is the distance CW.

Equations (4.9) and (4.18) define, generally speaking, different values of the tracing force T for almost all points of manifold (4.13), which completes the proof.

2. Case where the Moment of Nonconservative Forces Is Independent of the Angular Velocity

2.1. Reduced system. Similarly to the choice of the Chaplygin analytical functions (see [50, 51]), we take the dynamical functions s, y_N, and z_N as follows:

$$\begin{gathered} s(\alpha) = B\cos\alpha, \\ y_N\left(\alpha, \beta_1, \frac{\Omega}{v}\right) = y_0(\alpha, \beta_1) = A\sin\alpha\cos\beta_1, \\ z_N\left(\alpha, \beta_1, \frac{\Omega}{v}\right) = z_0(\alpha, \beta_1) = A\sin\alpha\sin\beta_1, \quad A, B > 0, \quad v \neq 0, \end{gathered} \tag{4.19}$$

which shows that in the considered system, the moment of nonconservative forces is independent of the angular velocity (and depends only on the angles α and β_1).

Then, owing to the nonintegrable constraint (4.7), outside manifold (4.13) (and only outside it), the dynamical part of the equations of motion (system (4.14)) has the following form:

$$
\begin{aligned}
\dot{\alpha} &= -z_2 + \frac{\sigma ABv}{I_2}\sin\alpha,\\
\dot{z}_2 &= \frac{ABv^2}{I_2}\sin\alpha\cos\alpha - z_1^2\frac{\cos\alpha}{\sin\alpha},\\
\dot{z}_1 &= z_1 z_2\frac{\cos\alpha}{\sin\alpha},\\
\dot{\beta}_1 &= z_1\frac{\cos\alpha}{\sin\alpha}.
\end{aligned}
\tag{4.20}
$$

If we introduce dimensionless variables, the parameters, and differentiation as follows:

$$
z_k \mapsto n_0 v z_k, \quad k = 1, 2, \quad n_0^2 = \frac{AB}{I_2}, \quad b = \sigma n_0, \quad \langle\cdot\rangle = n_0 v\langle'\rangle,
\tag{4.21}
$$

we reduce system (4.20) to the form

$$
\begin{aligned}
\alpha' &= -z_2 + b\sin\alpha,\\
z_2' &= \sin\alpha\cos\alpha - z_1^2\frac{\cos\alpha}{\sin\alpha},\\
z_1' &= z_1 z_2\frac{\cos\alpha}{\sin\alpha},
\end{aligned}
\tag{4.22}
$$

$$
\beta_1' = z_1\frac{\cos\alpha}{\sin\alpha}.
\tag{4.23}
$$

We see that in the fourth-order system (4.22), (4.23), an independent third-order system (4.22) with its own three-dimensional manifold is contained. In the sequel, we show that system (4.22), (4.23) can be considered as a system on the tangent bundle $T\mathbf{S}^2$ to the two-dimensional sphere $\mathbf{S}^2$.

2.2. Complete list of invariant relations. First, we compare the third-order system (4.22) with the nonautonomous second-order system

$$
\begin{aligned}
\frac{dz_2}{d\alpha} &= \frac{\sin\alpha\cos\alpha - z_1^2\cos\alpha/\sin\alpha}{-z_2 + b\sin\alpha},\\
\frac{dz_1}{d\alpha} &= \frac{z_1 z_2\cos\alpha/\sin\alpha}{-z_2 + b\sin\alpha}.
\end{aligned}
\tag{4.24}
$$

Using the substitution $\tau = \sin\alpha$, we rewrite system (4.24) in the algebraic form:

$$
\begin{aligned}
\frac{dz_2}{d\tau} &= \frac{\tau - z_1^2/\tau}{-z_2 + b\tau},\\
\frac{dz_1}{d\tau} &= \frac{z_1 z_2/\tau}{-z_2 + b\tau}.
\end{aligned}
\tag{4.25}
$$

Further, if we introduce the uniform variables by the formulas

$$
z_k = u_k\tau, \quad k = 1, 2,
\tag{4.26}
$$

we reduce system (4.25) to the following form:

$$
\begin{aligned}
\tau\frac{du_2}{d\tau} + u_2 &= \frac{1 - u_1^2}{-u_2 + b},\\
\tau\frac{du_1}{d\tau} + u_1 &= \frac{u_1 u_2}{-u_2 + b},
\end{aligned}
\tag{4.27}
$$

which is equivalent to

$$\begin{aligned}\tau\frac{du_2}{d\tau} &= \frac{1-u_1^2+u_2^2-bu_2}{-u_2+b},\\ \tau\frac{du_1}{d\tau} &= \frac{2u_1u_2-bu_1}{-u_2+b}.\end{aligned}\tag{4.28}$$

We compare the second-order system (4.28) with the nonautonomous first-order equation

$$\frac{du_2}{du_1} = \frac{1-u_1^2+u_2^2-bu_2}{2u_1u_2-bu_1},\tag{4.29}$$

which can be easily reduced to the exact differential equation

$$d\left(\frac{u_2^2+u_1^2-bu_2+1}{u_1}\right) = 0.\tag{4.30}$$

Therefore, Eq. (4.29) has the first integral

$$\frac{u_2^2+u_1^2-bu_2+1}{u_1} = C_1 = \text{const},\tag{4.31}$$

which in the old variables has the form

$$\frac{z_2^2+z_1^2-bz_2\sin\alpha+\sin^2\alpha}{z_1\sin\alpha} = C_1 = \text{const}\,.\tag{4.32}$$

Remark 4.1. We consider system (4.22) with variable dissipation with zero mean (see [67, 68, 243, 260, 262, 265, 282–286, 291, 295, 309, 324, 333, 340, 384, 391, 404–408, 412, 413, 421, 437, 438]), which becomes conservative for $b = 0$:

$$\begin{aligned}\alpha' &= -z_2,\\ z_2' &= \sin\alpha\cos\alpha - z_1^2\frac{\cos\alpha}{\sin\alpha},\\ z_1' &= z_1z_2\frac{\cos\alpha}{\sin\alpha}.\end{aligned}\tag{4.33}$$

It has two analytical first integrals of the form

$$z_2^2+z_1^2+\sin^2\alpha = C_1^* = \text{const},\tag{4.34}$$

$$z_1\sin\alpha = C_2^* = \text{const}\,.\tag{4.35}$$

It is obvious that the ratio of the first integrals (4.34) and (4.35) is also a first integral of system (4.33). However, for $b \neq 0$, both functions

$$z_2^2+z_1^2-bz_2\sin\alpha+\sin^2\alpha\tag{4.36}$$

and (4.35) are not first integrals of system (4.22), but their ratio is a first integral of system (4.22) for any b.

Later on, we find the obvious form of the additional first integral of the third-order system (4.22). For this, we transform the invariant relation (4.31) for $u_1 \neq 0$ as follows:

$$\left(u_2-\frac{b}{2}\right)^2 + \left(u_1-\frac{C_1}{2}\right)^2 = \frac{b^2+C_1^2}{4} - 1.\tag{4.37}$$

We see that the parameters of the given invariant relation must satisfy the condition

$$b^2+C_1^2-4 \geq 0,\tag{4.38}$$

and the phase space of system (4.22) is stratified into a family of surfaces defined by Eq. (4.37).

Thus, by virtue of relation (4.31), the first equation of system (4.28) has the form

$$\tau\frac{du_2}{d\tau} = \frac{2(1-bu_2+u_2^2) - C_1U_1(C_1,u_2)}{-u_2+b}, \tag{4.39}$$

where

$$U_1(C_1,u_2) = \frac{1}{2}\Big\{C_1 \pm \sqrt{C_1^2 - 4(u_2^2 - bu_2 + 1)}\Big\}, \tag{4.40}$$

and the integration constant C_1 is chosen from condition (4.38).

Therefore, the quadrature for the search of an additional first integral of system (4.22) has the form

$$\int\frac{d\tau}{\tau} = \int\frac{(b-u_2)du_2}{2(1-bu_2+u_2^2) - C_1\big\{C_1 \pm \sqrt{C_1^2 - 4(u_2^2 - bu_2 + 1)}\big\}/2}. \tag{4.41}$$

Obviously, the left-hand side up to an additive constant is equal to

$$\ln|\sin\alpha|. \tag{4.42}$$

If

$$u_2 - \frac{b}{2} = w_1, \quad b_1^2 = b^2 + C_1^2 - 4, \tag{4.43}$$

then the right-hand side of Eq. (4.41) has the form

$$-\frac{1}{4}\int\frac{d(b_1^2-4w_1^2)}{(b_1^2-4w_1^2) \pm C_1\sqrt{b_1^2-4w_1^2}} - b\int\frac{dw_1}{(b_1^2-4w_1^2) \pm C_1\sqrt{b_1^2-4w_1^2}}$$
$$= -\frac{1}{2}\ln\left|\frac{\sqrt{b_1^2-4w_1^2}}{C_1} \pm 1\right| \pm \frac{b}{2}I_1, \tag{4.44}$$

where

$$I_1 = \int\frac{dw_3}{\sqrt{b_1^2-w_3^2}(w_3 \pm C_1)}, \quad w_3 = \sqrt{b_1^2-4w_1^2}. \tag{4.45}$$

In the calculation of integral (4.45), the following three cases are possible.

I. $b > 2$:

$$I_1 = -\frac{1}{2\sqrt{b^2-4}}\ln\left|\frac{\sqrt{b^2-4}+\sqrt{b_1^2-w_3^2}}{w_3 \pm C_1} \pm \frac{C_1}{\sqrt{b^2-4}}\right|$$
$$+ \frac{1}{2\sqrt{b^2-4}}\ln\left|\frac{\sqrt{b^2-4}-\sqrt{b_1^2-w_3^2}}{w_3 \pm C_1} \mp \frac{C_1}{\sqrt{b^2-4}}\right| + \text{const}. \tag{4.46}$$

II. $b < 2$:

$$I_1 = \frac{1}{\sqrt{4-b^2}}\arcsin\frac{\pm C_1w_3 + b_1^2}{b_1(w_3 \pm C_1)} + \text{const}. \tag{4.47}$$

III. $b = 2$:

$$I_1 = \mp\frac{\sqrt{b_1^2-w_3^2}}{C_1(w_3 \pm C_1)} + \text{const}. \tag{4.48}$$

When we return to the variable

$$w_1 = \frac{z_2}{\sin\alpha} - \frac{b}{2}, \tag{4.49}$$

we obtain the final form for the value I_1:

I. $b > 2$:

$$I_1 = -\frac{1}{2\sqrt{b^2-4}} \ln \left| \frac{\sqrt{b^2-4} \pm 2w_1}{\sqrt{b_1^2 - 4w_1^2} \pm C_1} \pm \frac{C_1}{\sqrt{b^2-4}} \right| + \frac{1}{2\sqrt{b^2-4}} \ln \left| \frac{\sqrt{b^2-4} \mp 2w_1}{\sqrt{b_1^2 - 4w_1^2} \pm C_1} \mp \frac{C_1}{\sqrt{b^2-4}} \right| + \text{const}\,. \tag{4.50}$$

II. $b < 2$:

$$I_1 = \frac{1}{\sqrt{4-b^2}} \arcsin \frac{\pm C_1 \sqrt{b_1^2 - 4w_1^2} + b_1^2}{b_1\big(\sqrt{b_1^2 - 4w_1^2} \pm C_1\big)} + \text{const}\,. \tag{4.51}$$

III. $b = 2$:

$$I_1 = \mp \frac{2w_1}{C_1\big(\sqrt{b_1^2 - 4w_1^2} \pm C_1\big)} + \text{const}\,. \tag{4.52}$$

Thus, we have found an additional first integral for the third-order system (4.22), i.e., we have a complete set of first integrals that are transcendental functions of the phase variables.

Remark 4.2. In the expression of the found first integral, we must formally substitute the left-hand side of the first integral (4.31) instead of C_1. Then the obtained additional first integral has the following structure similar to the transcendental first integral from the planar dynamics):

$$\ln|\sin\alpha| + G_2\left(\sin\alpha, \frac{z_2}{\sin\alpha}, \frac{z_1}{\sin\alpha}\right) = C_2 = \text{const}\,. \tag{4.53}$$

Thus, we have already found two independent first integrals for the integration of the fourth-order system (4.22), (4.23). For its complete integrability, it suffices to find one additional first integral, which attaches Eq. (4.23).

Since

$$\frac{du_1}{d\tau} = \frac{u_1(2u_2 - b)}{(b-u_2)\tau}, \quad \frac{d\beta_1}{d\tau} = \frac{u_1}{(b-u_2)\tau}, \tag{4.54}$$

we have

$$\frac{du_1}{d\beta_1} = 2u_2 - b. \tag{4.55}$$

Obviously, for $u_1 \neq 0$, the following equality holds:

$$u_2 = \frac{1}{2}\left(b \pm \sqrt{b_1^2 - 4\left(u_1 - \frac{C_1}{2}\right)^2}\right), \quad b_1^2 = b^2 + C_1^2 - 4; \tag{4.56}$$

then integration of the quadrature

$$\beta_1 + \text{const} = \pm \int \frac{du_1}{\sqrt{b_1^2 - 4\left(u_1 - \frac{C_1}{2}\right)^2}} \tag{4.57}$$

yields the invariant relation

$$2(\beta_1 + C_3) = \pm \arcsin \frac{2u_1 - C_1}{\sqrt{b^2 + C_1^2 - 4}}, \quad C_3 = \text{const}\,. \tag{4.58}$$

In other words, the equality

$$\sin[2(\beta_1 + C_3)] = \pm \frac{2u_1 - C_1}{\sqrt{b^2 + C_1^2 - 4}} \tag{4.59}$$

holds and, returning to the old variables, we obtain

$$\sin\left[2(\beta_1 + C_3)\right] = \pm\frac{2z_1 - C_1\sin\alpha}{\sqrt{b^2 + C_1^2 - 4\sin\alpha}}. \tag{4.60}$$

Thus, we have obtained an additional invariant relation that "attaches" Eq. (4.23). However, we must formally substitute the left-hand side of (4.31) into the last expression instead of C_1.

We perform certain transformations which lead us to the following form of the additional first integral:

$$\tan^2[2(\beta_1 + C_3)] = \frac{(u_1^2 - u_2^2 + bu_2 - 1)^2}{u_1^2(4u_2^2 - 4bu_2 + b^2)}; \tag{4.61}$$

here Eq. (4.31) is used.

Returning to the old coordinates, we obtain an additional invariant relation of the form

$$\tan^2\left[2(\beta_1 + C_3)\right] = \frac{(z_1^2 - z_2^2 + bz_2\sin\alpha - \sin^2\alpha)^2}{z_1^2(4z_2^2 - 4bz_2\sin\alpha + b^2\sin^2\alpha)}, \tag{4.62}$$

or, finally,

$$-\beta_1 \pm \frac{1}{2}\arctan\frac{z_1^2 - z_2^2 + bz_2\sin\alpha - \sin^2\alpha}{z_1(2z_2 - b\sin\alpha)} = C_3 = \text{const}. \tag{4.63}$$

Therefore, in the considered case, the system of dynamical equations (4.1) under condition (4.19) has five invariant relations: an analytical nonintegrable constraint (4.7), the cyclic first integral (4.5), (4.6), the first integral (4.32), the first integral expressed by relations (4.46)–(4.53), which is a transcendental function of its phase variables (in the sense of the complex analysis) and is expressed as a finite combination of elementary functions, and, finally, the transcendental first integral (4.63).

Theorem 4.1. *System* (4.1) *under the conditions* (4.7), (4.5), (4.6), *and* (4.19) *possesses five invariant relations* (*the complete set*), *three of which are transcendental functions* (*in the sense of the complex analysis*). *All these relations are expressed as finite combinations of elementary functions.*

2.3. Topological analogies. We consider the following third-order system:

$$\begin{aligned} &\ddot{\xi} + b_*\dot{\xi}\cos\xi + \sin\xi\cos\xi - \dot{\eta}_1^2\frac{\sin\xi}{\cos\xi} = 0, \\ &\ddot{\eta}_1 + b_*\dot{\eta}_1\cos\xi + \dot{\xi}\dot{\eta}_1\frac{1+\cos^2\xi}{\cos\xi\sin\xi} = 0, \quad b_* > 0, \end{aligned} \tag{4.64}$$

which describes a fixed spherical pendulum in a flowing medium in the case where the moment of forces is independent of the angular velocity, i.e., a mechanical system is in a nonconservative field of the forces (see [120, 162, 188, 201, 203, 235, 238, 276, 316, 317, 319, 320, 338, 359, 360, 376, 377, 386, 392, 429, 442]). In general, the order of this system seems to be equal to 4, but the phase variable η_1 is cyclic, which leads to the stratification of the phase space and reduction of order.

The phase space is the tangent bundle

$$T\mathbf{S}^2\{\dot{\xi}, \dot{\eta}_1, \xi, \eta_1\} \tag{4.65}$$

of the two-dimensional sphere $\mathbf{S}^2\{\xi, \eta_1\}$, where the equation of large circles

$$\dot{\eta}_1 \equiv 0 \tag{4.66}$$

defines the family of integral manifolds.

It is easy to verify that system (4.64) is equivalent to the dynamical system with variable dissipation with zero mean on the tangent bundle (4.65) of the two-dimensional sphere. Moreover, the following theorem holds.

Theorem 4.2. *System* (4.1) *under the conditions* (4.7), (4.5), (4.6), *and* (4.19) *is equivalent to dynamical system* (4.64).

Indeed, it suffices to accept $\alpha = \xi$, $\beta_1 = \eta_1$, and $b = -b_*$.

On more general topological analogies, see also [253, 265, 267, 309, 340, 342].

3. Case where the Moment of Nonconservative Forces Depends on the Angular Velocity

3.1. Dependence of the moment of nonconservative forces on the angular velocity. We continue to study the dynamics of a three-dimensional rigid body in the three-dimensional space. This section is devoted to the study of the case of the motion where the moment of forces depends on the angular velocity. We introduce this dependence in the general case; this will allow us to generalize this dependence to higher-dimensional bodies.

Let $x = (x_{1N}, x_{2N}, x_{3N})$ be the coordinates of the point N of application of a nonconservative force (interaction with a medium) on a two-dimensional disk and $Q = (Q_1, Q_2, Q_3)$ be the components independent of the angular velocity. We introduce only the linear dependence of the functions $(x_{1N}, x_{2N}, x_{3N}) = (x_N, y_N, z_N)$ on the angular velocity since the introduction of this dependence itself is not a priori obvious (see [33, 34, 48, 49, 57–66, 120, 121, 169, 182, 203, 205, 249, 250, 261, 274–276, 440]).

Thus, we accept the following dependence:

$$x = Q + R, \tag{4.67}$$

where $R = (R_1, R_2, R_3)$ is a vector-valued function containing the angular velocity. Here, the dependence of the function R on the angular velocity is gyroscopic:

$$R = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = -\frac{1}{v} \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}, \tag{4.68}$$

where (h_1, h_2, h_3) are certain positive parameters (cf. [322, 330, 331, 345, 393]).

Now, for our problem, since $x_{1N} = x_N \equiv 0$, we have

$$x_{2N} = y_N = Q_2 - h_1 \frac{\Omega_3}{v}, \qquad x_{3N} = z_N = Q_3 + h_1 \frac{\Omega_2}{v}. \tag{4.69}$$

3.2. Reduced system. Similarly to the choice of the Chaplygin analytical functions (see [50, 51])

$$Q_2 = A \sin\alpha \cos\beta_1, \quad Q_3 = A \sin\alpha \sin\beta_1, \quad A > 0, \tag{4.70}$$

we take the dynamical functions s, y_N, and z_N of the following form:

$$\begin{gathered} s(\alpha) = B\cos\alpha, \quad B > 0, \\ y_N\left(\alpha, \beta_1, \frac{\Omega}{v}\right) = A\sin\alpha\cos\beta_1 - h\frac{\Omega_3}{v}, \quad h = h_1 > 0, \quad v \neq 0, \\ z_N\left(\alpha, \beta_1, \frac{\Omega}{v}\right) = A\sin\alpha\sin\beta_1 + h\frac{\Omega_2}{v}, \quad h = h_1 > 0, \quad v \neq 0, \end{gathered} \tag{4.71}$$

which shows that an additional dependence of the damping (or accelerating in some domains of the phase space) moment of the nonconservative forces is also present in the system considered (i.e., the moment depends on the angular velocity). Moreover, by virtue of the dynamical symmetry of the body, we have $h_2 = h_3$.

Then, owing to the nonintegrable constraint (4.7), outside the manifold (4.13) (and only outside it), the dynamical part of the equations of motion (system (4.14)) has the following form:

$$\begin{aligned}
\dot\alpha &= -\left(1+\frac{\sigma Bh}{I_2}\right)z_2 + \frac{\sigma ABv}{I_2}\sin\alpha,\\
\dot z_2 &= \frac{ABv^2}{I_2}\sin\alpha\cos\alpha - \left(1+\frac{\sigma Bh}{I_2}\right)z_1^2\frac{\cos\alpha}{\sin\alpha} - \frac{Bhv}{I_2}z_2\cos\alpha,\\
\dot z_1 &= \left(1+\frac{\sigma Bh}{I_2}\right)z_1z_2\frac{\cos\alpha}{\sin\alpha} - \frac{Bhv}{I_2}z_1\cos\alpha,\\
\dot\beta_1 &= \left(1+\frac{\sigma Bh}{I_2}\right)z_1\frac{\cos\alpha}{\sin\alpha}.
\end{aligned} \tag{4.72}$$

If we introduce the dimensionless variable, parameters, and differentiation as follows:

$$z_k \mapsto n_0vz_k, \quad k=1,2, \quad n_0^2=\frac{AB}{I_2}, \quad b=\sigma n_0, \quad H_1=\frac{Bh}{I_2n_0}, \quad \langle\cdot\rangle = n_0v\langle'\rangle, \tag{4.73}$$

we reduce system (4.72) to the form

$$\begin{aligned}
\alpha' &= -(1+bH_1)\,z_2 + b\sin\alpha,\\
z_2' &= \sin\alpha\cos\alpha - (1+bH_1)\,z_1^2\frac{\cos\alpha}{\sin\alpha} - H_1z_2\cos\alpha,\\
z_1' &= (1+bH_1)\,z_1z_2\frac{\cos\alpha}{\sin\alpha} - H_1z_1\cos\alpha,
\end{aligned} \tag{4.74}$$

$$\beta_1' = (1+bH_1)\,z_1\frac{\cos\alpha}{\sin\alpha}. \tag{4.75}$$

We see that in the fourth-order system (4.74), (4.75), an independent third-order system (4.74) with its own three-dimensional manifold is contained. In the sequel, we show that system (4.74), (4.75) can be considered as a system on the tangent bundle $T\mathbf{S}^2$ to the two-dimensional sphere $\mathbf{S}^2$.

3.3. Complete list of invariant relations. First, we compare the third-order system (4.74) with the nonautonomous second-order system

$$\begin{aligned}
\frac{dz_2}{d\alpha} &= \frac{\sin\alpha\cos\alpha - (1+bH_1)z_1^2\cos\alpha/\sin\alpha - H_1z_2\cos\alpha}{-(1+bH_1)z_2 + b\sin\alpha},\\
\frac{dz_1}{d\alpha} &= \frac{(1+bH_1)z_1z_2\cos\alpha/\sin\alpha - H_1z_1\cos\alpha}{-(1+bH_1)z_2 + b\sin\alpha}.
\end{aligned} \tag{4.76}$$

We rewrite system (4.76) in the algebraic form using the substitution $\tau = \sin\alpha$:

$$\begin{aligned}
\frac{dz_2}{d\tau} &= \frac{\tau - (1+bH_1)z_1^2/\tau - H_1z_2}{-(1+bH_1)z_2 + b\tau},\\
\frac{dz_1}{d\tau} &= \frac{(1+bH_1)z_1z_2/\tau - H_1z_1}{-(1+bH_1)z_2 + b\tau}.
\end{aligned} \tag{4.77}$$

Further, if we introduce the uniform variables by the formulas

$$z_k = u_k\tau, \quad k=1,2, \tag{4.78}$$

we reduce system (4.77) to the following form:

$$\begin{aligned}
\tau\frac{du_2}{d\tau} + u_2 &= \frac{1 - (1+bH_1)u_1^2 - H_1u_2}{-(1+bH_1)u_2 + b},\\
\tau\frac{du_1}{d\tau} + u_1 &= \frac{(1+bH_1)u_1u_2 - H_1u_1}{-(1+bH_1)u_2 + b},
\end{aligned} \tag{4.79}$$

which is equivalent to

$$\tau\frac{du_2}{d\tau} = \frac{(1+bH_1)(u_2^2-u_1^2)-(b+H_1)u_2+1}{-(1+bH_1)u_2+b},$$
$$\tau\frac{du_1}{d\tau} = \frac{2(1+bH_1)u_1u_2-(b+H_1)u_1}{-(1+bH_1)u_2+b}. \tag{4.80}$$

We compare the second-order system (4.80) with the nonautonomous first-order system

$$\frac{du_2}{du_1} = \frac{1-(1+bH_1)(u_1^2-u_2^2)-(b+H_1)u_2}{2(1+bH_1)u_1u_2-(b+H_1)u_1}, \tag{4.81}$$

which can be easily reduced to the exact differential equation

$$d\left(\frac{(1+bH_1)(u_2^2+u_1^2)-(b+H_1)u_2+1}{u_1}\right) = 0. \tag{4.82}$$

Thus, Eq. (4.81) has the following first integral:

$$\frac{(1+bH_1)(u_2^2+u_1^2)-(b+H_1)u_2+1}{u_1} = C_1 = \text{const}, \tag{4.83}$$

which in the old variables has the form

$$\frac{(1+bH_1)(z_2^2+z_1^2)-(b+H_1)z_2\sin\alpha+\sin^2\alpha}{z_1\sin\alpha} = C_1 = \text{const}\,. \tag{4.84}$$

Remark 4.3. We consider system (4.74) with variable dissipation with zero mean (see [67, 68, 243, 260, 262, 265, 282–286, 291, 295, 309, 324, 333, 340, 384, 391, 404–408, 412, 413, 421, 437, 438]), which becomes conservative for $b = H_1$:

$$\begin{aligned}
\alpha' &= -(1+b^2)z_2 + b\sin\alpha,\\
z_2' &= \sin\alpha\cos\alpha - (1+b^2)z_1^2\frac{\cos\alpha}{\sin\alpha} - bz_2\cos\alpha,\\
z_1' &= (1+b^2)z_1z_2\frac{\cos\alpha}{\sin\alpha} - bz_1\cos\alpha.
\end{aligned} \tag{4.85}$$

It has two analytical first integrals of the form

$$(1+b^2)(z_2^2+z_1^2) - 2bz_2\sin\alpha + \sin^2\alpha = C_1^* = \text{const}, \tag{4.86}$$
$$z_1\sin\alpha = C_2^* = \text{const}\,. \tag{4.87}$$

It is obvious that the ratio of the two first integrals (4.86) and (4.87) is also a first integral of system (4.85). However, for $b \neq H_1$, both functions

$$(1+bH_1)(z_2^2+z_1^2) - (b+H_1)z_2\sin\alpha + \sin^2\alpha \tag{4.88}$$

and (4.87) are not first integrals of system (4.74), but their ratio is a first integral of the system (4.74) for any b and H_1.

Now we find an explicit form of an additional first integral of the third-order system (4.74). First, we transform the invariant relation (4.83) for $u_1 \neq 0$ as follows:

$$\left(u_2 - \frac{b+H_1}{2(1+bH_1)}\right)^2 + \left(u_1 - \frac{C_1}{2(1+bH_1)}\right)^2 = \frac{(b-H_1)^2+C_1^2-4}{4(1+bH_1)^2}. \tag{4.89}$$

We see that the parameters of the given invariant relation must satisfy the condition

$$(b-H_1)^2 + C_1^2 - 4 \geq 0, \tag{4.90}$$

and the phase space of system (4.74) is stratified into a family of surfaces described by Eq. (4.89).

Thus, by virtue of relation (4.83), the first equation of system (4.80) has the form

$$\tau\frac{du_2}{d\tau} = \frac{2(1+bH_1)u_2^2 - 2(b+H_1)u_2 + 2 - C_1U_1(C_1,u_2)}{b-(1+bH_1)u_2}, \tag{4.91}$$

where

$$U_1(C_1,u_2) = \frac{1}{2(1+bH_1)}\{C_1 \pm U_2(C_1,u_2)\}, \tag{4.92}$$

$$U_2(C_1,u_2) = \sqrt{C_1^2 - 4(1+bH_1)(1-(b+H_1)u_2+(1+bH_1)u_2^2)},$$

and the integration constant C_1 is chosen from condition (4.90).

Therefore, the quadrature for the search for an additional first integral of system (4.74) has the form

$$\int\frac{d\tau}{\tau} = \int\frac{(b-(1+bH_1)u_2)du_2}{2(1-(b+H_1)u_2+(1+bH_1)u_2^2) - C_1\{C_1 \pm U_2(C_1,u_2)\}/(2(1+bH_1))}. \tag{4.93}$$

Obviously, the left-hand side (up to an additive constant) is equal to

$$\ln|\sin\alpha|. \tag{4.94}$$

If

$$u_2 - \frac{b+H_1}{2(1+bH_1)} = w_1, \quad b_1^2 = (b-H_1)^2 + C_1^2 - 4, \tag{4.95}$$

then the right-hand side of Eq. (4.93) has the form

$$-\frac{1}{4}\int\frac{d(b_1^2-4(1+bH_1)w_1^2)}{(b_1^2-4(1+bH_1)w_1^2) \pm C_1\sqrt{b_1^2-4(1+bH_1)w_1^2}}$$
$$- (b-H_1)(1+bH_1)\int\frac{dw_1}{(b_1^2-4(1+bH_1)w_1^2) \pm C_1\sqrt{b_1^2-4(1+bH_1)w_1^2}}$$
$$= -\frac{1}{2}\ln\left|\frac{\sqrt{b_1^2-4(1+bH_1)w_1^2}}{C_1} \pm 1\right| \pm \frac{b-H_1}{2}I_1, \tag{4.96}$$

where

$$I_1 = \int\frac{dw_3}{\sqrt{b_1^2-w_3^2}(w_3 \pm C_1)}, \quad w_3 = \sqrt{b_1^2-4(1+bH_1)w_1^2}. \tag{4.97}$$

In the calculation of integral (4.97), the following three cases are possible.

I. $|b-H_1| > 2$:

$$I_1 = -\frac{1}{2\sqrt{(b-H_1)^2-4}}\ln\left|\frac{\sqrt{(b-H_1)^2-4}+\sqrt{b_1^2-w_3^2}}{w_3 \pm C_1} \pm \frac{C_1}{\sqrt{(b-H_1)^2-4}}\right|$$
$$+ \frac{1}{2\sqrt{(b-H_1)^2-4}}\ln\left|\frac{\sqrt{(b-H_1)^2-4}-\sqrt{b_1^2-w_3^2}}{w_3 \pm C_1} \mp \frac{C_1}{\sqrt{(b-H_1)^2-4}}\right| + \text{const}; \tag{4.98}$$

II. $|b-H_1| < 2$:

$$I_1 = \frac{1}{\sqrt{4-(b-H_1)^2}}\arcsin\frac{\pm C_1w_3 + b_1^2}{b_1(w_3 \pm C_1)} + \text{const}; \tag{4.99}$$

III. $|b-H_1| = 2$:

$$I_1 = \mp\frac{\sqrt{b_1^2-w_3^2}}{C_1(w_3 \pm C_1)} + \text{const}. \tag{4.100}$$

When we return to the variable

$$w_1 = \frac{z_2}{\sin\alpha} - \frac{b+H_1}{2(1+bH_1)}, \tag{4.101}$$

we obtain the final form for the value I_1:

I. $|b-H_1|>2$:

$$\begin{aligned} I_1 = &-\frac{1}{2\sqrt{(b-H_1)^2-4}}\ln\left|\frac{\sqrt{(b-H_1)^2-4}\pm 2(1+bH_1)w_1}{\sqrt{b_1^2-4(1+bH_1)^2w_1^2}\pm C_1} \pm \frac{C_1}{\sqrt{(b-H_1)^2-4}}\right| \\ &+\frac{1}{2\sqrt{(b-H_1)^2-4}}\ln\left|\frac{\sqrt{(b-H_1)^2-4}\mp 2(1+bH_1)w_1}{\sqrt{b_1^2-4(1+bH_1)^2w_1^2}\pm C_1} \mp \frac{C_1}{\sqrt{(b-H_1)^2-4}}\right| + \text{const}; \end{aligned} \tag{4.102}$$

II. $|b-H_1|<2$:

$$I_1 = \frac{1}{\sqrt{4-(b-H_1)^2}}\arcsin\frac{\pm C_1\sqrt{b_1^2-4(1+bH_1)^2w_1^2}+b_1^2}{b_1(\sqrt{b_1^2-4(1+bH_1)^2w_1^2}\pm C_1)} + \text{const}; \tag{4.103}$$

III. $|b-H_1|=2$:

$$I_1 = \mp\frac{2(1+bH_1)w_1}{C_1(\sqrt{b_1^2-4(1+bH_1)^2w_1^2}\pm C_1)} + \text{const}\,. \tag{4.104}$$

Thus, we have found an additional first integral for the third-order system (4.74), i.e., we have a complete set of first integrals that are transcendental functions of their phase variables.

Remark 4.4. We must formally substitute the left-hand side of the first integral (4.83) into the expression of the found first integral instead of C_1. Then the additional first integral obtained has the following structure (which is similar to a transcendental first integral from plane dynamics):

$$\ln|\sin\alpha| + G_2\left(\sin\alpha, \frac{z_2}{\sin\alpha}, \frac{z_1}{\sin\alpha}\right) = C_2 = \text{const}\,. \tag{4.105}$$

We have already found two independent first integrals for integration of the fourth-order system (4.74), (4.75). For the complete integrability of the system, it suffices to find one additional first integral that “attaches” Eq. (4.75).

Since

$$\begin{aligned} \frac{du_1}{d\tau} &= \frac{u_1(2(1+bH_1)u_2-(b+H_1))}{(b-(1+bH_1)u_2)\tau}, \\ \frac{d\beta_1}{d\tau} &= \frac{(1+bH_1)u_1}{(b-(1+bH_1)u_2)\tau}, \end{aligned} \tag{4.106}$$

we have

$$\frac{du_1}{d\beta_1} = 2u_2 - \frac{b+H_1}{1+bH_1}. \tag{4.107}$$

Obviously, for $u_1 \neq 0$, the following equality holds:

$$u_2 = \frac{1}{2(1+bH_1)}\left((b+H_1)\pm\sqrt{b_1^2-(2(1+bH_1)u_1-C_1)^2}\right), \tag{4.108}$$

where

$$b_1^2 = (b-H_1)^2 + C_1^2 - 4.$$

Then integration of the quadrature

$$\beta_1 + \text{const} = \pm(1+bH_1)\int\frac{du_1}{\sqrt{b_1^2-(2(1+bH_1)u_1-C_1)^2}} \tag{4.109}$$

leads to the invariant relation

$$2(\beta_1 + C_3) = \pm \arcsin \frac{2(1+bH_1)u_1 - C_1}{\sqrt{(b-H_1)^2 + C_1^2 - 4}}, \quad C_3 = \text{const}\,. \tag{4.110}$$

In other words, the equality

$$\sin[2(\beta_1 + C_3)] = \pm \frac{2(1+bH_1)u_1 - C_1}{\sqrt{(b-H_1)^2 + C_1^2 - 4}} \tag{4.111}$$

is fulfilled and, returning to the old variables, we have

$$\sin[2(\beta_1 + C_3)] = \pm \frac{2(1+bH_1)z_1 - C_1 \sin\alpha}{\sqrt{(b-H_1)^2 + C_1^2 - 4\sin\alpha}}. \tag{4.112}$$

Thus, we have obtained an additional invariant relation that "attaches" Eq. (4.75). However, we must formally substitute the left-hand side of (4.83) into the last expression instead of C_1.

We make certain transformations that lead us to the following explicit form of the additional first integral:

$$\tan^2\left[2(\beta_1 + C_3)\right] = \frac{((1+bH_1)u_1^2 - (1+bH_1)u_2^2 + (b+H_1)u_2 - 1)^2}{u_1^2(2(1+bH_1)u_2 - (b+H_1))^2} \tag{4.113}$$

(here Eq. (4.83) is used).

Returning to the old coordinates, we obtain the additional invariant relation of the form

$$\tan^2\left[2(\beta_1 + C_3)\right] = \frac{((1+bH_1)z_1^2 - (1+bH_1)z_2^2 + (b+H_1)z_2\sin\alpha - \sin^2\alpha)^2}{z_1^2(2(1+bH_1)z_2 - (b+H_1)\sin\alpha)^2}, \tag{4.114}$$

or, finally,

$$-\beta_1 \pm \frac{1}{2}\arctan \frac{(1+bH_1)z_1^2 - (1+bH_1)z_2^2 + (b+H_1)z_2\sin\alpha - \sin^2\alpha}{z_1(2(1+bH_1)z_2 - (b+H_1)\sin\alpha)} = C_3 = \text{const}\,. \tag{4.115}$$

Therefore, in the considered case, the system of dynamical equations (4.1) under condition (4.71) has five invariant relations: the analytical nonintegrable constraint (4.7), the cyclic first integral (4.5), (4.6), the first integral (4.84), the first integral expressed by relations (4.98)–(4.105), which is a transcendental function of its phase variables (in the sense of complex analysis) and is expressed as a finite combination of elementary functions, and, finally, the transcendental first integral (4.115).

Theorem 4.3. *System* (4.1) *under conditions* (4.7), (4.5), (4.6), *and* (4.71) *possesses five invariant relations* (*a complete set*), *three of which are transcendental functions* (*in the sense of the complex analysis*). *All these relations are expressed as finite combinations of elementary functions.*

3.4. Topological analogies. We consider the following third-order system:

$$\begin{gathered}
\ddot{\xi} + (b_* - H_1^*)\dot{\xi}\cos\xi + \sin\xi\cos\xi - \dot{\eta}_1^2 \frac{\sin\xi}{\cos\xi} = 0, \\
\ddot{\eta}_1 + (b_* - H_1^*)\dot{\eta}_1\cos\xi + \dot{\xi}\dot{\eta}_1 \frac{1+\cos^2\xi}{\cos\xi\sin\xi} = 0, \quad b_* > 0, \quad H_1^* > 0,
\end{gathered} \tag{4.116}$$

which describes a fixed spherical pendulum in a flowing medium in the case where the moment of forces depends on the angular velocity, i.e., the mechanical system is in a nonconservative field of forces (see [120, 162, 188, 201, 203, 235, 238, 276, 316, 317, 319, 320, 338, 359, 360, 376, 377, 386, 392, 429, 442]). In general, the order of this system seems to be equal to 4, but the phase variable η_1 is cyclic, which leads to the stratification of the phase space and reduction of order.

The phase space of the system is the tangent bundle

$$T\mathbf{S}^2\{\dot{\xi}, \dot{\eta}_1, \xi, \eta_1\} \tag{4.117}$$

of the two-dimensional sphere $\mathbf{S}^2\{\xi, \eta_1\}$, where the equation of large circles

$$\dot{\eta}_1 \equiv 0 \tag{4.118}$$

defines a family of integral manifolds.

It is easy to verify that system (4.116) is equivalent to the dynamical system with variable dissipation with zero mean on the tangent bundle (4.117) of the two-dimensional sphere. Moreover, the following theorem holds.

Theorem 4.4. *System* (4.1) *under conditions* (4.7)*,* (4.5)*,* (4.6)*, and* (4.71) *is equivalent to the dynamical system* (4.116)*.*

Indeed, it suffices to take $\alpha = \xi$, $\beta_1 = \eta_1$, $b = -b_*$, and $H_1 = -H_1^*$.

On more general topological analogies, see also [253, 265, 267, 309, 340, 342].

Chapter 5

CASES OF INTEGRABILITY CORRESPONDING TO THE MOTION OF A RIGID BODY IN THE THREE-DIMENSIONAL SPACE, II

In this chapter, we systematize some earlier results and new results on the study of the equations of motion of dynamically symmetric three-dimensional (3D) rigid bodies in nonconservative force fields. The form of these equations is taken from the dynamics of real rigid bodies interacting with a resisting medium by laws of jet flows where a body is influenced by a nonconservative tracing force. Under action of this force, the center of mass of the body moves rectilinearly and uniformly; this means that there exists a nonconservative couple of forces in the system (see [1, 64, 70, 72, 119–121, 157, 164–167, 180, 181, 184, 191, 194, 212, 231, 258, 291, 353, 354, 374, 390, 414]).

Earlier (see [164–167]), the author already proved the complete integrability of the equations of a plane-parallel motion of a body in a resisting medium under the jet flow conditions when the system of dynamical equations possesses a first integral, which is a transcendental (in the sense of the theory of functions of a complex variable, i.e., it has essential singularities) function of quasi-velocities. It was assumed that the interaction of the medium with the body is concentrated on a part of the surface of the body that has the form of a (one-dimensional) plate.

In [193, 196, 202, 204, 208, 209, 218, 232, 241], the planar problem was generalized to the spatial (three-dimensional) case, where the system of dynamical equations has a complete set of transcendental first integrals. It was assumed that the interaction of the medium with the body is concentrated on a part of the body surface that has the form of a planar (two-dimensional) disk.

In this chapter, we review both new results and results obtained earlier. We systematize these results and present them in the invariant form. Moreover, we introduce an extra dependence of the moment of the nonconservative force on the angular velocity. This dependence can be further extended to cases of the motion in spaces of higher dimension.

1. General Problem on the Motion under the Tracing Force

Let us consider the spatial motion of a homogeneous axis-symmetrical rigid body with flat front end face (two-dimensional disk) in the field of a resisting force under the assumption of quasi-stationarity (see [28, 50, 51, 62–66, 98, 112, 119–121, 160–169, 171, 431, 432]). If (v, α, β_1) are the spherical coordinates of the velocity vector of a certain typical point D of a rigid body (D is the center of the disk lying on the symmetry axis of the body), $\{\Omega_1, \Omega_2, \Omega_3\}$ are the projections of its angular velocity Ω

onto the coordinate axes of the coordinate system $Dx_1x_2x_3$ attached to the body, where the symmetry axis CD coincides with the axis Dx_1 (C is the center of mass), and the axes Dx_2 and Dx_3 lie in the hyperplane containing the disk, I_1, I_2, $I_3 = I_2$, and m are the characteristics of inertia and mass, then the dynamical part of the equations of motion (including the case of Chaplygin analytical functions [50, 51], see below), where the tangent forces of the interaction of a medium with the body are absent, has the following form:

$$\begin{gathered}
\dot{v}\cos\alpha - \dot{\alpha}v\sin\alpha + \Omega_2 v\sin\alpha\sin\beta_1 - \Omega_3 v\sin\alpha\cos\beta_1 + \sigma(\Omega_2^2 + \Omega_3^2) = \frac{F_x}{m}, \\
\dot{v}\sin\alpha\cos\beta_1 + \dot{\alpha}v\cos\alpha\cos\beta_1 - \dot{\beta}_1 v\sin\alpha\sin\beta_1 + \Omega_3 v\cos\alpha \\
-\Omega_1 v\sin\alpha\sin\beta_1 - \sigma\Omega_1\Omega_2 - \sigma\dot{\Omega}_3 = 0, \\
\dot{v}\sin\alpha\sin\beta_1 + \dot{\alpha}v\cos\alpha\sin\beta_1 + \dot{\beta}_1 v\sin\alpha\cos\beta_1 + \Omega_1 v\sin\alpha\cos\beta_1 \\
-\Omega_2 v\cos\alpha - \sigma\Omega_1\Omega_3 + \sigma\dot{\Omega}_2 = 0, \\
\dot{\Omega}_1 = 0, \\
I_2\dot{\Omega}_2 + (I_1 - I_2)\Omega_1\Omega_3 = -z_N\left(\alpha, \beta_1, \frac{\Omega}{v}\right)s(\alpha)v^2, \\
I_2\dot{\Omega}_3 + (I_2 - I_1)\Omega_1\Omega_2 = y_N\left(\alpha, \beta_1, \frac{\Omega}{v}\right)s(\alpha)v^2,
\end{gathered} \tag{5.1}$$

where

$$F_x = -S, \qquad S = s(\alpha)v^2, \quad \sigma > 0, \quad v > 0. \tag{5.2}$$

The first two equations of (5.1) describe the motion of the center of mass in the three-dimensional Euclidean space $\mathbf{E}^3$ in the projections on the system of coordinates $Dx_1x_2x_3$. The second three equations of (5.1) are obtained from the theorem on the angular momentum of a rigid body in the König axes.

Thus, the direct product of the three-dimensional manifold by the Lie algebra so(3)

$$\mathbf{R}^1 \times \mathbf{S}^2 \times \mathrm{so}(3) \tag{5.3}$$

is the phase space of sixth-order system (5.1).

We note that system (5.1), by virtue of its dynamical symmetry

$$I_2 = I_3, \tag{5.4}$$

possesses the cyclic first integral

$$\Omega_1 \equiv \Omega_1^0 = \text{const}. \tag{5.5}$$

Therefore, we consider the dynamics of the system on the zero level:

$$\Omega_1^0 = 0. \tag{5.6}$$

If we consider a more general problem on the motion of a rigid body under the action of a tracing force $\mathbf{T}$ passing through the center of mass and providing the fulfillment of the condition

$$\mathbf{V}_C \equiv \text{const} \tag{5.7}$$

during the motion ($\mathbf{V}_C$ is the velocity of the center of mass, see also [164–167]), then F_x in system (5.1) must be replaced by zero since a nonconservative couple of forces acts on the body:

$$T - s(\alpha)v^2 \equiv 0, \quad \sigma = DC. \tag{5.8}$$

Obviously, we need to choose the magnitude of the tracing force T as follows:

$$T = T_v(\alpha, \Omega) = s(\alpha)v^2, \quad \mathbf{T} \equiv -\mathbf{S}. \tag{5.9}$$

The choice (5.9) of the magnitude of the tracing force T is a particular case in which an independent second-order subsystem can be extracted from the sixth-order system (5.1) after a certain transformation.

Indeed, let the following condition hold for the value T:

$$T = T_v(\alpha, \beta_1, \Omega) = \sum_{\substack{i,j=0 \\ i\leq j}}^{3} \tau_{i,j}\left(\alpha, \beta_1, \frac{\Omega}{v}\right)\Omega_i\Omega_j = T_1\left(\alpha, \beta_1, \frac{\Omega}{v}\right)v^2, \quad \Omega_0 = v. \tag{5.10}$$

First, we introduce the new quasi-velocities:

$$z_1 = \Omega_2 \cos\beta_1 + \Omega_3 \sin\beta_1, \quad z_2 = -\Omega_2 \sin\beta_1 + \Omega_3 \cos\beta_1. \tag{5.11}$$

We can rewrite system (5.1) in the cases (5.5) and (5.6) as follows:

$$\begin{gathered}
\dot v + \sigma(z_1^2+z_2^2)\cos\alpha - \sigma\frac{v^2}{I_2}s(\alpha)\sin\alpha\left[y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1 + z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1\right] \\
= \frac{T_1\left(\alpha,\beta_1,\frac{\Omega}{v}\right)v^2 - s(\alpha)v^2}{m}\cos\alpha, \\
\dot\alpha v + z_2 v - \sigma(z_1^2+z_2^2)\sin\alpha - \sigma\frac{v^2}{I_2}s(\alpha)\cos\alpha\left[y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1 + z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1\right] \\
= \frac{s(\alpha)v^2 - T_1\left(\alpha,\beta_1,\frac{\Omega}{v}\right)v^2}{m}\sin\alpha, \\
\dot\Omega_3 = \frac{v^2}{I_2}y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)s(\alpha), \qquad \dot\Omega_2 = -\frac{v^2}{I_2}z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)s(\alpha), \\
\dot\beta_1\sin\alpha - z_1\cos\alpha - \frac{\sigma v}{I_2}s(\alpha)\left[z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1 - y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1\right] = 0.
\end{gathered} \tag{5.12}$$

Introducing the new dimensionless phase variables and differentiation by the formulas

$$z_k = n_1 v Z_k, \quad k = 1,2, \quad \langle\cdot\rangle = n_1 v\langle'\rangle, \quad n_1 > 0, \quad n_1 = \text{const}, \tag{5.13}$$

we rewrite system (5.12) in the following form:

$$v' = v\Psi(\alpha, \beta_1, Z_1, Z_2), \tag{5.14}$$

$$\begin{aligned}
\alpha' = -Z_2 + \sigma n_1(Z_1^2+Z_2^2)\sin\alpha \\
+ \frac{\sigma}{I_2 n_1}s(\alpha)\cos\alpha\Big[y_N(\alpha,\beta_1,n_1 Z)\cos\beta_1 + z_N(\alpha,\beta_1,n_1 Z)\sin\beta_1\Big] \\
- \frac{T_1(\alpha,\beta_1,n_1 Z) - s(\alpha)}{m n_1}\sin\alpha,
\end{aligned} \tag{5.15}$$

$$\begin{aligned}
Z_2' = \frac{s(\alpha)}{I_2 n_1^2}\Big[1 - \sigma n_1 Z_2\sin\alpha\Big]\Big[y_N(\alpha,\beta_1,n_1 Z)\cos\beta_1 + z_N(\alpha,\beta_1,n_1 Z)\sin\beta_1\Big] - Z_1^2\frac{\cos\alpha}{\sin\alpha} \\
+ \sigma n_1 Z_2(Z_1^2+Z_2^2)\cos\alpha - \frac{\sigma}{I_2 n_1}Z_1\frac{s(\alpha)}{\sin\alpha}\Big[z_N(\alpha,\beta_1,n_1 Z)\cos\beta_1 - y_N(\alpha,\beta_1,n_1 Z)\sin\beta_1\Big] \\
- Z_2\frac{T_1(\alpha,\beta_1,n_1 Z) - s(\alpha)}{m n_1}\cos\alpha,
\end{aligned} \tag{5.16}$$

$$Z_1' = \frac{1}{I_2 n_1^2}\frac{s(\alpha)}{\sin\alpha}\Big[\sigma n_1 Z_2 \sin\alpha - 1\Big]\Big[z_N(\alpha,\beta_1,n_1Z)\cos\beta_1 - y_N(\alpha,\beta_1,n_1Z)\sin\beta_1\Big] + Z_1Z_2\frac{\cos\alpha}{\sin\alpha}$$
$$+ \sigma n_1 Z_1(Z_1^2+Z_2^2)\cos\alpha - \frac{\sigma}{I_2 n_1} Z_1 s(\alpha)\sin\alpha\Big[z_N(\alpha,\beta_1,n_1Z)\sin\beta_1 + y_N(\alpha,\beta_1,n_1Z)\cos\beta_1\Big]$$
$$- Z_1\frac{T_1(\alpha,\beta_1,n_1Z)-s(\alpha)}{mn_1}\cos\alpha, \quad (5.17)$$

$$\beta_1' = Z_1\frac{\cos\alpha}{\sin\alpha} + \frac{\sigma}{I_2 n_1}\frac{s(\alpha)}{\sin\alpha}\Big[z_N(\alpha,\beta_1,n_1Z)\cos\beta_1 - y_N(\alpha,\beta_1,n_1Z)\sin\beta_1\Big], \quad (5.18)$$

$$\Psi(\alpha,\beta_1,Z_1,Z_2) = -\sigma n_1(Z_1^2+Z_2^2)\cos\alpha$$
$$+ \frac{\sigma}{I_2 n_1}s(\alpha)\sin\alpha\Big[y_N(\alpha,\beta_1,n_1Z)\cos\beta_1 + z_N(\alpha,\beta_1,n_1Z)\sin\beta_1\Big]$$
$$+ \frac{T_1(\alpha,\beta_1,n_1Z)-s(\alpha)}{mn_1}\cos\alpha.$$

We see that the independent fourth-order subsystem (5.15)–(5.18) can be extracted from the fifth-order system (5.14)–(5.18); we can consider this subsystem on its own four-dimensional phase space.

In particular, under the condition (5.9), the extraction of an independent fourth-order subsystem is also possible.

2. Case where the Moment of Nonconservative Forces Is Independent of the Angular Velocity

2.1. Reduced system. Similarly to the choice of the Chaplygin analytical functions (see [50, 51]), we take the dynamical functions s, y_N, and z_N as follows:

$$s(\alpha) = B\cos\alpha, \quad y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right) = y_0(\alpha,\beta_1) = A\sin\alpha\cos\beta_1,$$
$$z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right) = z_0(\alpha,\beta_1) = A\sin\alpha\sin\beta_1, \quad A,B>0, \quad v\neq 0, \quad (5.19)$$

which shows that in the considered system, the moment of nonconservative forces is independent of the angular velocity (and depends only on the angles α and β_1).

Then, owing to the conditions (5.7) and (5.19), the transformed dynamical part of the equations of motion (system (5.14)–(5.18)) has the following form:

$$v' = v\Psi(\alpha,\beta_1,Z_1,Z_2), \quad (5.20)$$

$$\alpha' = -Z_2 + b(Z_1^2+Z_2^2)\sin\alpha + b\sin\alpha\cos^2\alpha,$$
$$Z_2' = \sin\alpha\cos\alpha - Z_1^2\frac{\cos\alpha}{\sin\alpha} + bZ_2(Z_1^2+Z_2^2)\cos\alpha - bZ_2\sin^2\alpha\cos\alpha, \quad (5.21)$$

$$Z_1' = Z_1Z_2\frac{\cos\alpha}{\sin\alpha} + bZ_1(Z_1^2+Z_2^2)\cos\alpha - bZ_1\sin^2\alpha\cos\alpha,$$
$$\beta_1' = Z_1\frac{\cos\alpha}{\sin\alpha}, \quad (5.22)$$

$$\Psi(\alpha,\beta_1,Z_1,Z_2) = -b(Z_1^2+Z_2^2)\cos\alpha + b\sin^2\alpha\cos\alpha,$$

where, as above, we choose the dimensionless variable, the parameter b, and the constant n_1 as follows:

$$b = \sigma n_0, \quad n_0^2 = \frac{AB}{I_2}, \quad n_1 = n_0. \quad (5.23)$$

Therefore, system (5.20)–(5.22) can be considered on its own five-dimensional phase manifold

$$W_1 = \mathbf{R}_+^1\{v\}\times T\mathbf{S}^2\{Z_1,\ Z_2,\ 0<\alpha<\pi,\ 0\le\beta_1<2\pi\}, \quad (5.24)$$

i.e., on the direct product of the real line on the tangent bundle of the two-dimensional sphere $\mathbf{S}^2\{0 < \alpha < \pi,\ 0 \le \beta_1 < 2\pi\}$.

2.2. Complete list of first integrals. The independent fourth-order system (5.21), (5.22) was extracted from system (5.20)–(5.22).

We note that the magnitude of the velocity of the center of mass is the first integral of system (5.1) by virtue of (5.7) and, therefore, the function of the phase variables

$$\Psi_0(v, \alpha, \beta_1, z_1, z_2) = v^2 + \sigma^2(z_1^2 + z_2^2) - 2\sigma z_2 v \sin\alpha = V_C^2 \tag{5.25}$$

is constant on its phase trajectories (where the values z_1 and z_2 are chosen as in (5.11)).

Performing a nondegenerate change of the independent variable (for $v \neq 0$), we see that system (5.20)–(5.22) also has an analytical integral and, therefore, the function of the phase variables

$$\Psi_1(v, \alpha, \beta_1, Z_1, Z_2) = v^2(1 + b^2(Z_1^2 + Z_2^2) - 2bZ_2 \sin\alpha) = V_C^2 \tag{5.26}$$

is constant on its phase trajectories.

Equality (5.26) allows one to find the dependence of the velocity of a certain point of the rigid body (namely, the center D of the disk) on other phase variables without solution of the system (5.20)–(5.22); namely, for $V_C \neq 0$, the equality

$$v^2 = \frac{V_C^2}{1 + b^2(Z_1^2 + Z_2^2) - 2bZ_2 \sin\alpha} \tag{5.27}$$

holds.

Since the phase space (5.24) of system (5.20)–(5.22) is five-dimensional and there exist asymptotic limit sets in it, we see that Eq. (5.26) defines a unique analytical (even continuous) first integral of system (5.20)–(5.22) in the whole phase space (cf. [23, 24, 43, 53, 103, 105, 137, 152–154, 156, 213, 427, 438]).

We consider in detail the problem of the existence of other additional first integrals of system (5.20)–(5.22). Its phase space is stratified into surfaces

$$\{(v, \alpha, \beta_1, Z_1, Z_2) \in W_1 : V_C = \text{const}\}, \tag{5.28}$$

on which the dynamics is defined by the first integral of system (5.21), (5.22).

First, we compare the third-order system (5.21) with the nonautonomous second-order system

$$\begin{aligned}
\frac{dZ_2}{d\alpha} &= \frac{\sin\alpha\cos\alpha + bZ_2(Z_1^2 + Z_2^2)\cos\alpha - bZ_2\sin^2\alpha\cos\alpha - Z_1^2\cos\alpha/\sin\alpha}{-Z_2 + b(Z_1^2 + Z_2^2)\sin\alpha + b\sin\alpha\cos^2\alpha},\\
\frac{dZ_1}{d\alpha} &= \frac{bZ_1(Z_1^2 + Z_2^2)\cos\alpha - bZ_1\sin^2\alpha\cos\alpha + Z_1Z_2\cos\alpha/\sin\alpha}{-Z_2 + b(Z_1^2 + Z_2^2)\sin\alpha + b\sin\alpha\cos^2\alpha}.
\end{aligned} \tag{5.29}$$

We rewrite system (5.29) in the algebraic form using the substitution $\tau = \sin\alpha$:

$$\begin{aligned}
\frac{dZ_2}{d\tau} &= \frac{\tau + bZ_2(Z_1^2 + Z_2^2) - bZ_2\tau^2 - Z_1^2/\tau}{-Z_2 + b\tau(1 - \tau^2) + b\tau(Z_1^2 + Z_2^2)},\\
\frac{dZ_1}{d\tau} &= \frac{bZ_1(Z_1^2 + Z_2^2) - bZ_1\tau^2 + Z_1Z_2/\tau}{-Z_2 + b\tau(1 - \tau^2) + b\tau(Z_1^2 + Z_2^2)}.
\end{aligned} \tag{5.30}$$

Further, if we introduce the homogeneous variables by the formulas

$$Z_k = u_k\tau, \quad k = 1, 2, \tag{5.31}$$

we reduce system (5.30) to the following form:

$$
\begin{aligned}
\tau\frac{du_2}{d\tau}+u_2&=\frac{1-bu_2\tau^2+bu_2(u_1^2+u_2^2)\tau^2-u_1^2}{-u_2+b\tau^2(u_1^2+u_2^2)+b(1-\tau^2)},\\
\tau\frac{du_1}{d\tau}+u_1&=\frac{bu_1(u_1^2+u_2^2)\tau^2-bu_1\tau^2+u_1u_2}{-u_2+b\tau^2(u_1^2+u_2^2)+b(1-\tau^2)},
\end{aligned}
\tag{5.32}
$$

which is equivalent to

$$
\begin{aligned}
\tau\frac{du_2}{d\tau}&=\frac{1-bu_2+u_2^2-u_1^2}{-u_2+b\tau^2(u_1^2+u_2^2)+b(1-\tau^2)},\\
\tau\frac{du_1}{d\tau}&=\frac{2u_1u_2-bu_1}{-u_2+b\tau^2(u_1^2+u_2^2)+b(1-\tau^2)}.
\end{aligned}
\tag{5.33}
$$

We compare the second-order system (5.33) with the nonautonomous first-order equation

$$
\frac{du_2}{du_1}=\frac{1-bu_2+u_2^2-u_1^2}{2u_1u_2-bu_1},
\tag{5.34}
$$

which can be easily reduced to the exact differential equation

$$
d\left(\frac{u_2^2+u_1^2-bu_2+1}{u_1}\right)=0.
\tag{5.35}
$$

Therefore, Eq. (5.34) has the following first integral:

$$
\frac{u_2^2+u_1^2-bu_2+1}{u_1}=C_1=\text{const},
\tag{5.36}
$$

which in the old variables has the form

$$
\frac{Z_2^2+Z_1^2-bZ_2\sin\alpha+\sin^2\alpha}{Z_1\sin\alpha}=C_1=\text{const}\,.
\tag{5.37}
$$

Remark 5.1. We consider system (5.21) with variable dissipation with zero mean (see [67, 68, 243, 260, 262, 265, 282–286, 291, 295, 309, 324, 333, 340, 384, 391, 404–408, 412, 413, 421, 437, 438]), which becomes conservative for $b=0$:

$$
\begin{aligned}
\alpha'&=-Z_2,\\
Z_2'&=\sin\alpha\cos\alpha-Z_1^2\frac{\cos\alpha}{\sin\alpha},\\
Z_1'&=Z_1Z_2\frac{\cos\alpha}{\sin\alpha}.
\end{aligned}
\tag{5.38}
$$

It has two analytical first integrals of the form

$$
Z_2^2+Z_1^2+\sin^2\alpha=C_1^*=\text{const},
\tag{5.39}
$$

$$
Z_1\sin\alpha=C_2^*=\text{const}\,.
\tag{5.40}
$$

Obviously, the ratio of the two first integrals (5.39) and (5.40) is also a first integral of system (5.38). However, for $b\neq 0$, both functions

$$
Z_2^2+Z_1^2-bZ_2\sin\alpha+\sin^2\alpha
\tag{5.41}
$$

and (5.40) are not first integrals of system (5.21), but their ratio is a first integral of system (5.21) for any b.

Further, we find an explicit form of an additional first integral of the third-order system (5.21). For this, first, we transform the invariant relation (5.36) for $u_1\neq 0$ as follows:

$$
\left(u_2-\frac{b}{2}\right)^2+\left(u_1-\frac{C_1}{2}\right)^2=\frac{b^2+C_1^2}{4}-1.
\tag{5.42}
$$

We see that the parameters of the given invariant relation must satisfy the condition

$$b^2 + C_1^2 - 4 \geq 0, \tag{5.43}$$

and the phase space of system (5.21) is stratified into a family of surfaces defined by Eq. (5.42).

Thus, by virtue of relation (5.36), the first equation of system (5.33) takes either the form

$$\tau \frac{du_2}{d\tau} = \frac{1 - bu_2 + u_2^2 - U_1^2(C_1, u_2)}{-u_2 + b(1-\tau^2) + b\tau^2(U_1^2(C_1, u_2) + u_2^2)}, \tag{5.44}$$

where

$$U_1(C_1, u_2) = \frac{1}{2}\left\{C_1 \pm \sqrt{C_1^2 - 4(u_2^2 - bu_2 + 1)}\right\} \tag{5.45}$$

and the integration constant C_1 is chosen according to the condition (5.43), or the form of a Bernoulli equation:

$$\frac{d\tau}{du_2} = \frac{(b - u_2)\tau - b\tau^3(1 - U_1^2(C_1, u_2) - u_2^2)}{1 - bu_2 + u_2^2 - U_1^2(C_1, u_2)}. \tag{5.46}$$

Using (5.45), we can easily reduce Eq. (5.46) to the linear nonhomogeneous equation

$$\frac{dp}{du_2} = \frac{2(u_2 - b)p + 2b(1 - U_1^2(C_1, u_2) - u_2^2)}{1 - bu_2 + u_2^2 - U_1^2(C_1, u_2)}, \quad p = \frac{1}{\tau^2}. \tag{5.47}$$

This fact means that we can find another transcendental first integral in the explicit form (i.e., as a finite combination of quadratures). Here, the general solution of Eq. (5.47) depends on an arbitrary constant C_2. We omit the calculation, but note that the general solution of the linear homogeneous equation obtained from (5.47) even in the particular case $b = C_1 = 2$ has the following solution:

$$p = p_0(u_2) = C\left[\sqrt{1 - (u_2 - 1)^2} \pm 1\right] \exp\left[\sqrt{\frac{1 \mp \sqrt{1 - (u_2 - 1)^2}}{1 \pm \sqrt{1 - (u_2 - 1)^2}}}\right], \quad C = \text{const}. \tag{5.48}$$

Remark 5.2. We must substitute formally the left-hand side of the first integral (5.36) into the expression of the found first integral instead of C_1. Then the obtained additional first integral has the following structural form (which is similar to the transcendental first integral from the plane-parallel dynamics):

$$K_1\left(\sin\alpha, Z_2, Z_1, \frac{Z_2}{\sin\alpha}, \frac{Z_1}{\sin\alpha}\right) = C_2 = \text{const}. \tag{5.49}$$

Thus, we have already found two independent first integrals for integration of the forth-order system (5.21), (5.22). For the complete integrability, it suffices find one additional first integral that "connects" Eq. (5.22).

Since

$$\frac{d\beta_1}{d\tau} = \frac{u_1}{-u_2\tau + b\tau^3(u_1^2 + u_2^2) + b\tau(1-\tau^2)}, \tag{5.50}$$

$$\frac{du_1}{d\tau} = \frac{u_1(2u_2 - b)}{-u_2\tau + b\tau^3(u_1^2 + u_2^2) + b\tau(1-\tau^2)}, \tag{5.51}$$

we have

$$\frac{du_1}{d\beta_1} = 2u_2 - b. \tag{5.52}$$

Obviously, for $u_1 \neq 0$, the following equality holds:

$$u_2 = \frac{1}{2}\left(b \pm \sqrt{b_1^2 - 4\left(u_1 - \frac{C_1}{2}\right)^2}\right), \quad b_1^2 = b^2 + C_1^2 - 4; \tag{5.53}$$

therefore, integration of the quadrature

$$\beta_1 + \text{const} = \pm \int \frac{du_1}{\sqrt{b_1^2 - 4\left(u_1 - \dfrac{C_1}{2}\right)^2}} \tag{5.54}$$

leads to the invariant relation

$$2(\beta_1 + C_3) = \pm \arcsin \frac{2u_1 - C_1}{\sqrt{b^2 + C_1^2 - 4}}, \quad C_3 = \text{const}\,. \tag{5.55}$$

In other words, the equality

$$\sin\left[2(\beta_1 + C_3)\right] = \pm \frac{2u_1 - C_1}{\sqrt{b^2 + C_1^2 - 4}} \tag{5.56}$$

holds; in the old variables, it has the form

$$\sin\left[2(\beta_1 + C_3)\right] = \pm \frac{2Z_1 - C_1 \sin\alpha}{\sqrt{b^2 + C_1^2 - 4}\,\sin\alpha}. \tag{5.57}$$

Thus, we have obtained an additional invariant relation that "attaches" Eq. (5.22). However, we must formally substitute the left-hand side of (5.37) into the last expression instead of C_1.

We perform certain transformations that lead us to the following explicit form of the additional first integral:

$$\tan^2\left[2(\beta_1 + C_3)\right] = \frac{(u_1^2 - u_2^2 + bu_2 - 1)^2}{u_1^2(4u_2^2 - 4bu_2 + b^2)}; \tag{5.58}$$

here Eq. (5.36) is used.

Returning to the old coordinates, we obtain the additional invariant relation of the form

$$\tan^2\left[2(\beta_1 + C_3)\right] = \frac{(Z_1^2 - Z_2^2 + bZ_2\sin\alpha - \sin^2\alpha)^2}{Z_1^2(4Z_2^2 - 4bZ_2\sin\alpha + b^2\sin^2\alpha)}, \tag{5.59}$$

or, finally,

$$-\beta_1 \pm \frac{1}{2}\arctan\frac{Z_1^2 - Z_2^2 + bZ_2\sin\alpha - \sin^2\alpha}{Z_1(2Z_2 - b\sin\alpha)} = C_3 = \text{const}\,. \tag{5.60}$$

Therefore, in the considered case, under condition (5.19), the system of dynamical equations (5.1) has five invariant relations: the analytical nonintegrable constraint (5.7), the cyclic first integral (5.5), (5.6), the first integral (5.37), the first integral expressed by relation (5.47) (see also (5.49)), which is a transcendental function of its phase variables (in the sense of the complex analysis), and, finally, the transcendental first integral (5.60).

Theorem 5.1. *Under the conditions* (5.7), (5.5), (5.6), *and* (5.19), *system* (5.1) *possesses five invariant relations* (*a complete set*), *three of which are transcendental functions* (*in the sense of the complex analysis*). *Moreover, at least four of them are expressed as finite combinations of elementary functions.*

2.3. Topological analogies. We show that there exists another mechanical and topological analogy.

Theorem 5.2. *Under the conditions* (5.7), (5.5), (5.6), *and* (5.19), *the first integral* (5.37) *of system* (5.1) *is constant on phase trajectories of system* (4.22), (4.23).

Proof. . Indeed, the first integral (5.37) can be obtained by a change of coordinates from (5.36), and the first integral (4.32) can be obtained by a change of coordinates from (4.31). However, relations (5.36) and (4.31) coincide. The theorem is proved. □

Thus, we have the following topological and mechanical analogies in the sense explained above.

(1) A motion of a free rigid body in a nonconservative force field under a tracing force (in the presence of a nonintegrable constraint).

(2) A motion of a fixed physical pendulum in a flowing medium (a nonconservative force field).

(3) A rotation of a rigid body in a nonconservative force about its center of mass, which moves rectilinearly and uniformly.

On more general topological analogies, see also [253, 265, 267, 309, 340, 342].

3. Case where the Moment of Nonconservative Forces Depends on the Angular Velocity

3.1. Dependence of the moment of nonconservative forces on the angular velocity and the reduced system. We continue to study the dynamics of a two-dimensional rigid body on the plane. This section (similarly to the corresponding section of Chap. 2) is devoted to the study of the case of the motion where the moment of forces depends on the angular velocity. We introduce this dependence just as was done in the previous chapter. We also recall that this point of view will also allow us to introduce this dependence for three-dimensional and higher-dimensional bodies.

Let $x = (x_{1N}, x_{2N})$ be the coordinates of the point N of application of a nonconservative force (interaction with a medium) to a two-dimensional disk and $Q = (Q_1, Q_2)$ be the components independent on the angular velocity. We consider only the linear dependence of the functions $(x_{1N}, x_{2N}) = (x_N, y_N)$ on the angular velocity since the introduction of this dependence itself is not a priori obvious (see [33, 34, 48, 49, 57–66, 120, 121, 169, 182, 203, 205, 249, 250, 261, 274–276, 440]).

Thus, we accept the following dependence:

$$x = Q + R, \tag{5.61}$$

where $R = (R_1, R_2, R_3)$ is the vector-valued function containing the angular velocity. Here, the dependence of the function R on the angular velocity is gyroscopic (see also the previous chapter):

$$R = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = -\frac{1}{v} \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}, \tag{5.62}$$

where (h_1, h_2, h_3) are certain positive parameters (cf. [322, 330, 331, 345, 393]).

Now, for our problem, since $x_{1N} = x_N \equiv 0$, we have

$$x_{2N} = y_N = Q_2 - h_1 \frac{\Omega_3}{v}, \quad x_{3N} = z_N = Q_3 + h_1 \frac{\Omega_2}{v}. \tag{5.63}$$

Similarly to the choice of the Chaplygin analytical functions (see [50, 51])

$$Q_2 = A \sin\alpha \cos\beta_1, \quad Q_3 = A \sin\alpha \sin\beta_1, \quad A > 0, \tag{5.64}$$

we take the dynamical functions s, y_N, and z_N as follows:

$$\begin{gathered} s(\alpha) = B\cos\alpha, \quad B > 0, \\ y_N\left(\alpha, \beta_1, \frac{\Omega}{v}\right) = A \sin\alpha \cos\beta_1 - h\frac{\Omega_3}{v}, \quad h = h_1 > 0, \quad v \neq 0, \\ z_N\left(\alpha, \beta_1, \frac{\Omega}{v}\right) = A \sin\alpha \sin\beta_1 + h\frac{\Omega_2}{v}, \quad h = h_1 > 0, \quad v \neq 0, \end{gathered} \tag{5.65}$$

which shows that an additional dependence of the damping (or accelerating in some domains of the phase space) moment of the nonconservative forces is also present in the system considered (i.e., the moment depends on the angular velocity). Moreover, by virtue of the dynamical symmetry of the body, we have $h_2 = h_3$.

Then, owing to the conditions (5.7) and (5.65), the transformed dynamical part of the equations of motion (system (5.14)–(5.18)) has the following form:

$$v' = v\Psi(\alpha, \beta_1, Z_1, Z_2), \tag{5.66}$$

$$\alpha' = -Z_2 + b(Z_1^2 + Z_2^2)\sin\alpha + b\sin\alpha\cos^2\alpha - bH_1 Z_2\cos^2\alpha,$$

$$Z_2' = \sin\alpha\cos\alpha - (1 + bH_1)Z_1^2\frac{\cos\alpha}{\sin\alpha} + bZ_2(Z_1^2 + Z_2^2)\cos\alpha \\ - bZ_2\sin^2\alpha\cos\alpha + bH_1 Z_2^2\sin\alpha\cos\alpha - H_1 Z_2\cos\alpha, \tag{5.67}$$

$$Z_1' = (1 + bH_1)Z_1 Z_2\frac{\cos\alpha}{\sin\alpha} + bZ_1(Z_1^2 + Z_2^2)\cos\alpha \\ - bZ_1\sin^2\alpha\cos\alpha + bH_1 Z_1 Z_2\sin\alpha\cos\alpha - H_1 Z_1\cos\alpha,$$

$$\beta_1' = (1 + bH_1)Z_1\frac{\cos\alpha}{\sin\alpha}, \tag{5.68}$$

$$\Psi(\alpha, \beta_1, Z_1, Z_2) = -b(Z_1^2 + Z_2^2)\cos\alpha + b\sin^2\alpha\cos\alpha - bH_1 Z_2\sin\alpha\cos\alpha.$$

We introduce the dimensionless parameters b and H_1 and the constant n_1 as follows:

$$b = \sigma n_0, \quad n_0^2 = \frac{AB}{I_2}, \quad H_1 = \frac{Bh}{I_2 n_0}, \quad n_1 = n_0. \tag{5.69}$$

We see that in the fifth-order system (5.66)–(5.68), the independent fourth-order system (5.67), (5.68) was isolated; this system can be considered on the tangent bundle $T\mathbf{S}^2$ of the two-dimensional sphere $\mathbf{S}^2$. Moreover, the independent third-order system (5.67) can be considered on its own three-dimensional manifold.

3.2. Complete list of first integrals. The independent fourth-order system (5.67), (5.68) was extracted from system (5.66)–(5.68).

Note that, by virtue of (5.7), the magnitude of the velocity of the center of mass is a first integral of system (5.1) and, therefore, the function of phase variables (5.25) is constant on its phase trajectories (here the values z_1, z_2 are chosen from (5.11)).

Using a nondegenerate change of the independent variable (for $v \neq 0$), we see that system (5.66)–(5.68) also has an analytical integral, namely, the function of the phase variables

$$\Psi_1(v, \alpha, \beta_1, Z_1, Z_2) = v^2(1 + b^2(Z_1^2 + Z_2^2) - 2bZ_2\sin\alpha) = V_C^2 \tag{5.70}$$

is constant on its phase trajectories.

Equality (5.70) allows one to find the dependence of the velocity of a certain point of a rigid body (namely, the center D of the disk) on the other phase variables without solution of system (5.66)–(5.68); therefore, for $V_C \neq 0$, Eq. (5.27) is fulfilled.

Since the phase space of system (5.66)–(5.68) is five-dimensional and there exist asymptotic limit sets in it, we see that Eq. (5.70) defines a unique analytical (even continuous) first integral of system (5.66)–(5.68) in the whole phase space (cf. [23, 24, 43, 53, 103, 105, 137, 152–154, 156, 213, 427, 438]).

We consider in detail the problem on the existence of other (additional) first integrals of system (5.66)–(5.68). Its phase space is stratified into surfaces (5.28) on which the dynamics is defined by the first integral of system (5.67), (5.68).

First, we compare the third-order system (5.67) with the nonautonomous second-order system

$$\frac{dZ_2}{d\alpha} = \frac{R_2(\alpha, Z_1, Z_2)}{-Z_2 + b(Z_1^2 + Z_2^2)\sin\alpha + b\sin\alpha\cos^2\alpha - bH_1 Z_2\cos^2\alpha}, \\ \frac{dZ_1}{d\alpha} = \frac{R_1(\alpha, Z_1, Z_2)}{-Z_2 + b(Z_1^2 + Z_2^2)\sin\alpha + b\sin\alpha\cos^2\alpha - bH_1 Z_2\cos^2\alpha}, \tag{5.71}$$

$$R_2(\alpha, Z_1, Z_2) = \sin\alpha\cos\alpha + bZ_2(Z_1^2 + Z_2^2)\cos\alpha - bZ_2\sin^2\alpha\cos\alpha \\ - (1 + bH_1)Z_1^2\frac{\cos\alpha}{\sin\alpha} + bH_1Z_2^2\sin\alpha\cos\alpha - H_1Z_2\cos\alpha,$$

$$R_1(\alpha, Z_1, Z_2) = bZ_1(Z_1^2 + Z_2^2)\cos\alpha - bZ_1\sin^2\alpha\cos\alpha \\ + (1 + bH_1)Z_1Z_2\frac{\cos\alpha}{\sin\alpha} + bH_1Z_1Z_2\sin\alpha\cos\alpha - H_1Z_1\cos\alpha.$$

Using the substitution $\tau = \sin\alpha$, we rewrite system (5.71) in the algebraic form:

$$\begin{aligned} \frac{dZ_2}{d\tau} &= \frac{\tau + bZ_2(Z_1^2 + Z_2^2) - bZ_2\tau^2 - (1 + bH_1)Z_1^2/\tau + bH_1Z_2^2\tau - H_1Z_2}{-Z_2 + b\tau(1 - \tau^2) + b\tau(Z_1^2 + Z_2^2) - bH_1Z_2(1 - \tau^2)}, \\ \frac{dZ_1}{d\tau} &= \frac{bZ_1(Z_1^2 + Z_2^2) - bZ_1\tau^2 + (1 + bH_1)Z_1Z_2/\tau + bH_1Z_1Z_2\tau - H_1Z_1}{-Z_2 + b\tau(1 - \tau^2) + b\tau(Z_1^2 + Z_2^2) - bH_1Z_2(1 - \tau^2)}. \end{aligned} \tag{5.72}$$

Further, introducing the homogeneous variables by the formulas

$$Z_k = u_k\tau, \quad k = 1, 2, \tag{5.73}$$

we reduce system (5.72) to the following form:

$$\begin{aligned} \tau\frac{du_2}{d\tau} + u_2 &= \frac{1 - bu_2\tau^2 + bu_2(u_1^2 + u_2^2)\tau^2 - (1 + bH_1)u_1^2 - H_1u_2 + bH_1u_2^2\tau^2}{-u_2 + b\tau^2(u_1^2 + u_2^2) + b(1 - \tau^2) - bH_1u_2(1 - \tau^2)}, \\ \tau\frac{du_1}{d\tau} + u_1 &= \frac{bu_1(u_1^2 + u_2^2)\tau^2 - bu_1\tau^2 + (1 + bH_1)u_1u_2 - H_1u_1 + bH_1u_1u_2}{-u_2 + b\tau^2(u_1^2 + u_2^2) + b(1 - \tau^2) - bH_1u_2(1 - \tau^2)}, \end{aligned} \tag{5.74}$$

which is equivalent to

$$\begin{aligned} \tau\frac{du_2}{d\tau} &= \frac{1 - (b + H_1)u_2 + (1 + bH_1)u_2^2 - (1 + bH_1)u_1^2}{-u_2 + b\tau^2(u_1^2 + u_2^2) + b(1 - \tau^2) - bH_1u_2(1 - \tau^2)}, \\ \tau\frac{du_1}{d\tau} &= \frac{2(1 + bH_1)u_1u_2 - (b + H_1)u_1}{-u_2 + b\tau^2(u_1^2 + u_2^2) + b(1 - \tau^2) - bH_1u_2(1 - \tau^2)}. \end{aligned} \tag{5.75}$$

We compare the second-order system (5.75) with the nonautonomous first-order equation

$$\frac{du_2}{du_1} = \frac{1 - (b + H_1)u_2 + (1 + bH_1)u_2^2 - (1 + bH_1)u_1^2}{2(1 + bH_1)u_1u_2 - (b + H_1)u_1}, \tag{5.76}$$

which can be easily reduced to the exact differential equation

$$d\left(\frac{(1 + bH_1)u_2^2 + (1 + bH_1)u_1^2 - (b + H_1)u_2 + 1}{u_1}\right) = 0. \tag{5.77}$$

Therefore, Eq. (5.76) has the following first integral:

$$\frac{(1 + bH_1)u_2^2 + (1 + bH_1)u_1^2 - (b + H_1)u_2 + 1}{u_1} = C_1 = \text{const}, \tag{5.78}$$

which in the old variables has the form

$$\frac{(1 + bH_1)Z_2^2 + (1 + bH_1)Z_1^2 - (b + H_1)Z_2\sin\alpha + \sin^2\alpha}{Z_1\sin\alpha} = C_1 = \text{const}. \tag{5.79}$$

Remark 5.3. Consider system (5.67) with variable dissipation with zero mean (see [67, 68, 243, 260, 262, 265, 282–286, 291, 295, 309, 324, 333, 340, 384, 391, 404–408, 412, 413, 421, 437, 438] which becomes conservative for $b = H_1$:

$$\alpha' = -Z_2 + b(Z_1^2 + Z_2^2)\sin\alpha + b\sin\alpha\cos^2\alpha - b^2Z_2\cos^2\alpha,$$

$$Z_2' = \sin\alpha\cos\alpha - (1+b^2)Z_1^2\frac{\cos\alpha}{\sin\alpha} + bZ_2(Z_1^2+Z_2^2)\cos\alpha - bZ_2\sin^2\alpha\cos\alpha + b^2Z_2^2\sin\alpha\cos\alpha - bZ_2\cos\alpha, \quad (5.80)$$

$$Z_1' = (1+b^2)Z_1Z_2\frac{\cos\alpha}{\sin\alpha} + bZ_1(Z_1^2+Z_2^2)\cos\alpha - bZ_1\sin^2\alpha\cos\alpha + b^2Z_1Z_2\sin\alpha\cos\alpha - bZ_1\cos\alpha.$$

It has two analytical first integrals of the form

$$(1+b^2)(Z_2^2+Z_1^2) - 2bZ_2\sin\alpha + \sin^2\alpha = C_1^* = \text{const}, \quad (5.81)$$

$$Z_1\sin\alpha = C_2^* = \text{const}\,. \quad (5.82)$$

Obbviously, the ratio of two first integrals (5.81) and (5.82) is also a first integral of system (5.80). However, for $b \neq H_1$, both functions

$$(1+bH_1)(Z_2^2+Z_1^2) - (b+H_1)Z_2\sin\alpha + \sin^2\alpha \quad (5.83)$$

and (5.82) are not first integrals of system (5.67), but their ratio is a first integral of system (5.67) for any b and H_1.

Further, we find the explicit form of the additional first integral of the third-order system (5.67). For this, we transform the invariant relation (5.78) for $u_1 \neq 0$ as follows:

$$\left(u_2 - \frac{b+H_1}{2(1+bH_1)}\right)^2 + \left(u_1 - \frac{C_1}{2(1+bH_1)}\right)^2 = \frac{(b-H_1)^2 + C_1^2 - 4}{4(1+bH_1)^2}. \quad (5.84)$$

We see that the parameters of this invariant relation must satisfy the condition

$$(b-H_1)^2 + C_1^2 - 4 \geq 0, \quad (5.85)$$

and the phase space of system (5.67) is stratified into a family of surfaces defined by Eq. (5.84).

Thus, by virtue of relation (5.78), the first equation of system (5.75) has either the form

$$\tau\frac{du_2}{d\tau} = \frac{1-(b+H_1)u_2 + (1+bH_1)u_2^2 - (1+bH_1)U_1^2(C_1,u_2)}{-u_2 + b(1-\tau^2) + b\tau^2(U_1^2(C_1,u_2)+u_2^2) - bH_1u_2(1-\tau^2)}, \quad (5.86)$$

where

$$U_1(C_1,u_2) = \frac{1}{2}\left\{C_1 \pm \sqrt{C_1^2 - 4(1+bH_1)\Big(1-(b+H_1)u_2 + (1+bH_1)u_2^2\Big)}\right\}, \quad (5.87)$$

and the integration constant C_1 is chosen from condition (5.85), or the form of the Bernoulli equation

$$\frac{d\tau}{du_2} = \frac{(b-(1+bH_1)u_2)\tau - b\tau^3(1-U_1^2(C_1,u_2) - u_2^2 - H_1u_2)}{1-(b+H_1)u_2 + (1+bH_1)u_2^2 - (1+bH_1)U_1^2(C_1,u_2)}. \quad (5.88)$$

Using (5.87), we can reduce Eq. (5.88) to the linear nonhomogeneous equation

$$\frac{dp}{du_2} = \frac{2((1+bH_1)u_2 - b)p + 2b(1-H_1u_2 - u_2^2 - U_1^2(C_1,u_2))}{1-(b+H_1)u_2 + (1+bH_1)u_2^2 - (1+bH_1)U_1^2(C_1,u_2)}, \quad p = \frac{1}{\tau^2}. \quad (5.89)$$

This fact means that we can find another transcendental first integral in the explicit form (i.e., as a finite combination of quadratures). Moreover, the general solution of Eq. (5.89) depends on an arbitrary constant C_2. We omit calculations but note that the general solution of the linear homogeneous equation obtained from (5.89)) even in the particular case where

$$|b-H_1| = 2, \quad C_1 = \frac{1-A_1^4}{1+A_1^4}, \quad A_1 = \frac{1}{2}(b+H_1),$$

has the following solution:

$$p = p_0(u_2) = C\big[1 - A_1u_2\big]^{2/(1+A_1^4)} \left| \frac{\sqrt{C_1^2 - 4A_1^2(1 - A_1u_2)^2} \pm C_1}{\sqrt{C_1^2 - 4A_1^2(1 - A_1u_2)^2} \mp C_1} \right|^{\pm A_1^4/(1+A_1^4)}$$
$$\times \exp \frac{2(A_1 - b)}{(1 + A_1^4)A_1(A_1u_2 - 1)}, \quad C = \text{const}\,. \quad (5.90)$$

Remark 5.4. We must formally substitute the left-hand side of the first integral (5.78) into the expression of the found first integral instead of C_1. Then the additional first integral obtained has the following structure (which is similar to a transcendental first integral from the planar dynamics):

$$K_1\left(\sin\alpha, Z_2, Z_1, \frac{Z_2}{\sin\alpha}, \frac{Z_1}{\sin\alpha} \right) = C_2 = \text{const}\,. \quad (5.91)$$

We have already found two independent first integrals for integration of the fourth-order system (5.67), (5.68). For the complete integrability, it suffices to find another additional first integral that attaches Eq. (5.68).

Since

$$\frac{d\beta_1}{d\tau} = \frac{(1 + bH_1)u_1}{-u_2\tau + b\tau^3(u_1^2 + u_2^2) + b\tau(1 - \tau^2) - bH_1\tau u_2(1 - \tau^2)}, \quad (5.92)$$
$$\frac{du_1}{d\tau} = \frac{2(1 + bH_1)u_1u_2 - (b + H_1)u_1}{-u_2\tau + b\tau^3(u_1^2 + u_2^2) + b\tau(1 - \tau^2) - bH_1\tau u_2(1 - \tau^2)}, \quad (5.93)$$

we have

$$\frac{du_1}{d\beta_1} = 2u_2 - \frac{b + H_1}{1 + bH_1}. \quad (5.94)$$

Obviously, for $u_1 \neq 0$, the following equality holds:

$$u_2 = \frac{1}{2(1 + bH_1)}\Big((b + H_1) \pm \sqrt{b_1^2 - (2(1 + bH_1)u_1 - C_1)^2}\Big), \quad (5.95)$$

where

$$b_1^2 = (b - H_1)^2 + C_1^2 - 4.$$

Then integration of the quadrature

$$\beta_1 + \text{const} = \pm(1 + bH_1)\int \frac{du_1}{\sqrt{b_1^2 - (2(1 + bH_1)u_1 - C_1)^2}} \quad (5.96)$$

leads to the invariant relation

$$2(\beta_1 + C_3) = \pm \arcsin \frac{2(1 + bH_1)u_1 - C_1}{\sqrt{(b - H_1)^2 + C_1^2 - 4}}, \quad C_3 = \text{const}\,. \quad (5.97)$$

In other words, the equality

$$\sin\big[2(\beta_1 + C_3)\big] = \pm\frac{2(1 + bH_1)u_1 - C_1}{\sqrt{(b - H_1)^2 + C_1^2 - 4}} \quad (5.98)$$

is fulfilled; in the old variables, it has the form

$$\sin\big[2(\beta_1 + C_3)\big] = \pm\frac{2(1 + bH_1)Z_1 - C_1\sin\alpha}{\sqrt{(b - H_1)^2 + C_1^2 - 4}\,\sin\alpha}. \quad (5.99)$$

Thus, we have obtained an additional invariant relation that “attaches” Eq. (5.68). However, we must formally substitute the left-hand side of (5.78) into the last expression instead of C_1.

However, we perform certain transformations that allow one to obtain the following explicit form of the additional first integral (here Eq. (5.78) is used):

$$\tan^2\left[2(\beta_1 + C_3)\right] = \frac{((1+bH_1)u_1^2 - (1+bH_1)u_2^2 + (b+H_1)u_2 - 1)^2}{u_1^2(2(1+bH_1)u_2 - (b+H_1))^2}. \tag{5.100}$$

Returning to the old coordinates, we obtain the additional invariant relation of the form

$$\tan^2\left[2(\beta_1 + C_3)\right] = \frac{((1+bH_1)Z_1^2 - (1+bH_1)Z_2^2 + (b+H_1)Z_2\sin\alpha - \sin^2\alpha)^2}{Z_1^2(2(1+bH_1)Z_2 - (b+H_1)\sin\alpha)^2}, \tag{5.101}$$

or, finally,

$$-\beta_1 \pm \frac{1}{2}\arctan\frac{(1+bH_1)Z_1^2 - (1+bH_1)Z_2^2 + (b+H_1)Z_2\sin\alpha - \sin^2\alpha}{Z_1(2(1+bH_1)Z_2 - (b+H_1)\sin\alpha)} = C_3 = \mathrm{const}\,. \tag{5.102}$$

Therefore, in the considered case, the system of dynamical equations (5.1) under condition (5.65) has five invariant relations: the analytical nonintegrable constraint (5.7), the cyclic first integral (5.5), (5.6), the first integral (5.79), the first integral expressed by relation (5.89) (see also (5.91)), which is a transcendental function of its phase variables (in the sense of complex analysis), and, finally, the transcendental first integral (5.102).

Theorem 5.3. *System* (5.1) *under conditions* (5.7), (5.5), (5.6), *and* (5.65) *possesses five invariant relations* (*a complete set*), *three of which are transcendental functions* (*in the sense of the complex analysis*). *Moreover, at least four of these five relations are expressed as finite combinations of elementary functions.*

3.3. Topological analogies. We show that there exists another mechanical and topological analogy.

Theorem 5.4. *The first integral* (5.79) *of system* (5.1) *under conditions* (5.7), (5.5), (5.6), *and* (5.65) *is constant on phase trajectories of system* (4.74), (4.75).

Proof. Indeed, the first integral (5.79) can be obtained by a change of coordinates from relation (5.78) and the first integral (4.84) can be obtained by a change of coordinates from relation (4.83). But relations (5.78) and (4.83) coincide. The theorem is proved. □

Thus, we have the following topological and mechanical analogies in the sense explained above.

(1) A motion of a free rigid body in a nonconservative force field under a tracing force (in the presence of a nonintegrable constraint).

(2) A motion of a fixed physical pendulum in a flowing medium (a nonconservative force field).

(3) A rotation of a rigid body in a nonconservative force about its center of mass, which moves rectilinearly and uniformly.

On more general topological analogies, see [253, 265, 267, 309, 340, 342].

Chapter 6

CASES OF INTEGRABILITY CORRESPONDING TO THE MOTION OF A RIGID BODY IN THE FOUR-DIMENSIONAL SPACE, I

In this chapter, we systematize some earlier results and new results on the study of the equations of motion of axially symmetric four-dimensional (4D) rigid bodies in nonconservative force fields. The form of these equations is taken from the dynamics of real lower-dimensional rigid bodies interacting with resisting medium by laws of jet flows where a body is influenced by a nonconservative tracing force; under action of this force, the velocity of some characteristic point of the body remains constant, which means that the system possesses a nonintegrable servo constraint (see [1, 64, 70, 72, 119–121, 157, 164–167, 182, 184, 191, 198, 237, 242, 346, 351, 352, 390]).

Earlier (see [164–167]), the author proved the complete integrability of the equations of a plane-parallel motion of a body in a resisting medium under the jet flow conditions when the system of dynamical equations possesses a first integral, which is a transcendental (in the sense of the theory of functions of a complex variable) function of quasi-velocities having essential singularities. It was assumed that the interaction of the medium with the body is concentrated on a part of the surface of the body that has the form of a (one-dimensional) plate.

In the sequel (see [193, 196, 202, 204, 208, 209, 218, 232, 241]), the planar problem was generalized to the spatial (three-dimensional) case, where the system of dynamical equations possesses a complete set of transcendental first integrals. In this case, it was assumed that the interaction of the medium with the body is concentrated on the part of the surface of the body that has the form of a planar (two-dimensional) disk.

In this chapter, we discuss results, both new and obtained earlier, concerning the case where the interaction of the medium with the body is concentrated on the part of the surface of the body that has the form of a three-dimensional disk and the force acts in the direction perpendicular to the disk. We systematize these results and formulate them in the invariant form. We also introduce the additional dependence of the moment of a nonconservative force on the angular velocity; this dependence can be generalized to the motion in higher-dimensional spaces.

1. General Discourse

1.1. Two cases of dynamical symmetry of a four-dimensional body. Let a four-dimensional rigid body Θ of mass m with smooth three-dimensional boundary $\partial\Theta$ be under the influence of a nonconservative force field; this can be interpreted as a motion of the body in a resisting medium that fills up the four-dimensional domain of Euclidean space $\mathbf{E}^4$. We assume that the body is dynamically symmetric. If the body has two independent principal moments of inertia, then in some coordinate system $Dx_1x_2x_3x_4$ attached to the body, the operator of inertia has the form

$$\operatorname{diag}\{I_1, I_2, I_2, I_2\} \tag{6.1}$$

or the form

$$\operatorname{diag}\{I_1, I_1, I_3, I_3\}. \tag{6.2}$$

In the first case, the body is dynamically symmetric in the hyperplane $Dx_2x_3x_4$ and in the second case, the two-dimensional planes Dx_1x_2 and Dx_3x_4 are planes of dynamical symmetry of the body.

1.2. Dynamics on $\mathrm{so}(4)$ **and** $\mathbb{R}^4$**.** The configuration space of a free, n-dimensional rigid body is the direct product

$$\mathbb{R}^n \times \mathrm{SO}(n) \tag{6.3}$$

of the space $\mathbb{R}^n$, which defines the coordinates of the center of mass of the body, and the rotation group $\mathrm{SO}(n)$, which defined rotations of the body about its center of mass and has dimension

$$n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

Therefore, the dynamical part of the equations of motion has the same dimension, whereas the dimension of the phase space is equal to $n(n+1)$.

In particular, if Ω is the tensor of angular velocity of a four-dimensional rigid body (it is a second-rank tensor, see [44, 52, 53, 74–77, 79–81, 85, 86, 89, 133, 433–439]), $\Omega \in \mathrm{so}(4)$, then the part of the dynamical equations of motion corresponding to the Lie algebra so(4) has the following form (see [69, 436, 438]):

$$\dot{\Omega}\Lambda + \Lambda\dot{\Omega} + [\Omega, \Omega\Lambda + \Lambda\Omega] = M, \tag{6.4}$$

where

$$\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}, \tag{6.5}$$

$$\lambda_1 = \frac{-I_1 + I_2 + I_3 + I_4}{2}, \quad \lambda_2 = \frac{I_1 - I_2 + I_3 + I_4}{2},$$
$$\lambda_3 = \frac{I_1 + I_2 - I_3 + I_4}{2}, \quad \lambda_4 = \frac{I_1 + I_2 + I_3 - I_4}{2},$$

$M = M_F$ is the natural projection of the moment of external forces $\mathbf{F}$ acting on the body in $\mathbb{R}^4$ on the natural coordinates of the Lie algebra so(4) and [ ] is the commutator in so(4). The skew-symmetric matrix corresponding to this second-rank tensor $\Omega \in \mathrm{so}(4)$ we represent in the form

$$\begin{pmatrix} 0 & -\omega_6 & \omega_5 & -\omega_3 \\ \omega_6 & 0 & -\omega_4 & \omega_2 \\ -\omega_5 & \omega_4 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}, \tag{6.6}$$

where ω_1, ω_2, ω_3, ω_4, ω_5, and ω_6 are the components of the tensor of angular velocity corresponding to the projections on the coordinates of the Lie algebra so(4).

Obviously, the following relations hold:

$$\lambda_i - \lambda_j = I_j - I_i, \quad i, j = 1, \dots, 4. \tag{6.7}$$

For the calculation of the moment of an external force acting on the body, we need to construct the mapping

$$\mathbb{R}^4 \times \mathbb{R}^4 \to \mathrm{so}(4) \tag{6.8}$$

than maps a pair of vectors

$$(\mathbf{DN}, \mathbf{F}) \in \mathbb{R}^4 \times \mathbb{R}^4 \tag{6.9}$$

to an element of the Lie algebra so(4), where

$$\mathbf{DN} = \{0, x_{2N}, x_{3N}, x_{4N}\}, \quad \mathbf{F} = \{F_1, F_2, F_3, F_4\}, \tag{6.10}$$

and $\mathbf{F}$ is an external force acting on the body. For this end, we construct the following auxiliary matrix

$$\begin{pmatrix} 0 & x_{2N} & x_{3N} & x_{4N} \\ F_1 & F_2 & F_3 & F_4 \end{pmatrix}. \tag{6.11}$$

Then the right-hand side of system (6.4) takes the form

$$M = \Big\{ x_{3N}F_4 - x_{4N}F_3,\ x_{4N}F_2 - x_{2N}F_4,\ -x_{4N}F_1,\ x_{2N}F_3 - x_{3N}F_2,\ x_{3N}F_1,\ -x_{2N}F_1 \Big\}. \tag{6.12}$$

Dynamical systems studied in the following chapters, generally speaking, are not conservative; they are dynamical systems with variable dissipation with zero mean (see [265]). We need to examine by direct methods a part of the main system of dynamical equations, namely, the Newton equation, which plays the role of the equation of motion of the center of mass, i.e., the part of the dynamical equations corresponding to the space $\mathbb{R}^4$:

$$m\mathbf{w}_C = \mathbf{F}, \tag{6.13}$$

where $\mathbf{w}_C$ is the acceleration of the center of mass C of the body and m is its mass. Moreover, due to the higher-dimensional Rivals formula (it can be obtained by the operator method) we have the following relations:

$$\mathbf{w}_C = \mathbf{w}_D + \Omega^2\mathbf{DC} + E\mathbf{DC}, \quad \mathbf{w}_D = \dot{\mathbf{v}}_D + \Omega\mathbf{v}_D, \quad E = \dot{\Omega}, \tag{6.14}$$

where $\mathbf{w}_D$ is the acceleration of the point D, $\mathbf{F}$ is the external force acting on the body (in our case, $\mathbf{F} = \mathbf{S}$), and E is the tensor of angular acceleration (second-rank tensor).

So, the system of equations (6.4) and (6.13) of tenth order on the manifold $\mathbb{R}^4 \times \mathrm{so}(4)$ is a *closed* system of dynamical equations of the motion of a free four-dimensional rigid body under the action of an external force $\mathbf{F}$. This system has been separated from the kinematic part of the equations of motion on the manifold (6.3) and can be examined independently.

2. General Problem on the Motion Under a Tracing Force

Consider a motion of a homogeneous, dynamically symmetric (case (6.1)), rigid body with front end face (a three-dimensional disk interacting with a medium that fills the four-dimensional space) in the field of a resistance force $\mathbf{S}$ under the quasi-stationarity conditions (see [28, 50, 51, 62–66, 98, 112, 119–121, 160–169, 171, 431, 432]).

Let $(v, \alpha, \beta_1, \beta_2)$ be the (generalized) spherical coordinates of the velocity vector of the center of the three-dimensional disk lying on the axis of symmetry of the body,

$$\Omega = \begin{pmatrix} 0 & -\omega_6 & \omega_5 & -\omega_3 \\ \omega_6 & 0 & -\omega_4 & \omega_2 \\ -\omega_5 & \omega_4 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

be the tensor of angular velocity of the body, $Dx_1x_2x_3x_4$ be the coordinate system attached to the body such that the axis of symmetry CD coincides with the axis Dx_1 (recall that C is the center of mass), and the axes Dx_2, Dx_3, and Dx_4 lie in the hyperplane of the disk, and I_1, I_2, $I_3 = I_2$, $I_4 = I_2$, and m are characteristics of inertia and mass.

We adopt the following expansions in the projections on the axes of the coordinate system $Dx_1x_2x_3x_4$:

$$\begin{gathered} \mathbf{DC} = \{-\sigma, 0, 0, 0\}, \\ \mathbf{v}_D = \Big\{v\cos\alpha,\ v\sin\alpha\cos\beta_1,\ v\sin\alpha\sin\beta_1\cos\beta_2,\ v\sin\alpha\sin\beta_1\sin\beta_2\Big\}. \end{gathered} \tag{6.15}$$

In the case (6.1) we additionally have the expansion for the function of the influence of the medium on the four-dimensional body:

$$\mathbf{S} = \{-S, 0, 0, 0\}, \tag{6.16}$$

i.e., in this case $\mathbf{F} = \mathbf{S}$.

Then the part of the dynamical equations of motion (including the analytic Chaplygin functions [50, 51]; see below) that describes the motion of the center of mass and corresponds to the space $\mathbb{R}^4$, in which tangential forces of the influence of the medium on the three-dimensional disk vanish, takes the form

$$\begin{gathered} \dot{v}\cos\alpha - \dot{\alpha}v\sin\alpha - \omega_6 v\sin\alpha\cos\beta_1 + \omega_5 v\sin\alpha\sin\beta_1\cos\beta_2 - \omega_3 v\sin\alpha\sin\beta_1\sin\beta_2 \\ +\sigma\left(\omega_6^2 + \omega_5^2 + \omega_3^2\right) = -\frac{S}{m}, \end{gathered} \tag{6.17}$$

$$\dot{v}\sin\alpha\cos\beta_1 + \dot{\alpha}v\cos\alpha\cos\beta_1 - \dot{\beta}_1 v\sin\alpha\sin\beta_1 + \omega_6 v\cos\alpha - \omega_4 v\sin\alpha\sin\beta_1\cos\beta_2 \\ +\omega_2 v\sin\alpha\sin\beta_1\sin\beta_2 - \sigma(\omega_4\omega_5 + \omega_2\omega_3) - \sigma\dot{\omega}_6 = 0, \tag{6.18}$$

$$\dot{v}\sin\alpha\sin\beta_1\cos\beta_2 + \dot{\alpha}v\cos\alpha\sin\beta_1\cos\beta_2 + \dot{\beta}_1 v\sin\alpha\cos\beta_1\cos\beta_2 - \dot{\beta}_2 v\sin\alpha\sin\beta_1\sin\beta_2 \\ -\omega_5 v\cos\alpha + \omega_4 v\sin\alpha\cos\beta_1 - \omega_1 v\sin\alpha\sin\beta_1\sin\beta_2 - \sigma(-\omega_1\omega_2 + \omega_4\omega_6) + \sigma\dot{\omega}_5 = 0, \tag{6.19}$$

$$\dot{v}\sin\alpha\sin\beta_1\sin\beta_2 + \dot{\alpha}v\cos\alpha\sin\beta_1\sin\beta_2 + \dot{\beta}_1 v\sin\alpha\cos\beta_1\sin\beta_2 + \dot{\beta}_2 v\sin\alpha\sin\beta_1\cos\beta_2 \\ +\omega_3 v\cos\alpha - \omega_2 v\sin\alpha\cos\beta_1 + \omega_1 v\sin\alpha\sin\beta_1\cos\beta_2 + \sigma(\omega_2\omega_6 + \omega_1\omega_5) - \sigma\dot{\omega}_3 = 0, \tag{6.20}$$

where

$$S = s(\alpha)v^2, \quad \sigma = CD, \quad v > 0. \tag{6.21}$$

Further, the auxiliary matrix (6.11) for the calculation of the moment of the resistance force has the form

$$\begin{pmatrix} 0 & x_{2N} & x_{3N} & x_{4N} \\ -S & 0 & 0 & 0 \end{pmatrix}; \tag{6.22}$$

then the part of the dynamical equations of motion that describes the motion of the body about the center of mass and corresponds to the Lie algebra so(4), becomes

$$(\lambda_4 + \lambda_3)\dot{\omega}_1 + (\lambda_3 - \lambda_4)(\omega_3\omega_5 + \omega_2\omega_4) = 0, \tag{6.23}$$

$$(\lambda_2 + \lambda_4)\dot{\omega}_2 + (\lambda_2 - \lambda_4)(\omega_3\omega_6 - \omega_1\omega_4) = 0, \tag{6.24}$$

$$(\lambda_4 + \lambda_1)\dot{\omega}_3 + (\lambda_4 - \lambda_1)(\omega_2\omega_6 + \omega_1\omega_5) = x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)v^2, \tag{6.25}$$

$$(\lambda_3 + \lambda_2)\dot{\omega}_4 + (\lambda_2 - \lambda_3)(\omega_5\omega_6 + \omega_1\omega_2) = 0, \tag{6.26}$$

$$(\lambda_1 + \lambda_3)\dot{\omega}_5 + (\lambda_3 - \lambda_1)(\omega_4\omega_6 - \omega_1\omega_3) = -x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)v^2, \tag{6.27}$$

$$(\lambda_1 + \lambda_2)\dot{\omega}_6 + (\lambda_1 - \lambda_2)(\omega_4\omega_5 + \omega_2\omega_3) = x_{2N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)v^2. \tag{6.28}$$

Thus, the phase space of system (6.17)–(6.20), (6.23)–(6.28) of tenth order is the direct product of the four-dimensional manifold and the Lie algebra so(4):

$$\mathbb{R}^1 \times \mathbf{S}^3 \times \mathrm{so}(4). \tag{6.29}$$

We note that system (6.17)–(6.20), (6.23)–(6.28), due to the existing dynamical symmetry

$$I_2 = I_3 = I_4, \tag{6.30}$$

possesses cyclic first integrals

$$\omega_1 \equiv \omega_1^0 = \mathrm{const}, \quad \omega_2 \equiv \omega_2^0 = \mathrm{const}, \quad \omega_4 \equiv \omega_4^0 = \mathrm{const}\,. \tag{6.31}$$

In the sequel, we consider the dynamics of the system on zero levels:

$$\omega_1^0 = \omega_2^0 = \omega_4^0 = 0. \tag{6.32}$$

If one considers a more general problem on the motion of a body under a tracing force $\mathbf{T}$ that lies on the straight line $CD = Dx_1$ and provides the fulfillment of the relation

$$v \equiv \mathrm{const} \tag{6.33}$$

throughout the motion (see [164–167])), then instead of F_1 system (6.17)–(6.20), (6.23)–(6.28) contains

$$T - s(\alpha)v^2, \quad \sigma = DC. \tag{6.34}$$

Choosing the value T of the tracing force appropriately, one can achieve the equality (6.33) throughout the motion. Indeed, expressing T due to system (6.17)–(6.20), (6.23)–(6.28), we obtain for $\cos\alpha \neq 0$ the relation

$$T = T_v(\alpha, \beta_1, \beta_2, \Omega) = m\sigma\left(\omega_3^2 + \omega_5^2 + \omega_6^2\right) + s(\alpha)v^2\left[1 - \frac{m\sigma}{2I_2}\frac{\sin\alpha}{\cos\alpha}\Gamma_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)\right], \tag{6.35}$$

where

$$\Gamma_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)\sin\beta_1\sin\beta_2 + x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)\sin\beta_1\cos\beta_2 + x_{2N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)\cos\beta_1; \tag{6.36}$$

here we used conditions (6.31)–(6.33).

This procedure can be interpreted in two ways. First, we have transformed the system using the tracing force (control) that provides the consideration of the class (6.33) of motions interesting for us. Second, we can treat this as an order-reduction procedure. Indeed, system (6.17)–(6.20), (6.23)–(6.28) generates the following independent system of sixth order:

$$\dot\alpha v\cos\alpha\cos\beta_1 - \dot\beta_1 v\sin\alpha\sin\beta_1 + \omega_6 v\cos\alpha - \sigma\dot\omega_6 = 0, \tag{6.37}$$

$$\dot\alpha v\cos\alpha\sin\beta_1\cos\beta_2 + \dot\beta_1 v\sin\alpha\cos\beta_1\cos\beta_2 - \dot\beta_2 v\sin\alpha\sin\beta_1\sin\beta_2 - \omega_5 v\cos\alpha + \sigma\dot\omega_5 = 0, \tag{6.38}$$

$$\dot\alpha v\cos\alpha\sin\beta_1\sin\beta_2 + \dot\beta_1 v\sin\alpha\cos\beta_1\sin\beta_2 + \dot\beta_2 v\sin\alpha\sin\beta_1\cos\beta_2 + \omega_3 v\cos\alpha - \sigma\dot\omega_3 = 0, \tag{6.39}$$

$$2I_2\dot\omega_3 = x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha)v^2, \tag{6.40}$$

$$2I_2\dot\omega_5 = -x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha)v^2, \tag{6.41}$$

$$2I_2\dot\omega_6 = x_{2N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha)v^2, \tag{6.42}$$

which, in addition to the permanent parameters specified above, contains the parameter v.

System (6.37)–(6.42) is equivalent to the system

$$\begin{aligned}\dot\alpha v\cos\alpha + v\cos\alpha\left[\omega_6\cos\beta_1 - \omega_5\sin\beta_1\cos\beta_2 + \omega_3\sin\beta_1\sin\beta_2\right]\\ +\sigma\left[-\dot\omega_6\cos\beta_1 + \dot\omega_5\sin\beta_1\cos\beta_2 - \dot\omega_3\sin\beta_1\sin\beta_2\right] = 0,\end{aligned} \tag{6.43}$$

$$\begin{aligned}\dot\beta_1 v\sin\alpha - v\cos\alpha\left[\omega_5\cos\beta_1\cos\beta_2 + \omega_6\sin\beta_1 - \omega_3\cos\beta_1\sin\beta_2\right]\\ +\sigma\left[\dot\omega_5\cos\beta_1\cos\beta_2\dot\omega_6\sin\beta_1 - \dot\omega_3\cos\beta_1\sin\beta_2\right] = 0,\end{aligned} \tag{6.44}$$

$$\dot\beta_2 v\sin\alpha\sin\beta_1 + v\cos\alpha\left[\omega_3\cos\beta_2 + \omega_5\sin\beta_2\right] + \sigma\left[-\dot\omega_3\cos\beta_2 - \dot\omega_5\sin\beta_2\right] = 0, \tag{6.45}$$

$$\dot\omega_3 = \frac{v^2}{2I_2}x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha), \tag{6.46}$$

$$\dot\omega_5 = -\frac{v^2}{2I_2}x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha), \tag{6.47}$$

$$\dot\omega_6 = \frac{v^2}{2I_2}x_{2N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha). \tag{6.48}$$

Introduce the new quasi-velocities. For this, we transform ω_3, ω_5, and ω_6 by two rotations:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = T_1(-\beta_1) \circ T_3(-\beta_2) \begin{pmatrix} \omega_3 \\ \omega_5 \\ \omega_6 \end{pmatrix}, \tag{6.49}$$

where

$$T_1(\beta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta_1 & -\sin\beta_1 \\ 0 & \sin\beta_1 & \cos\beta_1 \end{pmatrix}, \quad T_3(\beta_2) = \begin{pmatrix} \cos\beta_2 & -\sin\beta_2 & 0 \\ \sin\beta_2 & \cos\beta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.50}$$

Therefore, the following relations hold:

$$\begin{aligned} z_1 &= \omega_3 \cos\beta_2 + \omega_5 \sin\beta_2, \\ z_2 &= -\omega_3 \cos\beta_1 \sin\beta_2 + \omega_5 \cos\beta_1 \cos\beta_2 + \omega_6 \sin\beta_1, \\ z_3 &= \omega_3 \sin\beta_1 \sin\beta_2 - \omega_5 \sin\beta_1 \cos\beta_2 + \omega_6 \cos\beta_1. \end{aligned} \tag{6.51}$$

As we see from (6.43)–(6.48), we cannot solve the system with respect to $\dot\alpha$, $\dot\beta_1$, and $\dot\beta_2$ on the manifold

$$O_1 = \left\{ (\alpha, \beta_1, \beta_2, \omega_3, \omega_5, \omega_6) \in \mathbb{R}^6 : \alpha = \frac{\pi}{2}k,\ \beta_1 = \pi l,\ k, l \in \mathbb{Z} \right\}. \tag{6.52}$$

Therefore, on the manifold (6.52) the uniqueness theorem formally is violated. Moreover, for even k and any l, an indeterminate form appears due to the degeneration of the spherical coordinates $(v, \alpha, \beta_1, \beta_2)$. For odd k, the uniqueness theorem is obviously violated since the first equation (6.43) degenerates.

This implies that system (6.43)–(6.48) outside (and only outside) the manifold (6.52) is equivalent to the system

$$\dot\alpha = -z_3 + \frac{\sigma v}{2I_2} \frac{s(\alpha)}{\cos\alpha} \Gamma_v \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right), \tag{6.53}$$

$$\begin{aligned} \dot z_3 = {} & \frac{v^2}{2I_2} s(\alpha) \Gamma_v \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right) - \left(z_1^2 + z_2^2 \right) \frac{\cos\alpha}{\sin\alpha} - \frac{\sigma v}{2I_2} \frac{s(\alpha)}{\sin\alpha} z_2 \Delta_v \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right) \\ & + \frac{\sigma v}{2I_2} \frac{s(\alpha)}{\sin\alpha} z_1 \Theta_v \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right), \end{aligned} \tag{6.54}$$

$$\begin{aligned} \dot z_2 = {} & -\frac{v^2}{2I_2} s(\alpha) \Delta_v \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right) + z_2 z_3 \frac{\cos\alpha}{\sin\alpha} + z_1^2 \frac{\cos\alpha}{\sin\alpha} \frac{\cos\beta_1}{\sin\beta_1} + \\ & + \frac{\sigma v}{2I_2} \frac{s(\alpha)}{\sin\alpha} z_3 \Delta_v \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right) - -\frac{\sigma v}{2I_2} \frac{s(\alpha)}{\sin\alpha} \frac{\cos\beta_1}{\sin\beta_1} z_1 \Theta_v \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right), \end{aligned} \tag{6.55}$$

$$\begin{aligned} \dot z_1 = {} & \frac{v^2}{2I_2} s(\alpha) \Theta_v \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right) + z_1 z_3 \frac{\cos\alpha}{\sin\alpha} - z_1 z_2 \frac{\cos\alpha}{\sin\alpha} \frac{\cos\beta_1}{\sin\beta_1} - \\ & - \frac{\sigma v}{2I_2} \frac{s(\alpha)}{\sin\alpha} z_3 \Theta_v \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right) + \frac{\sigma v}{2I_2} \frac{s(\alpha)}{\sin\alpha} \frac{\cos\beta_1}{\sin\beta_1} z_2 \Theta_v \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right), \end{aligned} \tag{6.56}$$

$$\dot\beta_1 = z_2 \frac{\cos\alpha}{\sin\alpha} + \frac{\sigma v}{2I_2} \frac{s(\alpha)}{\sin\alpha} \Delta_v \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right), \tag{6.57}$$

$$\dot\beta_2 = -z_1 \frac{\cos\alpha}{\sin\alpha \sin\beta_1} + \frac{\sigma v}{2I_2} \frac{s(\alpha)}{\sin\alpha \sin\beta_1} \Theta_v \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right), \tag{6.58}$$

where

$$\Delta_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)=x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\cos\beta_1\sin\beta_2 + x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\cos\beta_1\cos\beta_2 - x_{2N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\sin\beta_1, \tag{6.59}$$

$$\Theta_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)=x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\cos\beta_2 - x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\sin\beta_2, \tag{6.60}$$

and the function $\Gamma_v(\alpha,\beta_1,\beta_2,\Omega/v)$ can be represented in the form (6.36).

Here and in the sequel, the dependence on the group of variables $(\alpha,\beta_1,\beta_2,\Omega/v)$ is meant as the composite dependence on $(\alpha,\beta_1,\beta_2,z_1/v,z_2/v,z_3/v)$ due to (6.51).

The uniqueness theorem for system (6.43)–(6.48) on the manifold (6.52) for odd k is violated in the following sense: for odd k, through almost all points of the manifold (6.52), passes a nonsingular phase trajectory of system (6.43)–(6.48) intersecting the manifold (6.52) at right angle and there exists a phase trajectory that at any time instants completely coincides with the point specified. However, physically these trajectories are different since they correspond to different values of the tracing force. Prove this.

As was shown above, to maintain the constraint of the form (6.33), we must take a value of T for $\cos\alpha\neq 0$ according to (6.35).

Let

$$\lim_{\alpha\to\pi/2}\frac{s(\alpha)}{\cos\alpha}\Gamma_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)=L\left(\beta_1,\beta_2,\frac{\Omega}{v}\right). \tag{6.61}$$

Note that $|L|<+\infty$ if and only if

$$\lim_{\alpha\to\pi/2}\left|\frac{\partial}{\partial\alpha}\left(\Gamma_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)\right)\right|<+\infty. \tag{6.62}$$

For $\alpha=\pi/2$, the required value of the tracing force is defined by the equation

$$T=T_v\left(\frac{\pi}{2},\beta_1,\beta_2,\Omega\right)=m\sigma\left(\omega_3^2+\omega_5^2+\omega_6^2\right)-\frac{m\sigma Lv^2}{2I_2}. \tag{6.63}$$

where ω_3, ω_5, and ω_6 are arbitrary.

On the other hand, maintaining the rotation about some point W by the tracing force, we must choose this force according to the relation

$$T=T_v\left(\frac{\pi}{2},\beta_1,\beta_2,\Omega\right)=\frac{mv^2}{R_0}, \tag{6.64}$$

where R_0 is the distance between C and W.

Relations (6.35) and (6.64) define, in general, different values of the tracing force T for almost all points of the manifold (6.52), which proves our assertion.

3. Case Where the Moment of a Nonconservative Force Is Independent of the Angular Velocity

3.1. Reduced system. Similarly to the choice of Chaplygin analytic functions (see [50, 51]), we take the dynamical functions s, x_{2N}, x_{3N}, and x_{4N} in the following form:

$$
\begin{gathered}
s(\alpha) = B\cos\alpha, \\
x_{2N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) = x_{2N0}(\alpha,\beta_1,\beta_2) = A\sin\alpha\cos\beta_1, \\
x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) = x_{3N0}(\alpha,\beta_1,\beta_2) = A\sin\alpha\sin\beta_1\cos\beta_2, \\
x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) = x_{4N0}(\alpha,\beta_1,\beta_2) = A\sin\alpha\sin\beta_1\sin\beta_2,
\end{gathered}
\tag{6.65}
$$

where $A, B > 0$ and $v \neq 0$. We see that in the system considered, the moment of nonconservative forces is independent of the angular velocity (but depends on the angles α, β_1, and β_2). Herewith, the functions $\Gamma_v(\alpha,\beta_1,\beta_2,\Omega/v)$, $\Delta_v(\alpha,\beta_1,\beta_2,\Omega/v)$, and $\Theta_v(\alpha,\beta_1,\beta_2,\Omega/v)$ in system (6.53)–(6.58) take the following form:

$$
\Gamma_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) = A\sin\alpha, \quad \Delta_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) \equiv \Theta_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) \equiv 0. \tag{6.66}
$$

Then, due to the nonintegrable constraint (6.33), outside the manifold (6.52), the dynamical part of the equations of motion (system (6.53)–(6.58)) has the form of the following analytic system:

$$
\dot\alpha = -z_3 + \frac{\sigma ABv}{2I_2}\sin\alpha, \tag{6.67}
$$

$$
\dot z_3 = \frac{ABv^2}{2I_2}\sin\alpha\cos\alpha - \left(z_1^2+z_2^2\right)\frac{\cos\alpha}{\sin\alpha}, \tag{6.68}
$$

$$
\dot z_2 = z_2z_3\frac{\cos\alpha}{\sin\alpha} + z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}, \tag{6.69}
$$

$$
\dot z_1 = z_1z_3\frac{\cos\alpha}{\sin\alpha} - z_1z_2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}, \tag{6.70}
$$

$$
\dot\beta_1 = z_2\frac{\cos\alpha}{\sin\alpha}, \tag{6.71}
$$

$$
\dot\beta_2 = -z_1\frac{\cos\alpha}{\sin\alpha\sin\beta_1}. \tag{6.72}
$$

Further, introducing the dimensionless variables, parameters, and the differentiation as follows:

$$
z_k \mapsto n_0vz_k,\ k=1,2,3, \quad n_0^2 = \frac{AB}{2I_2}, \quad b = \sigma n_0, \quad \langle\,\dot{}\,\rangle = n_0v\langle\,'\,\rangle, \tag{6.73}
$$

we reduce system (6.67)–(6.72) to the form

$$
\alpha' = -z_3 + b\sin\alpha, \tag{6.74}
$$

$$
z_3' = \sin\alpha\cos\alpha - \left(z_1^2+z_2^2\right)\frac{\cos\alpha}{\sin\alpha}, \tag{6.75}
$$

$$
z_2' = z_2z_3\frac{\cos\alpha}{\sin\alpha} + z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}, \tag{6.76}
$$

$$
z_1' = z_1z_3\frac{\cos\alpha}{\sin\alpha} - z_1z_2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}, \tag{6.77}
$$

$$
\beta_1' = z_2\frac{\cos\alpha}{\sin\alpha}, \tag{6.78}
$$

$$\beta_2' = -z_1 \frac{\cos\alpha}{\sin\alpha \sin\beta_1}. \tag{6.79}$$

We see that the sixth-order system (6.74)–(6.79) (which can be considered as a system on the tangent bundle $T\mathbf{S}^3$ of the three-dimensional sphere $\mathbf{S}^3$, see below) contains the independent fifth-order system (6.74)–(6.78) on its own five-dimensional manifold.

For the complete integration of system (6.74)–(6.79), in general, we need five independent first integrals. However, after the change of variables

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to \begin{pmatrix} z \\ z_* \end{pmatrix}, \quad z = \sqrt{z_1^2 + z_2^2}, \quad z_* = z_2/z_1, \tag{6.80}$$

system (6.74)–(6.79) splits as follows:

$$\alpha' = -z_3 + b\sin\alpha, \tag{6.81}$$

$$z_3' = \sin\alpha\cos\alpha - z^2 \frac{\cos\alpha}{\sin\alpha}, \tag{6.82}$$

$$z' = zz_3 \frac{\cos\alpha}{\sin\alpha}, \tag{6.83}$$

$$z_*' = (\pm) z\sqrt{1 + z_*^2} \frac{\cos\alpha}{\sin\alpha} \frac{\cos\beta_1}{\sin\beta_1}, \tag{6.84}$$

$$\beta_1' = (\pm) \frac{zz_*}{\sqrt{1 + z_*^2}} \frac{\cos\alpha}{\sin\alpha}, \tag{6.85}$$

$$\beta_2' = (\mp) \frac{z}{\sqrt{1 + z_*^2}} \frac{\cos\alpha}{\sin\alpha\sin\beta_1}. \tag{6.86}$$

We see that the sixth-order system splits into independent subsystems of lower order: system (6.81)–(6.83) has order three and system (6.84), (6.85) (after the change of the independent variable) has order two. Thus, for the complete integration of system (6.81)–(6.86) it suffices to specify two independent first integrals of system (6.81)–(6.83), one first integral of system (6.84), (6.85), and an additional first integral that "attaches" Eq. (6.86).

Note that system (6.81)–(6.83) can be considered on the tangent bundle $T\mathbf{S}^2$ of the two-dimensional sphere $\mathbf{S}^2$.

3.2. Complete list of invariant relations. System (6.81)–(6.83) has the form of a system that appears in the dynamics of a three-dimensional (3D) rigid body in a field of nonconservative forces.

First, to the third-order system (6.81)–(6.83), we put in correspondence the nonautonomous second-order system

$$\begin{aligned} \frac{dz_3}{d\alpha} &= \frac{\sin\alpha\cos\alpha - z^2\cos\alpha/\sin\alpha}{-z_3 + b\sin\alpha}, \\ \frac{dz}{d\alpha} &= \frac{zz_3\cos\alpha/\sin\alpha}{-z_3 + b\sin\alpha}. \end{aligned} \tag{6.87}$$

Applying the substitution $\tau = \sin\alpha$, we rewrite system (6.87) in the algebraic form

$$\begin{aligned} \frac{dz_3}{d\tau} &= \frac{\tau - z^2/\tau}{-z_3 + b\tau}, \\ \frac{dz}{d\tau} &= \frac{zz_3/\tau}{-z_3 + b\tau}. \end{aligned} \tag{6.88}$$

Introducing the homogeneous variables by the formulas

$$z = u_1\tau, \quad z_3 = u_2\tau, \tag{6.89}$$

we reduce system (6.88) to the following form:

$$\begin{aligned}\tau\frac{du_2}{d\tau}+u_2&=\frac{1-u_1^2}{-u_2+b},\\ \tau\frac{du_1}{d\tau}+u_1&=\frac{u_1u_2}{-u_2+b},\end{aligned}\tag{6.90}$$

which is equivalent to the system

$$\begin{aligned}\tau\frac{du_2}{d\tau}&=\frac{1-u_1^2+u_2^2-bu_2}{-u_2+b},\\ \tau\frac{du_1}{d\tau}&=\frac{2u_1u_2-bu_1}{-u_2+b}.\end{aligned}\tag{6.91}$$

To the second-order system (6.91), we put in correspondence the nonautonomous first-order equation

$$\frac{du_2}{du_1}=\frac{1-u_1^2+u_2^2-bu_2}{2u_1u_2-bu_1},\tag{6.92}$$

which can be easily reduced to the exact differential equation

$$d\left(\frac{u_2^2+u_1^2-bu_2+1}{u_1}\right)=0.\tag{6.93}$$

Thus, Eq. (6.92) has the following first integral:

$$\frac{u_2^2+u_1^2-bu_2+1}{u_1}=C_1=\text{const},\tag{6.94}$$

which in the previous variables has the form

$$\frac{z_3^2+z^2-bz_3\sin\alpha+\sin^2\alpha}{z\sin\alpha}=C_1=\text{const}\,.\tag{6.95}$$

Remark 6.1. Consider system (6.81)–(6.83) with variable dissipation with zero mean (see [67, 68, 243, 260, 262, 265, 282, 283, 285, 286, 291, 295, 309, 324, 333, 340, 384, 391, 404–408, 412, 413, 421, 437, 438]) that becomes conservative for $b=0$:

$$\begin{aligned}\alpha'&=-z_3,\\ z_3'&=\sin\alpha\cos\alpha-z^2\frac{\cos\alpha}{\sin\alpha},\\ z'&=zz_3\frac{\cos\alpha}{\sin\alpha}.\end{aligned}\tag{6.96}$$

It possesses two analytic first integrals of the form

$$z_3^2+z^2+\sin^2\alpha=C_1^*=\text{const},\tag{6.97}$$

$$z\sin\alpha=C_2^*=\text{const}\,.\tag{6.98}$$

Obviously, the ratio of the first two integrals (6.97) and (6.98) is also a first integral of system (6.96). However, for $b\neq 0$, each of the functions

$$z_3^2+z^2-bz_3\sin\alpha+\sin^2\alpha\tag{6.99}$$

and (6.98) is not a first integral of system (6.81)–(6.83) but their ratio is a first integral for any b.

Further, we find the explicit form of the additional first integral of the third-order system (6.81)–(6.83). For this, we transform the invariant relation (6.94) for $u_1\neq 0$ as follows:

$$\left(u_2-\frac{b}{2}\right)^2+\left(u_1-\frac{C_1}{2}\right)^2=\frac{b^2+C_1^2}{4}-1.\tag{6.100}$$

We see that the parameters of this invariant relation satisfy the condition

$$b^2 + C_1^2 - 4 \geq 0, \tag{6.101}$$

and the phase space of system (6.81)–(6.83) is stratified into the family of surfaces defined by Eq. (6.100).

Thus, by relation (6.94), the first equation of system (6.91) has the form

$$\tau \frac{du_2}{d\tau} = \frac{2\left(1 - bu_2 + u_2^2\right) - C_1 U_1(C_1, u_2)}{-u_2 + b}, \tag{6.102}$$

where

$$U_1(C_1, u_2) = \frac{1}{2}\left\{C_1 \pm \sqrt{C_1^2 - 4\left(u_2^2 - bu_2 + 1\right)}\right\}; \tag{6.103}$$

the integration constant C_1 is defined by condition (6.101).

Therefore, the quadrature for the search for the additional first integral of system (6.81)–(6.83) becomes

$$\int \frac{d\tau}{\tau} = \int \frac{(b - u_2)du_2}{2\left(1 - bu_2 + u_2^2\right) - C_1\left\{C_1 \pm \sqrt{C_1^2 - 4\left(u_2^2 - bu_2 + 1\right)}\right\}/2}. \tag{6.104}$$

Obviously, the left-hand side (up to an additive constant) equals

$$\ln|\sin\alpha|. \tag{6.105}$$

If

$$u_2 - \frac{b}{2} = w_1, \quad b_1^2 = b^2 + C_1^2 - 4, \tag{6.106}$$

then the right-hand side of Eq. (6.104) has the form

$$-\frac{1}{4}\int \frac{d\left(b_1^2 - 4w_1^2\right)}{\left(b_1^2 - 4w_1^2\right) \pm C_1\sqrt{b_1^2 - 4w_1^2}} - b\int \frac{dw_1}{\left(b_1^2 - 4w_1^2\right) \pm C_1\sqrt{b_1^2 - 4w_1^2}}$$
$$= -\frac{1}{2}\ln\left|\frac{\sqrt{b_1^2 - 4w_1^2}}{C_1} \pm 1\right| \pm \frac{b}{2}I_1, \tag{6.107}$$

where

$$I_1 = \int \frac{dw_3}{\sqrt{b_1^2 - w_3^2}(w_3 \pm C_1)}, \quad w_3 = \sqrt{b_1^2 - 4w_1^2}. \tag{6.108}$$

In the calculation of integral (6.108), the following three cases are possible.

I. $b > 2$:

$$I_1 = -\frac{1}{2\sqrt{b^2 - 4}}\ln\left|\frac{\sqrt{b^2 - 4} + \sqrt{b_1^2 - w_3^2}}{w_3 \pm C_1} \pm \frac{C_1}{\sqrt{b^2 - 4}}\right| +$$
$$+ \frac{1}{2\sqrt{b^2 - 4}}\ln\left|\frac{\sqrt{b^2 - 4} - \sqrt{b_1^2 - w_3^2}}{w_3 \pm C_1} \mp \frac{C_1}{\sqrt{b^2 - 4}}\right| + \text{const}; \tag{6.109}$$

II. $b < 2$:

$$I_1 = \frac{1}{\sqrt{4 - b^2}}\arcsin\frac{\pm C_1 w_3 + b_1^2}{b_1(w_3 \pm C_1)} + \text{const}; \tag{6.110}$$

III. $b = 2$:

$$I_1 = \mp\frac{\sqrt{b_1^2 - w_3^2}}{C_1(w_3 \pm C_1)} + \text{const}. \tag{6.111}$$

Returning to the variable

$$w_1 = \frac{z_3}{\sin\alpha} - \frac{b}{2}, \tag{6.112}$$

we obtain the final expression for I_1:

I. $b > 2$:

$$I_1 = -\frac{1}{2\sqrt{b^2-4}} \ln\left|\frac{\sqrt{b^2-4} \pm 2w_1}{\sqrt{b_1^2-4w_1^2} \pm C_1} \pm \frac{C_1}{\sqrt{b^2-4}}\right| + \frac{1}{2\sqrt{b^2-4}} \ln\left|\frac{\sqrt{b^2-4} \mp 2w_1}{\sqrt{b_1^2-4w_1^2} \pm C_1} \mp \frac{C_1}{\sqrt{b^2-4}}\right| + \text{const}; \tag{6.113}$$

II. $b < 2$:

$$I_1 = \frac{1}{\sqrt{4-b^2}} \arcsin \frac{\pm C_1\sqrt{b_1^2-4w_1^2} + b_1^2}{b_1\left(\sqrt{b_1^2-4w_1^2} \pm C_1\right)} + \text{const}; \tag{6.114}$$

III. $b = 2$:

$$I_1 = \mp\frac{2w_1}{C_1\left(\sqrt{b_1^2-4w_1^2} \pm C_1\right)} + \text{const}\,. \tag{6.115}$$

Thus, we have found an additional first integral for the third-order system (6.81)–(6.83) and we have the complete set of first integrals that are transcendental functions of their phase variables.

Remark 6.2. We must substitute the left-hand side of the first integral (6.94) in the expression of this first integral instead of C_1. Then the additional first integral obtained has the following structure (similar to the transcendental first integral in planar dynamics):

$$\ln|\sin\alpha| + G_2\left(\sin\alpha,\ \frac{z_3}{\sin\alpha},\ \frac{z}{\sin\alpha}\right) = C_2 = \text{const}\,. \tag{6.116}$$

Thus, for the integration of the sixth-order system (6.81)–(6.86), we have found two independent first integrals. For the complete integration, as was mentioned above, it suffices to find one first integral for (potentially separated) system (6.84), (6.85), and an additional first integral that "attaches" Eq. (6.86).

To find a first integral for (potentially separated) system (6.84), (6.85), we put in correspondence the following nonautonomous first-order equation:

$$\frac{dz_*}{d\beta_1} = \frac{1+z_*^2}{z_*}\frac{\cos\beta_1}{\sin\beta_1}. \tag{6.117}$$

After integration, this leads to the invariant relation

$$\frac{\sqrt{1+z_*^2}}{\sin\beta_1} = C_3 = \text{const}, \tag{6.118}$$

which in the variables z_1 and z_2 has the form

$$\frac{\sqrt{z_1^2+z_2^2}}{z_1\sin\beta_1} = C_3 = \text{const}\,. \tag{6.119}$$

Further, for the search for an additional first integral that "attaches" Eq. (6.86), to Eqs. (6.86) and (6.84) we put in correspondence the following nonautonomous equation:

$$\frac{dz_*}{d\beta_2} = -\left(1+z_*^2\right)\cos\beta_1. \tag{6.120}$$

Since, by (6.118),

$$C_3\cos\beta_1 = \pm\sqrt{C_3^2-1-z_*^2}, \tag{6.121}$$

we have

$$\frac{dz_*}{d\beta_2} = \mp \frac{1}{C_3}\left(1 + z_*^2\right)\sqrt{C_3^2 - 1 - z_*^2}. \tag{6.122}$$

Integrating the last relation, we arrive at the following quadrature:

$$\mp(\beta_2 + C_4) = \int \frac{C_3 dz_*}{(1 + z_*^2)\sqrt{C_3^2 - 1 - z_*^2}}, \quad C_4 = \text{const}\,. \tag{6.123}$$

Integrating this relation we obtain

$$\mp \tan(\beta_2 + C_4) = \frac{C_3 z_*}{\sqrt{C_3^2 - 1 - z_*^2}}, \quad C_4 = \text{const}\,. \tag{6.124}$$

In the variables z_1 and z_2 the last invariant relation has the form

$$\mp \tan(\beta_2 + C_4) = \frac{C_3 z_2}{\sqrt{\left(C_3^2 - 1\right) z_1^2 - z_2^2}}, \quad C_4 = \text{const}\,. \tag{6.125}$$

Finally, we have the following form of the additional first integral that “attaches” Eq. (6.86):

$$\arctan \frac{C_3 z_*}{\sqrt{C_3^2 - 1 - z_*^2}} \pm \beta_2 = C_4, \quad C_4 = \text{const} \tag{6.126}$$

or

$$\arctan \frac{C_3 z_2}{\sqrt{\left(C_3^2 - 1\right) z_1^2 - z_2^2}} \pm \beta_2 = C_4, \quad C_4 = \text{const}\,. \tag{6.127}$$

Thus, in the case considered, the system of dynamical equations (6.17)–(6.20) and (6.23)–(6.28) under condition (6.65) has eight invariant relations: the nonintegrable analytic constraint of the form (6.33); the cyclic first integrals of the form (6.31), (6.32); the first integral of the form (6.95); the first integral expressed by relations (6.109)–(6.116), which is a transcendental function of the phase variables (in the sense of complex analysis) expressed through a finite combination of elementary functions; and, finally, the transcendental first integrals of the form (6.118) (or (6.119)) and (6.126) (or (6.127)).

Theorem 6.1. *System* (6.17)–(6.20), (6.23)–(6.28) *under conditions* (6.33), (6.65), (6.32) *possesses eight invariant relations* (*complete set*), *four of which are transcendental functions from the point of view of complex analysis. Herewith, all relations are expressed through finite combinations of elementary functions.*

3.3. Topological analogies. Consider the following fifth-order system:

$$\begin{gathered}
\ddot{\xi} + b_* \dot{\xi} \cos\xi + \sin\xi \cos\xi - \left[\dot{\eta}_1^{\,2} + \dot{\eta}_2^{\,2} \sin^2 \eta_1\right] \frac{\sin\xi}{\cos\xi} = 0, \\
\ddot{\eta}_1 + b_* \dot{\eta}_1 \cos\xi + \dot{\xi}\dot{\eta}_1 \frac{1 + \cos^2\xi}{\cos\xi \sin\xi} - \dot{\eta}_2^{\,2} \sin\eta_1 \cos\eta_1 = 0, \\
\ddot{\eta}_2 + b_* \dot{\eta}_2 \cos\xi + \dot{\xi}\dot{\eta}_2 \frac{1 + \cos^2\xi}{\cos\xi \sin\xi} + 2\dot{\eta}_1\dot{\eta}_2 \frac{\cos\eta_1}{\cos\eta_1} = 0, \quad b_* > 0,
\end{gathered} \tag{6.128}$$

which describes a fixed four-dimensional pendulum in a flowing medium for which the moment of forces is independent of the angular velocity, i.e., a mechanical system in a nonconservative field (see [48, 49, 439]). In general the order of such a system is equal to 6, but the phase variable η_2 is a cyclic variable, which leads to the stratification of the phase space and reduces the order of the system.

The phase space of this system is the tangent bundle

$$T\mathbf{S}^3\left\{\dot{\xi}, \dot{\eta}_1, \dot{\eta}_2, \xi, \eta_1, \eta_2\right\} \tag{6.129}$$

of the three-dimensional sphere $\mathbf{S}^3\{\xi, \eta_1, \eta_2\}$. The equation that transforms system (6.128) into the system on the tangent bundle of the two-dimensional sphere

$$\dot{\eta}_2 \equiv 0, \tag{6.130}$$

and the equations of great circles

$$\dot{\eta}_1 \equiv 0, \quad \dot{\eta}_2 \equiv 0 \tag{6.131}$$

define families of integral manifolds.

It is easy to verify that system (6.128) is equivalent to the dynamical system with variable dissipation with zero mean on the tangent bundle (6.129) of the three-dimensional sphere. Moreover, the following theorem holds.

Theorem 6.2. *System* (6.17)–(6.20), (6.23)–(6.28) *under conditions* (6.33), (6.65), *and* (6.32) *is equivalent to the dynamical system* (6.128).

Proof. Indeed, it suffices to set $\alpha = \xi$, $\beta_1 = \eta_1$, $\beta_2 = \eta_2$, and $b = -b_*$. □

On more general topological analogies, see [253, 265, 267, 309, 340, 342].

4. Case Where the Moment of a Nonconservative Force Depends on the Angular Velocity

4.1. Introduction of the dependence on the angular velocity. This chapter is devoted to the dynamics of a four-dimensional rigid body in the four-dimensional space. Since the present section is devoted to the study of the motion in the case where the moment of forces depends on the tensor of angular velocity, we introduce this dependence in a more general situation. This also allows us to introduce this dependence for multi-dimensional bodies.

Let $x = (x_{1N}, x_{2N}, x_{3N}, x_{4N})$ be the coordinates of the point N of application of a nonconservative force (influence of the medium) acting on the three-dimensional disk and $Q = (Q_1, Q_2, Q_3, Q_4)$ be the components of the force $\mathbf{S}$ of the influence of the medium independent of the tensor of the angular velocity. We consider only linear dependence of the functions $(x_{1N}, x_{2N}, x_{3N}, x_{4N})$ on the tensor of angular velocity since this introduction itself is not obvious (see [33, 34, 48, 49, 57–66, 120, 121, 169, 182, 203, 205, 249, 250, 261, 274–276, 440]).

We adopt the following dependence:

$$x = Q + R, \tag{6.132}$$

where $R = (R_1, R_2, R_3, R_4)$ is a vector-valued function containing the components of the tensor of angular velocity. The dependence of the function R on the components of the tensor of angular velocity is gyroscopic:

$$R = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} = -\frac{1}{v} \begin{pmatrix} 0 & -\omega_6 & \omega_5 & -\omega_3 \\ \omega_6 & 0 & -\omega_4 & \omega_2 \\ -\omega_5 & \omega_4 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}, \tag{6.133}$$

where (h_1, h_2, h_3, h_4) are some positive parameters (cf. [322, 330, 331, 345, 393]).

Since $x_{1N} \equiv 0$, we have

$$x_{2N} = Q_2 - h_1 \frac{\omega_6}{v}, \quad x_{3N} = Q_3 + h_1 \frac{\omega_5}{v}, \quad x_{4N} = Q_4 - h_1 \frac{\omega_3}{v}. \tag{6.134}$$

4.2. Reduced system. Similarly to the choice of the Chaplygin analytic functions (see [50, 51])

$$\begin{aligned} Q_2 &= A\sin\alpha\cos\beta_1, \\ Q_3 &= A\sin\alpha\sin\beta_1\cos\beta_2, \\ Q_4 &= A\sin\alpha\sin\beta_1\sin\beta_2, \quad A>0, \end{aligned} \tag{6.135}$$

we take the dynamical functions s, x_{2N}, x_{3N}, and x_{4N} in the following form:

$$\begin{aligned} s(\alpha) &= B\cos\alpha,\ B>0, \\ x_{2N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) &= A\sin\alpha\cos\beta_1 - h\frac{\omega_6}{v}, \quad h=h_1>0,\ v\neq 0, \\ x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) &= A\sin\alpha\sin\beta_1\cos\beta_2 + h\frac{\omega_5}{v}, \quad h=h_1>0,\ v\neq 0, \\ x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) &= A\sin\alpha\sin\beta_1\sin\beta_2 - h\frac{\omega_3}{v}, \quad h=h_1>0,\ v\neq 0. \end{aligned} \tag{6.136}$$

This shows that in the problem considered, there is an additional damping (but accelerating in certain domains of the phase space) moment of a nonconservative force (i.e., there is a dependence of the moment on the components of the tensor of angular velocity). Moreover, $h_2 = h_3 = h_4$ due to the dynamical symmetry of the body.

In this case, the functions $\Gamma_v(\alpha,\beta_1,\beta_2,\Omega/v)$, $\Delta_v(\alpha,\beta_1,\beta_2,\Omega/v)$, and $\Theta_v(\alpha,\beta_1,\beta_2,\Omega/v)$ in system (6.53)–(6.58) have the following form:

$$\begin{aligned} \Gamma_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) &= A\sin\alpha - \frac{h}{v}z_3, \\ \Delta_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) &= \frac{h}{v}z_2, \\ \Theta_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) &= -\frac{h}{v}z_1. \end{aligned} \tag{6.137}$$

Then, due to the nonintegrable constraint (6.33), outside the manifold (6.52) the dynamical part of the equations of motion (system (6.53)–(6.58)) takes the form of the analytic system

$$\dot\alpha = -\left(1+\frac{\sigma Bh}{2I_2}\right)z_3 + \frac{\sigma ABv}{2I_2}\sin\alpha, \tag{6.138}$$

$$\dot z_3 = \frac{ABv^2}{2I_2}\sin\alpha\cos\alpha - \left(1+\frac{\sigma Bh}{2I_2}\right)\left(z_1^2+z_2^2\right)\frac{\cos\alpha}{\sin\alpha} - \frac{Bhv}{2I_2}z_3\cos\alpha, \tag{6.139}$$

$$\dot z_2 = \left(1+\frac{\sigma Bh}{2I_2}\right)z_2z_3\frac{\cos\alpha}{\sin\alpha} + \left(1+\frac{\sigma Bh}{2I_2}\right)z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} - \frac{Bhv}{2I_2}z_2\cos\alpha, \tag{6.140}$$

$$\dot z_1 = \left(1+\frac{\sigma Bh}{2I_2}\right)z_1z_3\frac{\cos\alpha}{\sin\alpha} - \left(1+\frac{\sigma Bh}{2I_2}\right)z_1z_2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} - \frac{Bhv}{2I_2}z_1\cos\alpha, \tag{6.141}$$

$$\dot\beta_1 = \left(1+\frac{\sigma Bh}{2I_2}\right)z_2\frac{\cos\alpha}{\sin\alpha}, \tag{6.142}$$

$$\dot\beta_2 = -\left(1+\frac{\sigma Bh}{2I_2}\right)z_1\frac{\cos\alpha}{\sin\alpha\sin\beta_1}. \tag{6.143}$$

Introducing the dimensionless variables, parameters, and the differentiation as follows:

$$z_k \mapsto n_0 v z_k,\ k=1,2,3, \quad n_0^2 = \frac{AB}{2I_2}, \quad b=\sigma n_0, \quad H_1 = \frac{Bh}{2I_2 n_0}, \quad \langle\dot{\ }\rangle = n_0 v\langle\ '\rangle, \tag{6.144}$$

we reduce system (6.138)–(6.143) to the form

$$\dot{\alpha} = -\left(1 + bH_1\right) z_3 + b\sin\alpha, \tag{6.145}$$

$$\dot{z_3} = \sin\alpha\cos\alpha - \left(1 + bH_1\right)\left(z_1^2 + z_2^2\right)\frac{\cos\alpha}{\sin\alpha} - H_1 z_3\cos\alpha, \tag{6.146}$$

$$\dot{z_2} = \left(1 + bH_1\right) z_2 z_3\frac{\cos\alpha}{\sin\alpha} + \left(1 + bH_1\right) z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} - H_1 z_2\cos\alpha, \tag{6.147}$$

$$\dot{z_1} = \left(1 + bH_1\right) z_1 z_3\frac{\cos\alpha}{\sin\alpha} - \left(1 + bH_1\right) z_1 z_2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} - H_1 z_1\cos\alpha, \tag{6.148}$$

$$\dot{\beta_1} = \left(1 + bH_1\right) z_2\frac{\cos\alpha}{\sin\alpha}, \tag{6.149}$$

$$\dot{\beta_2} = -\left(1 + bH_1\right) z_1\frac{\cos\alpha}{\sin\alpha\sin\beta_1}. \tag{6.150}$$

We see that the sixth-order system (6.145)–(6.150) (which can be considered on the tangent bundle $T\mathbf{S}^3$ of the three-dimensional sphere $\mathbf{S}^3$), contains an independent fifth-order system (6.145)–(6.149) on its own five-dimensional manifold.

For the complete integration of system (6.145)–(6.150), we need, in general, five independent first integrals. However, after the change of variables

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to \begin{pmatrix} z \\ z_* \end{pmatrix}, \quad z = \sqrt{z_1^2 + z_2^2}, \quad z_* = z_2/z_1, \tag{6.151}$$

system (6.145)–(6.150) splits as follows:

$$\alpha' = -(1 + bH_1)z_3 + b\sin\alpha, \tag{6.152}$$

$$z_3' = \sin\alpha\cos\alpha - (1 + bH_1)z^2\frac{\cos\alpha}{\sin\alpha} - H_1 z_3\cos\alpha, \tag{6.153}$$

$$z' = (1 + bH_1)zz_3\frac{\cos\alpha}{\sin\alpha} - H_1 z\cos\alpha, \tag{6.154}$$

$$z_*' = (\pm)(1 + bH_1)z\sqrt{1 + z_*^2}\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}, \tag{6.155}$$

$$\beta_1' = (\pm)(1 + bH_1)\frac{zz_*}{\sqrt{1 + z_*^2}}\frac{\cos\alpha}{\sin\alpha}, \tag{6.156}$$

$$\beta_2' = (\mp)(1 + bH_1)\frac{z}{\sqrt{1 + z_*^2}}\frac{\cos\alpha}{\sin\alpha\sin\beta_1}. \tag{6.157}$$

We see that the sixth-order system splits into independent subsystems of lower orders: system (6.152)–(6.154) of order 3 and system (6.155), (6.156) (certainly, after a choice of the independent variables) of order 2. Thus, for the complete integration of system (6.152)–(6.157), it suffices to find two independent first integrals of system (6.152)–(6.154), one first integral of system (6.155), (6.156), and an additional first integral that "attaches" Eq. (6.157).

Note that system (6.152)–(6.154) can be considered on the tangent bundle $T\mathbf{S}^2$ of the two-dimensional sphere $\mathbf{S}^2$.

4.3. Complete list of invariant relations. System (6.152)–(6.154) has the form of a system of equations that appears in the dynamics of a three-dimensional (3D) rigid body in a nonconservative field. First, to the third-order system (6.152)–(6.154), we put in correspondence the nonautonomous second-order system

$$\begin{aligned}\frac{dz_3}{d\alpha} &= \frac{\sin\alpha\cos\alpha - (1+bH_1)z^2\cos\alpha/\sin\alpha - H_1 z_3\cos\alpha}{-(1+bH_1)z_3 + b\sin\alpha},\\ \frac{dz}{d\alpha} &= \frac{(1+bH_1)zz_3\cos\alpha/\sin\alpha - H_1 z\cos\alpha}{-(1+bH_1)z_3 + b\sin\alpha}.\end{aligned} \tag{6.158}$$

Using the substitution $\tau = \sin\alpha$, we rewrite system (6.158) in the algebraic form:

$$\begin{aligned}\frac{dz_3}{d\tau} &= \frac{\tau - (1+bH_1)z^2/\tau - H_1 z_3}{-(1+bH_1)z_3 + b\tau},\\ \frac{dz}{d\tau} &= \frac{(1+bH_1)zz_3/\tau - H_1 z}{-(1+bH_1)z_3 + b\tau}.\end{aligned} \tag{6.159}$$

Further, introducing the homogeneous variables by the formulas

$$z = u_1\tau, \quad z_3 = u_2\tau, \tag{6.160}$$

we reduce system (6.159) to the following form:

$$\begin{aligned}\tau\frac{du_2}{d\tau} + u_2 &= \frac{1 - (1+bH_1)u_1^2 - H_1u_2}{-(1+bH_1)u_2 + b},\\ \tau\frac{du_1}{d\tau} + u_1 &= \frac{(1+bH_1)u_1u_2 - H_1u_1}{-(1+bH_1)u_2 + b},\end{aligned} \tag{6.161}$$

which is equivalent to

$$\begin{aligned}\tau\frac{du_2}{d\tau} &= \frac{(1+bH_1)\left(u_2^2 - u_1^2\right) - (b+H_1)u_2 + 1}{-(1+bH_1)u_2 + b},\\ \tau\frac{du_1}{d\tau} &= \frac{2(1+bH_1)u_1u_2 - (b+H_1)u_1}{-(1+bH_1)u_2 + b}.\end{aligned} \tag{6.162}$$

To the second-order system (6.162), we put in correspondence the nonautonomous first-order equation

$$\frac{du_2}{du_1} = \frac{1 - (1+bH_1)\left(u_1^2 - u_2^2\right) - (b+H_1)u_2}{2(1+bH_1)u_1u_2 - (b+H_1)u_1}, \tag{6.163}$$

which can be easily reduce to the exact differential equation

$$d\left(\frac{(1+bH_1)\left(u_2^2 + u_1^2\right) - (b+H_1)u_2 + 1}{u_1}\right) = 0. \tag{6.164}$$

Thus, Eq. (6.163) has the following first integral:

$$\frac{(1+bH_1)\left(u_2^2 + u_1^2\right) - (b+H_1)u_2 + 1}{u_1} = C_1 = \text{const}, \tag{6.165}$$

which in the original variables has the form

$$\frac{(1+bH_1)\left(z_3^2 + z^2\right) - (b+H_1)z_3\sin\alpha + \sin^2\alpha}{z\sin\alpha} = C_1 = \text{const}\,. \tag{6.166}$$

Remark 6.3. Consider system (6.152)–(6.154) with variable dissipation with zero mean (see [265]), which becomes conservative for $b = H_1$:

$$\begin{aligned}\alpha' &= -\left(1+b^2\right)z_3 + b\sin\alpha,\\ z_3' &= \sin\alpha\cos\alpha - \left(1+b^2\right)z^2\frac{\cos\alpha}{\sin\alpha} - bz_3\cos\alpha,\\ z' &= \left(1+b^2\right)zz_3\frac{\cos\alpha}{\sin\alpha} - bz\cos\alpha.\end{aligned} \tag{6.167}$$

It possesses the following two analytic first integrals:

$$\left(1+b^2\right)\left(z_3^2+z^2\right)-2bz_3\sin\alpha+\sin^2\alpha=C_1^*=\text{const}, \tag{6.168}$$

$$z\sin\alpha=C_2^*=\text{const}\,. \tag{6.169}$$

Obviously, the ratio of the two first integrals (6.168) and (6.169) is also a first integral of system (6.167). However, for $b\neq H_1$, none of the functions

$$(1+bH_1)\left(z_3^2+z^2\right)-(b+H_1)z_3\sin\alpha+\sin^2\alpha \tag{6.170}$$

and (6.169) is a first integral of system (6.152)–(6.154), but their ratio is a first integral of system (6.152)–(6.154) for any b and H_1.

We find the explicit form of the additional first integral of the third-order system (6.152)–(6.154). First, we transform the invariant relation (6.165) for $u_1\neq 0$ as follows:

$$\left(u_2-\frac{b+H_1}{2(1+bH_1)}\right)^2+\left(u_1-\frac{C_1}{2(1+bH_1)}\right)^2=\frac{(b-H_1)^2+C_1^2-4}{4(1+bH_1)^2}. \tag{6.171}$$

We see that the parameters of this invariant relation must satisfy the condition

$$(b-H_1)^2+C_1^2-4\geq 0, \tag{6.172}$$

and the phase space of system (6.152)–(6.154) is stratified into the family of surfaces defined by Eq. (6.171).

Thus, due to relation (6.165), the first equation of system (6.162) has the form

$$\tau\frac{du_2}{d\tau}=\frac{2(1+bH_1)u_2^2-2(b+H_1)u_2+2-C_1U_1(C_1,u_2)}{b-(1+bH_1)u_2}, \tag{6.173}$$

where

$$\begin{aligned} U_1(C_1,u_2)&=\frac{1}{2(1+bH_1)}\{C_1\pm U_2(C_1,u_2)\},\\ U_2(C_1,u_2)&=\sqrt{C_1^2-4(1+bH_1)\left(1-(b+H_1)u_2+(1+bH_1)u_2^2\right)}, \end{aligned} \tag{6.174}$$

and the integration constant C_1 is defined by condition (6.172).

Therefore, the quadrature for the search for an additional first integral of system (6.152)–(6.154) becomes

$$\int\frac{d\tau}{\tau}=\int\frac{(b-(1+bH_1)u_2)du_2}{2(1-(b+H_1)u_2+(1+bH_1)u_2^2)-C_1\{C_1\pm U_2(C_1,u_2)\}/(2(1+bH_1))}. \tag{6.175}$$

Obviously, the left-hand side(up to an additive constant) is equal to

$$\ln|\sin\alpha|. \tag{6.176}$$

If

$$u_2-\frac{b+H_1}{2(1+bH_1)}=w_1,\quad b_1^2=(b-H_1)^2+C_1^2-4, \tag{6.177}$$

then the right-hand side of Eq. (6.175) becomes

$$\begin{aligned} &-\frac{1}{4}\int\frac{d\left(b_1^2-4(1+bH_1)w_1^2\right)}{\left(b_1^2-4(1+bH_1)w_1^2\right)\pm C_1\sqrt{b_1^2-4(1+bH_1)w_1^2}}\\ &\qquad-(b-H_1)(1+bH_1)\int\frac{dw_1}{\left(b_1^2-4(1+bH_1)w_1^2\right)\pm C_1\sqrt{b_1^2-4(1+bH_1)w_1^2}}\\ &\qquad\qquad=-\frac{1}{2}\ln\left|\frac{\sqrt{b_1^2-4(1+bH_1)w_1^2}}{C_1}\pm 1\right|\pm\frac{b-H_1}{2}I_1, \end{aligned} \tag{6.178}$$

where

$$I_1 = \int \frac{dw_3}{\sqrt{b_1^2 - w_3^2}(w_3 \pm C_1)}, \quad w_3 = \sqrt{b_1^2 - 4(1 + bH_1)w_1^2}. \tag{6.179}$$

In the calculation of integral (6.179), the following three cases are possible:

I. $|b - H_1| > 2$:

$$I_1 = -\frac{1}{2\sqrt{(b - H_1)^2 - 4}} \ln \left| \frac{\sqrt{(b - H_1)^2 - 4} + \sqrt{b_1^2 - w_3^2}}{w_3 \pm C_1} \pm \frac{C_1}{\sqrt{(b - H_1)^2 - 4}} \right| + \frac{1}{2\sqrt{(b - H_1)^2 - 4}} \ln \left| \frac{\sqrt{(b - H_1)^2 - 4} - \sqrt{b_1^2 - w_3^2}}{w_3 \pm C_1} \mp \frac{C_1}{\sqrt{(b - H_1)^2 - 4}} \right| + \text{const}; \tag{6.180}$$

II. $|b - H_1| < 2$:

$$I_1 = \frac{1}{\sqrt{4 - (b - H_1)^2}} \arcsin \frac{\pm C_1 w_3 + b_1^2}{b_1(w_3 \pm C_1)} + \text{const}; \tag{6.181}$$

III. $|b - H_1| = 2$:

$$I_1 = \mp \frac{\sqrt{b_1^2 - w_3^2}}{C_1(w_3 \pm C_1)} + \text{const}. \tag{6.182}$$

Returning to the variable

$$w_1 = \frac{z_2}{\sin\alpha} - \frac{b + H_1}{2(1 + bH_1)}, \tag{6.183}$$

we have the following final form of I_1:

I. $|b - H_1| > 2$:

$$I_1 = -\frac{1}{2\sqrt{(b - H_1)^2 - 4}} \ln \left| \frac{\sqrt{(b - H_1)^2 - 4} \pm 2(1 + bH_1)w_1}{\sqrt{b_1^2 - 4(1 + bH_1)^2 w_1^2} \pm C_1} \pm \frac{C_1}{\sqrt{(b - H_1)^2 - 4}} \right| + \frac{1}{2\sqrt{(b - H_1)^2 - 4}} \ln \left| \frac{\sqrt{(b - H_1)^2 - 4} \mp 2(1 + bH_1)w_1}{\sqrt{b_1^2 - 4(1 + bH_1)^2 w_1^2} \pm C_1} \mp \frac{C_1}{\sqrt{(b - H_1)^2 - 4}} \right| + \text{const}; \tag{6.184}$$

II. $|b - H_1| < 2$:

$$I_1 = \frac{1}{\sqrt{4 - (b - H_1)^2}} \arcsin \frac{\pm C_1 \sqrt{b_1^2 - 4(1 + bH_1)^2 w_1^2} + b_1^2}{b_1 \left(\sqrt{b_1^2 - 4(1 + bH_1)^2 w_1^2} \pm C_1 \right)} + \text{const}; \tag{6.185}$$

III. $|b - H_1| = 2$:

$$I_1 = \mp \frac{2(1 + bH_1)w_1}{C_1 \left(\sqrt{b_1^2 - 4(1 + bH_1)^2 w_1^2} \pm C_1 \right)} + \text{const}. \tag{6.186}$$

Thus, we have found an additional first integral for the third-order system (6.152)–(6.154) and we have the complete set of first integrals that are transcendental functions of their phase variables.

Remark 6.4. Formally, in the expression of the found first integral, we must substitute instead of C_1 the left-hand side of the first integral (6.165).

Then the obtained additional first integral has the following structure (similar to the transcendental first integral from planar dynamics):

$$\ln|\sin\alpha| + G_2 \left(\sin\alpha, \frac{z_3}{\sin\alpha}, \frac{z}{\sin\alpha} \right) = C_2 = \text{const}. \tag{6.187}$$

Thus, to integrate the sixth-order system (6.152)–(6.157), we have already found two independent first integrals. For the complete integration, as was mentioed above, it suffices to find one first integral for the (potentially separated) system (6.155), (6.156) and an additional first integral that "attaches" Eq. (6.157).

To find a first integral of the (potentially separated) system (6.155), (6.156), we put in correspondence the following nonautonomous first-order equation:

$$\frac{dz_*}{d\beta_1} = \frac{1+z_*^2}{z_*}\frac{\cos\beta_1}{\sin\beta_1}. \tag{6.188}$$

After integration we obtain the required invariant relation

$$\frac{\sqrt{1+z_*^2}}{\sin\beta_1} = C_3 = \text{const}, \tag{6.189}$$

which in the variables z_1 and z_2 has the form

$$\frac{\sqrt{z_1^2+z_2^2}}{z_1\sin\beta_1} = C_3 = \text{const}\,. \tag{6.190}$$

Further, to obtain an additional first integral that "attaches" Eq. (6.157), to Eqs. (6.157) and (6.155) we put in correspondence the following nonautonomous equation:

$$\frac{dz_*}{d\beta_2} = -\left(1+z_*^2\right)\cos\beta_1. \tag{6.191}$$

Since

$$C_3\cos\beta_1 = \pm\sqrt{C_3^2-1-z_*^2} \tag{6.192}$$

by (6.189), we have

$$\frac{dz_*}{d\beta_2} = \mp\frac{1}{C_3}\left(1+z_*^2\right)\sqrt{C_3^2-1-z_*^2}. \tag{6.193}$$

Integrating this relation, we arrive at the following quadrature:

$$\mp(\beta_2+C_4) = \int\frac{C_3dz_*}{(1+z_*^2)\sqrt{C_3^2-1-z_*^2}}, \quad C_4 = \text{const}\,. \tag{6.194}$$

Integration leads to the relation

$$\mp\tan(\beta_2+C_4) = \frac{C_3z_*}{\sqrt{C_3^2-1-z_*^2}}, \quad C_4 = \text{const}\,. \tag{6.195}$$

In the variables z_1 and z_2 this invariant relation has the form

$$\mp\tan(\beta_2+C_4) = \frac{C_3z_2}{\sqrt{\left(C_3^2-1\right)z_1^2-z_2^2}}, \quad C_4 = \text{const}\,. \tag{6.196}$$

Finally, we have the following additional first integral that "attaches" Eq. (6.157):

$$\arctan\frac{C_3z_*}{\sqrt{C_3^2-1-z_*^2}} \pm \beta_2 = C_4, \quad C_4 = \text{const} \tag{6.197}$$

or

$$\arctan\frac{C_3z_2}{\sqrt{\left(C_3^2-1\right)z_1^2-z_2^2}} \pm \beta_2 = C_4, \quad C_4 = \text{const}\,. \tag{6.198}$$

Thus, in the case considered, the system of dynamical equations (6.17)–(6.20), (6.23)–(6.28) under condition (6.136) has eight invariant relations: the analytic nonintegrable constraint of the form (6.33), the cyclic first integrals of the form (6.31) and (6.32), the first integral of the form (6.166), the first integral expressed by relations (6.180)–(6.187), which is a transcendental function of the phase variables (in the sense of complex analysis) expressed through a finite combination of functions, and the transcendental first integrals of the form (6.189) (or (6.190)) and (6.197) (or (6.198)).

Theorem 6.3. *System* (6.17)–(6.20), (6.23)–(6.28) *under conditions* (6.33), (6.136), *and* (6.32) *possesses eight invariant relations (complete set); four of them are transcendental functions from the point of view of complex analysis. All relations are expressed through finite combinations of elementary functions.*

4.4. Topological analogies. Consider the following fifth-order system:

$$\begin{gathered}\ddot{\xi} + (b_* - H_{1*})\dot{\xi}\cos\xi + \sin\xi\cos\xi - \left[\dot{\eta}_1{}^2 + \dot{\eta}_2{}^2\sin^2\eta_1\right]\frac{\sin\xi}{\cos\xi} = 0,\\ \ddot{\eta}_1 + (b_* - H_{1*})\dot{\eta}_1\cos\xi + \dot{\xi}\dot{\eta}_1\frac{1+\cos^2\xi}{\cos\xi\sin\xi} - \dot{\eta}_2{}^2\sin\eta_1\cos\eta_1 = 0,\\ \ddot{\eta}_2 + (b_* - H_{1*})\dot{\eta}_2\cos\xi + \dot{\xi}\dot{\eta}_2\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + 2\dot{\eta}_1\dot{\eta}_2\frac{\cos\eta_1}{\cos\eta_1} = 0,\end{gathered} \tag{6.199}$$

where $b_* > 0$ and $H_{1*} > 0$. This system describes a fixed four-dimensional pendulum in a flowing medium for which the moment of forces depends on the angular velocity, i.e., a mechanical system in a nonconservative field (see [120, 162, 188, 201, 203, 235, 238, 276, 316, 317, 319, 320, 338, 359, 360, 376, 377, 386, 392, 429, 442]). Generally speaking, the order of this system must be equal to 6, but the phase variable η_2 is a cyclic variable, which leads to stratification of the phase space and reduction of the order of the system.

The phase space of this system is the tangent bundle

$$T\mathbf{S}^3\left\{\dot{\xi}, \dot{\eta}_1, \dot{\eta}_2, \xi, \eta_1, \eta_2\right\} \tag{6.200}$$

of the three-dimensional sphere $\mathbf{S}^3\{\xi, \eta_1, \eta_2\}$. The equation that transforms system (6.128) into the system on the tangent bundle of the two-dimensional sphere

$$\dot{\eta}_2 \equiv 0 \tag{6.201}$$

and the equations of great circles

$$\dot{\eta}_1 \equiv 0, \quad \dot{\eta}_2 \equiv 0 \tag{6.202}$$

define families of integral manifolds.

It is easy to verify that system (6.199) is equivalent to the dynamical system with variable dissipation with zero mean on the tangent bundle (6.200) of the three-dimensional sphere. Moreover, the following theorem holds.

Theorem 6.4. *System* (6.17)–(6.20), (6.23)–(6.28) *under conditions* (6.33), (6.136), *and* (6.32) *is equivalent to the dynamical system* (6.199).

Proof. Indeed, it suffices to set $\alpha = \xi$, $\beta_1 = \eta_1$, $\beta_2 = \eta_2$, $b = -b_*$, and $H_1 = -H_{1*}$. □

On more general topological analogies, see [253, 265, 267, 309, 340, 342].

Chapter 7

CASES OF INTEGRABILITY CORRESPONDING TO THE MOTION OF A RIGID BODY IN THE FOUR-DIMENSIONAL SPACE, II

In this chapter, we systematize results, both new and obtained earlier, concerning the study of equations of motion of an axis-symmetric four-dimensional (4D) rigid body in a field of nonconservative forces. These equations are taken from the dynamics of realistic rigid bodies of lesser dimension that interact with a resisting medium by laws of jet flow when the body is subjected to a nonconservative tracing force such that throughout the motion, the center of mass of the body moves rectilinearly and uniformly; this means that in the system there exists a nonconservative couple of forces (see [1, 64, 70, 72, 119–121, 157, 164–167, 180, 181, 184, 191, 194, 212, 231, 258, 291, 353, 354, 374, 390, 414]).

Earlier, in [164–167] the author proved the complete integrability of the equations of a plane-parallel motion of a body in a resisting medium under the conditions of jet flow in the case where the system of dynamical equations possesses a first integral, which is a transcendental (in the sense of the theory of functions of a complex variable) function of quasi-velocities. It was assumed that the interaction of the body with the medium is concentrated on a part of the surface of the body that has the form of a (one-dimensional) plate.

In the sequel (see [193, 196, 202, 204, 208, 209, 218, 232, 241]), the planar problem was generalized to the spatial (three-dimensional) case, where the system of dynamical equations possesses a complete set of transcendental first integrals. In this case, it was assumed that the interaction of the medium with the body is concentrated on the part of the surface of the body that has the form of a planar (two-dimensional) disk.

In this chapter, we discuss results, both new and obtained earlier, concerning the case where the interaction of the medium with the body is concentrated on the part of the surface of the body that has the form of a three-dimensional disk and the force acts in a direction perpendicular to the disk. We systematize these results and formulate them in the invariant form. We also introduce the additional dependence of the moment of a nonconservative force on the angular velocity; this dependence can be generalized to the motion in higher-dimensional spaces.

1. General Problem on the Motion Under a Tracing Force

Consider the motion of a homogeneous, dynamically symmetric (case (6.1)), rigid body with front end face (a three-dimensional disk interacting with a medium that fills the four-dimensional space) in the field of a resistance force $\mathbf{S}$ under the quasi-stationarity conditions (see [28, 50, 51, 62–66, 98, 112, 119–121, 160–169, 171, 431, 432]).

Let $(v, \alpha, \beta_1, \beta_2)$ be the (generalized) spherical coordinates of the velocity vector of the center D of the three-dimensional disk lying on the axis of symmetry of the body,

$$\Omega = \begin{pmatrix} 0 & -\omega_6 & \omega_5 & -\omega_3 \\ \omega_6 & 0 & -\omega_4 & \omega_2 \\ -\omega_5 & \omega_4 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

be the tensor of angular velocity of the body, $Dx_1x_2x_3x_4$ be the coordinate system attached to the body such that the axis of symmetry CD coincides with the axis Dx_1 (recall that C is the center of mass), the axes Dx_2, Dx_3, and Dx_4 lie in the hyperplane of the disk, and I_1, I_2, $I_3 = I_2$, $I_4 = I_2$, and m are characteristics of inertia and mass.

We adopt the following expansions in the projections to the axes of the coordinate system $Dx_1x_2x_3x_4$:

$$\begin{aligned} \mathbf{DC} &= \{-\sigma, 0, 0, 0\}, \\ \mathbf{v}_D &= \Big\{v\cos\alpha,\ v\sin\alpha\cos\beta_1,\ v\sin\alpha\sin\beta_1\cos\beta_2,\ v\sin\alpha\sin\beta_1\sin\beta_2\Big\}. \end{aligned} \tag{7.1}$$

In the case (6.1) we additionally have the expansion for the function of the influence of the medium on the four-dimensional body:

$$\mathbf{S} = \{-S, 0, 0, 0\} \tag{7.2}$$

i.e., in this case $\mathbf{F} = \mathbf{S}$.

Then the part of dynamical equations of motion of the body (including the Chaplygin analytic functions, [50, 51], see below) that describes the motion of the center of mass and corresponds to the space $\mathbb{R}^4$, in which tangent forces of the influence of the medium on the three-dimensional disk vanish, takes the form

$$\begin{aligned} \dot v\cos\alpha - \dot\alpha v\sin\alpha - \omega_6 v\sin\alpha\cos\beta_1 + \omega_5 v\sin\alpha\sin\beta_1\cos\beta_2 - \omega_3 v\sin\alpha\sin\beta_1\sin\beta_2 \\ +\sigma\left(\omega_6^2+\omega_5^2+\omega_3^2\right) = -\frac{S}{m}, \end{aligned} \tag{7.3}$$

$$\begin{aligned} \dot v\sin\alpha\cos\beta_1 + \dot\alpha v\cos\alpha\cos\beta_1 - \dot\beta_1 v\sin\alpha\sin\beta_1 + \omega_6 v\cos\alpha - \omega_4 v\sin\alpha\sin\beta_1\cos\beta_2 \\ +\omega_2 v\sin\alpha\sin\beta_1\sin\beta_2 - \sigma(\omega_4\omega_5+\omega_2\omega_3) - \sigma\dot\omega_6 = 0, \end{aligned} \tag{7.4}$$

$$\begin{aligned} \dot v\sin\alpha\sin\beta_1\cos\beta_2 + \dot\alpha v\cos\alpha\sin\beta_1\cos\beta_2 + \dot\beta_1 v\sin\alpha\cos\beta_1\cos\beta_2 - \dot\beta_2 v\sin\alpha\sin\beta_1\sin\beta_2 \\ -\omega_5 v\cos\alpha + \omega_4 v\sin\alpha\cos\beta_1 - \omega_1 v\sin\alpha\sin\beta_1\sin\beta_2 - \sigma(-\omega_1\omega_2+\omega_4\omega_6) + \sigma\dot\omega_5 = 0, \end{aligned} \tag{7.5}$$

$$\begin{aligned} \dot v\sin\alpha\sin\beta_1\sin\beta_2 + \dot\alpha v\cos\alpha\sin\beta_1\sin\beta_2 + \dot\beta_1 v\sin\alpha\cos\beta_1\sin\beta_2 + \dot\beta_2 v\sin\alpha\sin\beta_1\cos\beta_2 \\ +\omega_3 v\cos\alpha - \omega_2 v\sin\alpha\cos\beta_1 + \omega_1 v\sin\alpha\sin\beta_1\cos\beta_2 + \sigma(\omega_2\omega_6+\omega_1\omega_5) - \sigma\dot\omega_3 = 0, \end{aligned} \tag{7.6}$$

where

$$S = s(\alpha)v^2, \quad \sigma = CD, \quad v > 0. \tag{7.7}$$

Further, the auxiliary matrix (6.11) for the calculation of the moment of the resistance force takes the form

$$\begin{pmatrix} 0 & x_{2N} & x_{3N} & x_{4N} \\ -S & 0 & 0 & 0 \end{pmatrix}; \tag{7.8}$$

then the part of dynamical equations that describes the motion of the body about the center of mass and corresponds to the Lie algebra so(4) takes the form

$$(\lambda_4+\lambda_3)\dot\omega_1 + (\lambda_3-\lambda_4)(\omega_3\omega_5+\omega_2\omega_4) = 0, \tag{7.9}$$

$$(\lambda_2+\lambda_4)\dot\omega_2 + (\lambda_2-\lambda_4)(\omega_3\omega_6-\omega_1\omega_4) = 0, \tag{7.10}$$

$$(\lambda_4+\lambda_1)\dot\omega_3 + (\lambda_4-\lambda_1)(\omega_2\omega_6+\omega_1\omega_5) = x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)v^2, \tag{7.11}$$

$$(\lambda_3+\lambda_2)\dot\omega_4 + (\lambda_2-\lambda_3)(\omega_5\omega_6+\omega_1\omega_2) = 0, \tag{7.12}$$

$$(\lambda_1+\lambda_3)\dot\omega_5 + (\lambda_3-\lambda_1)(\omega_4\omega_6-\omega_1\omega_3) = -x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)v^2, \tag{7.13}$$

$$(\lambda_1+\lambda_2)\dot\omega_6 + (\lambda_1-\lambda_2)(\omega_4\omega_5+\omega_2\omega_3) = x_{2N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)v^2. \tag{7.14}$$

Thus, the phase space of tenth-order system (7.3)–(7.6), (7.9)–(7.14) is the direct product of the four-dimensional manifold and the Lie algebra so(4):

$$\mathbb{R}^1 \times \mathbf{S}^3 \times \mathrm{so}(4). \tag{7.15}$$

Note that system (7.3)–(7.6), (7.9)–(7.14), due to the existing dynamical symmetry

$$I_2 = I_3 = I_4, \tag{7.16}$$

possesses the cyclic first integrals

$$\omega_1 \equiv \omega_1^0 = \text{const}, \quad \omega_2 \equiv \omega_2^0 = \text{const}, \quad \omega_4 \equiv \omega_4^0 = \text{const}\,. \tag{7.17}$$

In the sequel, we will consider the dynamics of the system on zero levels:

$$\omega_1^0 = \omega_2^0 = \omega_4^0 = 0. \tag{7.18}$$

If we consider a more general problem on the motion of a body under a tracing force $\mathbf{T}$ lying on the straight line $CD = Dx_1$ that provides throughout the motion the fulfillment of the condition

$$\mathbf{V}_C \equiv \text{const} \tag{7.19}$$

(here $\mathbf{V}_C$ is the velocity of the center of mass, see also [164–167]), then system (7.3)–(7.6), (7.9)–(7.14), equals zero instead of F_x, since a nonconservative couple of forces acts on the body:

$$T - s(\alpha)v^2 \equiv 0, \quad \sigma = DC. \tag{7.20}$$

For this, obviously, we must take the value of the tracing force T in the form

$$T = T_v(\alpha, \Omega) = s(\alpha)v^2, \quad \mathbf{T} \equiv -\mathbf{S}. \tag{7.21}$$

The case (7.21) of the choice of the value T of the tracing force is a particular case of the separation of an independent fifth-order subsystem after a certain transformation of the sixth-order system (7.3)–(7.6), (7.9)–(7.14).

Indeed, let the following condition for T hold:

$$T = T_v(\alpha, \beta_1, \beta_2, \Omega) = \sum_{\substack{i,j=0,\\ i\leq j}}^{4} \tau_{i,j}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)\Omega_i\Omega_j = T_1\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)v^2, \quad \Omega_0 = v. \tag{7.22}$$

Introduce the new quasi-velocities in the system. For this, we transform ω_3, ω_5, and ω_6 by a composition of two rotations:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \mathbf{T}_1(-\beta_1) \circ \mathbf{T}_3(-\beta_2) \begin{pmatrix} \omega_3 \\ \omega_5 \\ \omega_6 \end{pmatrix}, \tag{7.23}$$

where

$$\mathbf{T}_1(\beta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta_1 & -\sin\beta_1 \\ 0 & \sin\beta_1 & \cos\beta_1 \end{pmatrix}, \quad \mathbf{T}_3(\beta_2) = \begin{pmatrix} \cos\beta_2 & -\sin\beta_2 & 0 \\ \sin\beta_2 & \cos\beta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.24}$$

Thus, the following relations hold:

$$\begin{aligned} z_1 &= \omega_3\cos\beta_2 + \omega_5\sin\beta_2, \\ z_2 &= -\omega_3\cos\beta_1\sin\beta_2 + \omega_5\cos\beta_1\cos\beta_2 + \omega_6\sin\beta_1, \\ z_3 &= \omega_3\sin\beta_1\sin\beta_2 - \omega_5\sin\beta_1\cos\beta_2 + \omega_6\cos\beta_1. \end{aligned} \tag{7.25}$$

System (7.3)–(7.6), (7.9)–(7.14) in the cases (7.16)–(7.18) and (7.22) can be rewritten in the form

$$\dot{v} + \sigma\left(z_1^2 + z_2^2 + z_3^2\right)\cos\alpha - \sigma\frac{v^2}{2I_2}s(\alpha)\sin\alpha\cdot\Gamma_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = \frac{T_1\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)v^2 - s(\alpha)v^2}{m}\cos\alpha, \tag{7.26}$$

$$\dot{\alpha}v + z_3 v - \sigma\left(z_1^2 + z_2^2 + z_3^2\right)\sin\alpha - \sigma\frac{v^2}{2I_2}s(\alpha)\cos\alpha\cdot\Gamma_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = \frac{s(\alpha)v^2 - T_1\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)v^2}{m}\sin\alpha, \tag{7.27}$$

$$\dot{\beta}_1\sin\alpha - z_2\cos\alpha - \frac{\sigma v}{2I_2}s(\alpha)\cdot\Delta_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = 0, \tag{7.28}$$

$$\dot{\beta}_2\sin\alpha\sin\beta_1 + z_1\cos\alpha - \frac{\sigma v}{2I_2}s(\alpha)\cdot\Theta_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = 0, \tag{7.29}$$

$$\dot{\omega}_3 = \frac{v^2}{2I_2}x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha), \tag{7.30}$$

$$\dot{\omega}_5 = -\frac{v^2}{2I_2}x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha), \tag{7.31}$$

$$\dot{\omega}_6 = \frac{v^2}{2I_2}x_{2N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha). \tag{7.32}$$

Introducing the new dimensionless phase variables and differentiation by the formulas

$$z_k = n_1 v Z_k,\ k = 1, 2, 3, \quad \langle\dot{\ }\rangle = n_1 v\langle\ '\rangle,\ n_1 > 0,\ n_1 = \text{const}, \tag{7.33}$$

we reduce system (7.26)–(7.32) to the following form:

$$v' = v\Psi(\alpha, \beta_1, \beta_2, Z), \tag{7.34}$$

$$\alpha' = -Z_3 + \sigma n_1\left(Z_1^2 + Z_2^2 + Z_3^2\right)\sin\alpha + \frac{\sigma}{2I_2 n_1}s(\alpha)\cos\alpha\cdot\Gamma_v(\alpha, \beta_1, \beta_2, n_1 Z) - \frac{T_1(\alpha, \beta_1, \beta_2, n_1 Z) - s(\alpha)}{m n_1}\sin\alpha, \tag{7.35}$$

$$Z_3' = \frac{s(\alpha)}{2I_2 n_1^2}\cdot\Gamma_v(\alpha, \beta_1, \beta_2, n_1 Z) - \left(Z_1^2 + Z_2^2\right)\frac{\cos\alpha}{\sin\alpha} - \frac{\sigma}{2I_2 n_1}Z_2\frac{s(\alpha)}{\sin\alpha}\cdot\Delta_v(\alpha, \beta_1, \beta_2, n_1 Z) + \frac{\sigma}{2I_2 n_1}Z_1\frac{s(\alpha)}{\sin\alpha}\cdot\Theta_v(\alpha, \beta_1, \beta_2, n_1 Z) - Z_3\cdot\Psi(\alpha, \beta_1, \beta_2, Z), \tag{7.36}$$

$$
\begin{aligned}
Z_2' = & -\frac{s(\alpha)}{2I_2n_1^2}\cdot\Delta_v\left(\alpha,\beta_1,\beta_2,n_1Z\right) + Z_2Z_3\frac{\cos\alpha}{\sin\alpha} + Z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} \\
& + \frac{\sigma}{2I_2n_1}Z_3\frac{s(\alpha)}{\sin\alpha}\cdot\Delta_v\left(\alpha,\beta_1,\beta_2,n_1Z\right) - \frac{\sigma}{2I_2n_1}Z_1\frac{s(\alpha)}{\sin\alpha}\cdot\Theta_v\left(\alpha,\beta_1,\beta_2,n_1Z\right) \\
& - Z_2\cdot\Psi\left(\alpha,\beta_1,\beta_2,Z\right),
\end{aligned}
\tag{7.37}
$$

$$
\begin{aligned}
Z_1' = & \frac{s(\alpha)}{2I_2n_1^2}\cdot\Theta_v\left(\alpha,\beta_1,\beta_2,n_1Z\right) + Z_1Z_3\frac{\cos\alpha}{\sin\alpha} - Z_1Z_2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} \\
& - \frac{\sigma}{2I_2n_1}\frac{s(\alpha)}{\sin\alpha\sin\beta_1}\cdot\Theta_v\left(\alpha,\beta_1,\beta_2,n_1Z\right)\cdot\left[Z_3\sin\beta_1 - Z_2\cos\beta_1\right] \\
& - Z_1\cdot\Psi\left(\alpha,\beta_1,\beta_2,Z\right),
\end{aligned}
\tag{7.38}
$$

$$
\beta_1' = Z_2\frac{\cos\alpha}{\sin\alpha} + \frac{\sigma}{2I_2n_1}\frac{s(\alpha)}{\sin\alpha}\cdot\Delta_v\left(\alpha,\beta_1,\beta_2,n_1Z\right), \tag{7.39}
$$

$$
\beta_2' = -Z_1\frac{\cos\alpha}{\sin\alpha\sin\beta_1} + \frac{\sigma}{2I_2n_1}\frac{s(\alpha)}{\sin\alpha\sin\beta_1}\cdot\Theta_v\left(\alpha,\beta_1,\beta_2,n_1Z\right), \tag{7.40}
$$

where

$$
\begin{aligned}
\Psi(\alpha,\beta_1,\beta_2,Z) = & -\sigma n_1\left(Z_1^2+Z_2^2+Z_3^2\right)\cos\alpha + \frac{\sigma}{2I_2n_1}s(\alpha)\sin\alpha\cdot\Gamma_v\left(\alpha,\beta_1,\beta_2,n_1Z\right) \\
& + \frac{T_1\left(\alpha,\beta_1,\beta_2,n_1Z\right) - s(\alpha)}{mn_1}\cos\alpha,
\end{aligned}
\tag{7.41}
$$

$$
\begin{aligned}
\Gamma_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) = & \, x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\sin\beta_1\sin\beta_2 + x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\sin\beta_1\cos\beta_2 \\
& + x_{2N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\cos\beta_1,
\end{aligned}
\tag{7.42}
$$

$$
\begin{aligned}
\Delta_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) = & \, x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\cos\beta_1\sin\beta_2 + x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\cos\beta_1\cos\beta_2 \\
& - x_{2N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\sin\beta_1,
\end{aligned}
\tag{7.43}
$$

$$
\Theta_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) = x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\cos\beta_2 - x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\sin\beta_2. \tag{7.44}
$$

We see that the seventh-order system (7.34)–(7.40) contains an independent sixth-order subsystem (7.35)–(7.40), which can be separately examined on its own six-dimensional phase space.

In particular, this method of separation of an independent sixth-order subsystem can be also applied under condition (7.21).

Here and in what follows, the dependence on the group of variables $(\alpha,\beta_1,\beta_2,\Omega/v)$ is meant as the composite dependence on $(\alpha,\beta_1,\beta_2,z_1/v,z_2/v,z_3/v)$ (and further of $(\alpha,\beta_1,\beta_2,n_1Z_1,n_1Z_2,n_1Z_3)$) due to (7.25) and (7.33).

2. Case Where the Moment of a Nonconservative Force Is Independent of the Angular Velocity

2.1. Reduced system. Similarly to the choice of Chaplygin analytic functions (see [50, 51]), we take the dynamical functions s, x_{2N}, x_{3N}, and x_{4N} in the following form:

$$
\begin{gathered}
s(\alpha) = B\cos\alpha, \\
x_{2N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) = x_{2N0}(\alpha,\beta_1,\beta_2) = A\sin\alpha\cos\beta_1, \\
x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) = x_{3N0}(\alpha,\beta_1) = A\sin\alpha\sin\beta_1\cos\beta_2, \\
x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) = x_{4N0}(\alpha,\beta_1,\beta_2) = A\sin\alpha\sin\beta_1\sin\beta_2,
\end{gathered}
\tag{7.45}
$$

where $A, B > 0$ and $v \neq 0$. We see that in the system considered, the moment of nonconservative forces is independent of the angular velocity but depends only on the angles α, β_1, and β_2. The functions $\Gamma_v(\alpha,\beta_1,\beta_2,\Omega/v)$, $\Delta_v(\alpha,\beta_1,\beta_2,\Omega/v)$, and $\Theta_v(\alpha,\beta_1,\beta_2,\Omega/v)$ in system (7.34)–(7.40) have the following form:

$$
\Gamma_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) = A\sin\alpha, \quad \Delta_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) \equiv \Theta_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) \equiv 0. \tag{7.46}
$$

Then, due to conditions (7.19) and (7.45), the transformed dynamical part of the equations of motion (system (7.34)–(7.40)) becomes the analytic system

$$ v' = v\Psi(\alpha,\beta_1,\beta_2,Z), \tag{7.47} $$

$$ \alpha' = -Z_3 + b\left(Z_1^2+Z_2^2+Z_3^2\right)\sin\alpha + b\sin\alpha\cos^2\alpha, \tag{7.48} $$

$$ Z_3' = \sin\alpha\cos\alpha - \left(Z_1^2+Z_2^2\right)\frac{\cos\alpha}{\sin\alpha} + bZ_3\left(Z_1^2+Z_2^2+Z_3^2\right)\cos\alpha - bZ_3\sin^2\alpha\cos\alpha, \tag{7.49} $$

$$ Z_2' = Z_2Z_3\frac{\cos\alpha}{\sin\alpha} + Z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} + bZ_2\left(Z_1^2+Z_2^2+Z_3^2\right)\cos\alpha - bZ_2\sin^2\alpha\cos\alpha, \tag{7.50} $$

$$ Z_1' = Z_1Z_3\frac{\cos\alpha}{\sin\alpha} - Z_1Z_2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} + bZ_1\left(Z_1^2+Z_2^2+Z_3^2\right)\cos\alpha - bZ_1\sin^2\alpha\cos\alpha, \tag{7.51} $$

$$ \beta_1' = Z_2\frac{\cos\alpha}{\sin\alpha}, \tag{7.52} $$

$$ \beta_2' = -Z_1\frac{\cos\alpha}{\sin\alpha\sin\beta_1}, \tag{7.53} $$

where

$$ \Psi(\alpha,\beta_1,\beta_2,Z) = -b\left(Z_1^2+Z_2^2+Z_3^2\right)\cos\alpha + b\sin^2\alpha\cos\alpha $$

and the dimensionless parameter b and the constant n_1 are chosen as follows:

$$ b = \sigma n_0, \quad n_0^2 = \frac{AB}{2I_2}, \quad n_1 = n_0. \tag{7.54} $$

Thus, system (7.47)–(7.53) can be considered on its own seven-dimensional phase manifold

$$ W_1 = \mathbb{R}_+^1\{v\} \times T\mathbf{S}^3\Big\{Z_1,\ Z_2,\ Z_3,\ 0<\alpha<\pi,\ 0<\beta_1<\pi,\ 0\le\beta_2<2\pi\Big\}, \tag{7.55} $$

i.e., on the direct product of the number half-line and the tangent bundle of the three-dimensional sphere $\mathbf{S}^3\{0<\alpha<\pi,\ 0<\beta_1<\pi,\ 0\le\beta_2<2\pi\}$.

We see that the seven-dimensional system (7.47)–(7.53) contains the independent sixth-order system (7.48)–(7.53) on its own six-dimensional manifold.

For the complete integration of system (7.47)–(7.53) we need, in general, six independent first integrals. However, after the change of variables

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \to \begin{pmatrix} Z \\ Z_* \end{pmatrix}, \quad Z = \sqrt{Z_1^2 + Z_2^2}, \quad Z_* = Z_2/Z_1, \tag{7.56}$$

system (7.48)–(7.53) splits as follows:

$$\alpha' = -Z_3 + b\left(Z^2 + Z_3^2\right)\sin\alpha + b\sin\alpha\cos^2\alpha, \tag{7.57}$$

$$Z_3' = \sin\alpha\cos\alpha - Z^2\frac{\cos\alpha}{\sin\alpha} + bZ_3\left(Z^2 + Z_3^2\right)\cos\alpha - bZ_3\sin^2\alpha\cos\alpha, \tag{7.58}$$

$$Z' = ZZ_3\frac{\cos\alpha}{\sin\alpha} + bZ\left(Z^2 + Z_3^2\right)\cos\alpha - bZ\sin^2\alpha\cos\alpha, \tag{7.59}$$

$$Z_*' = (\pm)Z\sqrt{1 + Z_*^2}\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}, \tag{7.60}$$

$$\beta_1' = (\pm)\frac{ZZ_*}{\sqrt{1 + Z_*^2}}\frac{\cos\alpha}{\sin\alpha}, \tag{7.61}$$

$$\beta_2' = (\mp)\frac{Z}{\sqrt{1 + Z_*^2}}\frac{\cos\alpha}{\sin\alpha\sin\beta_1}. \tag{7.62}$$

We see that the sixth-order system also splits into independent subsystems of lower orders: system (7.57)–(7.59) of order 3 and system (7.60), (7.61) (after the change of the independent variable) of order 2. Thus, for the complete integrability of system (7.47), (7.57)–(7.62) it suffices to specify two independent first integrals of system (7.57)–(7.59), one first integral of system (7.60), (7.61), and two additional first integrals that "attach" Eqs. (7.62) and (7.47).

Note that system (7.57)–(7.59) can be considered on the tangent bundle $T\mathbf{S}^2$ of the two-dimensional sphere $\mathbf{S}^2$.

2.2. Complete list of first integrals. System (7.57)–(7.59) has the form of a system that appears in the dynamics of a three-dimensional (3D) rigid body in a nonconservative field.

Note that, by (7.19), the value of the velocity of the center of mass is a first integral of system (7.26)–(7.32) (under condition (7.21)); namely, the function of phase variables

$$\Psi_0(v, \alpha, \beta_1, \beta_2, z_1, z_2, z_3) = v^2 + \sigma^2\left(z_1^2 + z_2^2 + z_3^2\right) - 2\sigma z_3 v\sin\alpha = V_C^2 \tag{7.63}$$

is constant on phase trajectories of the system (here z_1, z_2, and z_3 are chosen due to (7.25)).

Due to a nondegenerate change of the independent variable (for $v \neq 0$), system (7.47), (7.57)–(7.62) also possesses an analytic integral, namely, the function of phase variables

$$\Psi_1(v, \alpha, \beta_1, \beta_2, Z, Z_*, Z_3) = v^2\left(1 + b^2\left(Z^2 + Z_3^2\right) - 2bZ_3\sin\alpha\right) = V_C^2 \tag{7.64}$$

is constant on phase trajectories of the system.

Equality (7.64) allows one to find the dependence of the velocity of the characteristic point of the rigid body (the center D of the disk) on the other phase variables without solving system (7.47), (7.57)–(7.62); namely, for $V_C \neq 0$ we have the relation

$$v^2 = \frac{V_C^2}{1 + b^2\left(Z^2 + Z_3^2\right) - 2bZ_3\sin\alpha}. \tag{7.65}$$

Since the phase space

$$W_2 = \mathbb{R}_+^1\{v\} \times T\mathbf{S}^3\Big\{Z,\ Z_*,\ Z_3,\ 0 < \alpha < \pi,\ 0 < \beta_1 < \pi,\ 0 \leq \beta_2 < 2\pi\Big\} \tag{7.66}$$

of system (7.47), (7.57)–(7.62) has dimension 7 and contains asymptotic limit sets, Eq. (7.64) defined a unique analytic (even continuous) first integral of system (7.47), (7.57)–(7.62) in the whole phase space (see [23, 24, 43, 53, 103, 105, 137, 152–154, 156, 213, 427, 438]).

We examine the existence of other (additional) first integrals of system (7.47), (7.57)–(7.62). Its phase space is stratified into surfaces

$$\Big\{(v, \alpha, \beta_1, \beta_2, Z, Z_*, Z_3) \in W_2 : V_C = \text{const}\Big\}; \tag{7.67}$$

dynamics on these surfaces is determined by the first integrals of system (7.47), (7.57)–(7.62).

First, to the independent third-order subsystem (7.57)–(7.59), we put in correspondence the nonautonomous second-order system

$$\begin{aligned}\frac{dZ_3}{d\alpha} &= \frac{\sin\alpha\cos\alpha + bZ_3\left(Z^2+Z_3^2\right)\cos\alpha - bZ_3\sin^2\alpha\cos\alpha - Z^2\cos\alpha/\sin\alpha}{-Z_3 + b\left(Z^2+Z_3^2\right)\sin\alpha + b\sin\alpha\cos^2\alpha},\\ \frac{dZ}{d\alpha} &= \frac{bZ\left(Z^2+Z_3^2\right)\cos\alpha - bZ\sin^2\alpha\cos\alpha + ZZ_3\cos\alpha/\sin\alpha}{-Z_3 + b\left(Z^2+Z_3^2\right)\sin\alpha + b\sin\alpha\cos^2\alpha}.\end{aligned} \tag{7.68}$$

Applying the substitution $\tau = \sin\alpha$, we rewrite system (7.68) in the algebraic form:

$$\begin{aligned}\frac{dZ_3}{d\tau} &= \frac{\tau + bZ_3\left(Z^2+Z_3^2\right) - bZ_3\tau^2 - Z^2/\tau}{-Z_3 + b\tau\left(1-\tau^2\right) + b\tau\left(Z^2+Z_3^2\right)},\\ \frac{dZ}{d\tau} &= \frac{bZ\left(Z^2+Z_3^2\right) - bZ\tau^2 + ZZ_3/\tau}{-Z_3 + b\tau\left(1-\tau^2\right) + b\tau\left(Z^2+Z_3^2\right)}.\end{aligned} \tag{7.69}$$

Further, introducing the homogeneous variables by the formulas

$$Z = u_1\tau, \quad Z_3 = u_2\tau, \tag{7.70}$$

we reduce system (7.69)) to the following form:

$$\begin{aligned}\tau\frac{du_2}{d\tau} + u_2 &= \frac{1 - bu_2\tau^2 + bu_2\left(u_1^2+u_2^2\right)\tau^2 - u_1^2}{-u_2 + b\tau^2\left(u_1^2+u_2^2\right) + b\left(1-\tau^2\right)},\\ \tau\frac{du_1}{d\tau} + u_1 &= \frac{bu_1\left(u_1^2+u_2^2\right)\tau^2 - bu_1\tau^2 + u_1u_2}{-u_2 + b\tau^2\left(u_1^2+u_2^2\right) + b\left(1-\tau^2\right)},\end{aligned} \tag{7.71}$$

which is equivalent to

$$\begin{aligned}\tau\frac{du_2}{d\tau} &= \frac{1 - bu_2 + u_2^2 - u_1^2}{-u_2 + b\tau^2\left(u_1^2+u_2^2\right) + b\left(1-\tau^2\right)},\\ \tau\frac{du_1}{d\tau} &= \frac{2u_1u_2 - bu_1}{-u_2 + b\tau^2\left(u_1^2+u_2^2\right) + b\left(1-\tau^2\right)}.\end{aligned} \tag{7.72}$$

To the second-order system (7.72), we put in correspondence the nonautonomous first-order equation

$$\frac{du_2}{du_1} = \frac{1 - bu_2 + u_2^2 - u_1^2}{2u_1u_2 - bu_1}, \tag{7.73}$$

which is easily transformed to the exact differential equation

$$d\left(\frac{u_2^2 + u_1^2 - bu_2 + 1}{u_1}\right) = 0. \tag{7.74}$$

Thus, Eq. (7.73) has the first integral

$$\frac{u_2^2 + u_1^2 - bu_2 + 1}{u_1} = C_1 = \text{const}, \tag{7.75}$$

which in the previous variables has the form

$$\frac{Z_3^2 + Z^2 - bZ_3 \sin\alpha + \sin^2\alpha}{Z \sin\alpha} = C_1 = \text{const}\,. \tag{7.76}$$

Remark 7.1. Consider system (7.57)–(7.59) with variable dissipation with zero mean (see [67, 68, 243, 260, 262, 265, 282–286, 291, 295, 309, 324, 333, 340, 384, 391, 404–408, 412, 413, 421, 437, 438]), which becomes conservative for $b = 0$:

$$\begin{aligned} \alpha' &= -Z_3, \\ Z_3' &= \sin\alpha\cos\alpha - Z^2\frac{\cos\alpha}{\sin\alpha}, \\ Z' &= ZZ_3\frac{\cos\alpha}{\sin\alpha}. \end{aligned} \tag{7.77}$$

This system possesses two analytic first integrals of the form

$$Z_3^2 + Z^2 + \sin^2\alpha = C_1^* = \text{const}, \tag{7.78}$$

$$Z\sin\alpha = C_2^* = \text{const}\,. \tag{7.79}$$

Obviously, the ratio of two first integrals (7.78) and (7.79) is also a first integral of system (7.77). However, for $b \neq 0$, none of the functions

$$Z_3^2 + Z^2 - bZ_3\sin\alpha + \sin^2\alpha \tag{7.80}$$

and (7.79) is a first integral of system (7.57)–(7.59), but their ratio is a first integral system (7.57)–(7.59) for any b.

Further, we find an additional first integral of the third-order system (7.57)–(7.59). First, we transform the invariant relation (7.75) for $u_1 \neq 0$ as follows:

$$\left(u_2 - \frac{b}{2}\right)^2 + \left(u_1 - \frac{C_1}{2}\right)^2 = \frac{b^2 + C_1^2}{4} - 1. \tag{7.81}$$

We see that the parameters of this invariant relation must satisfy the condition

$$b^2 + C_1^2 - 4 \geq 0, \tag{7.82}$$

and the phase space of system (7.57)–(7.59) is stratified into the family of surfaces defined by (7.81).

Thus, due to relation (7.75), the first equation of system (7.72) takes the form

$$\tau\frac{du_2}{d\tau} = \frac{1 - bu_2 + u_2^2 - U_1^2(C_1, u_2)}{-u_2 + b\left(1 - \tau^2\right) + b\tau^2\left(U_1^2(C_1, u_2) + u_2^2\right)}, \tag{7.83}$$

where

$$U_1(C_1, u_2) = \frac{1}{2}\left\{C_1 \pm \sqrt{C_1^2 - 4\left(u_2^2 - bu_2 + 1\right)}\right\} \tag{7.84}$$

and the integration constant C_1 is defined by condition (7.82), or the form of the Bernoulli equation:

$$\frac{d\tau}{du_2} = \frac{(b - u_2)\tau - b\tau^3\left(1 - U_1^2(C_1, u_2) - u_2^2\right)}{1 - bu_2 + u_2^2 - U_1^2(C_1, u_2)}. \tag{7.85}$$

Using (7.84), we can transform Eq. (7.85) to the form of a nonhomogeneous linear equation:

$$\frac{dp}{du_2} = \frac{2(u_2 - b)p + 2b\left(1 - U_1^2(C_1, u_2) - u_2^2\right)}{1 - bu_2 + u_2^2 - U_1^2(C_1, u_2)}, \quad p = \frac{1}{\tau^2}. \tag{7.86}$$

This means that we can find another transcendental first integral in the explicit form (i.e., in the form of a finite combination of quadratures). Herewith, the general solution of Eq. (7.86) depends on an arbitrary constant C_2. We omit complete calculations but note that the general solution of the homogeneous linear equation obtained from (7.86) in the particular case $b = C_1 = 2$ has the form

$$p = p_0(u_2) = C\left[\sqrt{1-(u_2-1)^2} \pm 1\right] \exp\left[\sqrt{\frac{1 \mp \sqrt{1-(u_2-1)^2}}{1 \pm \sqrt{1-(u_2-1)^2}}}\right], \quad C = \text{const}. \tag{7.87}$$

Remark 7.2. Formally, in the expression of the found first integral, we must substitute instead of C_1 the left-hand side of the first integral (7.75).

Then the obtained additional first integral has the following structure (similar to the transcendental first integral from planar dynamics):

$$\ln|\sin\alpha| + G_2\left(\sin\alpha, \frac{z_3}{\sin\alpha}, \frac{z}{\sin\alpha}\right) = C_2 = \text{const}. \tag{7.88}$$

Thus, for integration of the sixth-order system (7.57)–(7.62) we already have two independent first integrals. For the complete integration, it suffices to find one first integral for the (potentially separated) system (7.60), (7.61) and an additional first integral that "attaches" Eq. (7.62).

To find a first integral of the (potentially separated) system (7.60), (7.61), we put into correspondence the following nonautonomous first-order equation:

$$\frac{dZ_*}{d\beta_1} = \frac{1+Z_*^2}{Z_*}\frac{\cos\beta_1}{\sin\beta_1}. \tag{7.89}$$

After integration we obtain the required invariant relation

$$\frac{\sqrt{1+Z_*^2}}{\sin\beta_1} = C_3 = \text{const}; \tag{7.90}$$

in the variables Z_1 and Z_2 it has the form

$$\frac{\sqrt{Z_1^2+Z_2^2}}{Z_1\sin\beta_1} = C_3 = \text{const}. \tag{7.91}$$

Further, to find an additional first integral that "attaches" Eq. (7.62), to Eqs. (7.62) and (7.60) we put into correspondence the following nonautonomous equation:

$$\frac{dZ_*}{d\beta_2} = -\left(1+Z_*^2\right)\cos\beta_1. \tag{7.92}$$

Since, due to (7.90),

$$C_3\cos\beta_1 = \pm\sqrt{C_3^2-1-Z_*^2}, \tag{7.93}$$

we have

$$\frac{dZ_*}{d\beta_2} = \mp\frac{1}{C_3}\left(1+Z_*^2\right)\sqrt{C_3^2-1-Z_*^2}. \tag{7.94}$$

Integrating this relation, we obtain the following quadrature:

$$\mp(\beta_2+C_4) = \int\frac{C_3 dZ_*}{(1+Z_*^2)\sqrt{C_3^2-1-Z_*^2}}, \quad C_4 = \text{const}. \tag{7.95}$$

Another integration leads to the relation

$$\mp\tan(\beta_2+C_4) = \frac{C_3 Z_*}{\sqrt{C_3^2-1-Z_*^2}}, \quad C_4 = \text{const}. \tag{7.96}$$

In the variables Z_1 and Z_2, this invariant relation has the form

$$\mp\tan(\beta_2+C_4) = \frac{C_3 Z_2}{\sqrt{\left(C_3^2-1\right)Z_1^2-Z_2^2}}, \quad C_4 = \text{const}. \tag{7.97}$$

Finally, we have the following form of the additional first integral that "attaches" Eq. (7.62):

$$\arctan \frac{C_3 Z_*}{\sqrt{C_3^2 - 1 - Z_*^2}} \pm \beta_2 = C_4, \quad C_4 = \text{const}, \tag{7.98}$$

or

$$\arctan \frac{C_3 Z_2}{\sqrt{\left(C_3^2 - 1\right) Z_1^2 - Z_2^2}} \pm \beta_2 = C_4, \quad C_4 = \text{const}\,. \tag{7.99}$$

Thus, in the case considered the system of dynamical equations (7.3)–(7.6), (7.9)–(7.14) under condition (7.45) has 8 invariant relations: the analytic nonintegrable constraint of the form (7.19) corresponding to the analytic first integral (7.63), the cyclic first integrals of the form (7.17) and (7.18), the first integral of the form (7.76). Moreover, there exists a first integral that can be found from Eq. (7.86); it is a transcendental function of phase variables (in the sense of complex analysis). Finally, we have the transcendental first integrals of the form (7.90) (or (7.91)) and (7.98) (or (7.99)).

Theorem 7.1. *System* (7.3)–(7.6), (7.9)–(7.14) *under conditions* (7.19), (7.45), (7.18), *and* (7.17) *possesses* 8 *invariant relations* (*complete set*), *four of which are transcendental functions* (*from the point of view of complex analysis*). *Herewith, seven of these eight relations are expressed through finite combinations of elementary function.*

2.3. Topological analogies. We show that there exists another mechanical and topological analogy.

Theorem 7.2. *The first integral* (7.76) *of system* (7.3)–(7.6), (7.9)–(7.14) *under conditions* (7.19), (7.45), (7.18), *and* (7.17) *is constant on the phase trajectories of system* (6.74)–(6.79).

Proof. Indeed, the first integral (7.76) can be obtained by the change of coordinates by relation (7.75), whereas the first integral (6.95) can be obtained by the change of coordinates by relation (6.94). But relations (7.75) and (6.94) coincide. The theorem is proved. ☐

Thus, we have the following topological and mechanical analogies in the sense explained above:

(1) a motion of a free rigid body in a nonconservative field with a tracing force (under a nonintegrable constraint);
(2) a motion of a fixed physical pendulum in flowing medium (a nonconservative field);
(3) a rotation of a rigid body about the center of mass, which, in its turn, moves rectilinearly and uniformly in a nonconservative field.

On more general topological analogies, see also [253, 265, 267, 309, 340, 342].

3. Case Where the Moment of a Nonconservative Force Depends on the Angular Velocity

3.1. Introduction of the dependence on the angular velocity and the reduced system. In this section, we continue to study the dynamics of a four-dimensional rigid body in the four-dimensional space. The present section, similarly to the previous section, is devoted to the study of the motion in the case where the moment of forces depends on the tensor of angular velocity. Thus, we introduce this dependence similarly to the previous chapter. This also allows us to introduce this dependence for multi-dimensional bodies.

Let $x = (x_{1N}, x_{2N}, x_{3N}, x_{4N})$ be the coordinates of the application point N of the nonconservative force (influence of the medium) on the three-dimensional disk and $Q = (Q_1, Q_2, Q_3, Q_4)$ be the components of the force $\mathbf{S}$ of the influence of the medium independent of the tensor of angular velocity. We consider only linear dependence of the function $(x_{1N}, x_{2N}, x_{3N}, x_{4N})$ on the tensor of angular velocity since this introduction itself is not obvious (see [33, 34, 48, 49, 57–66, 120, 121, 169, 182, 203, 205, 249, 250, 261, 274–276, 440]).

We adopt the following dependence:

$$x = Q + R, \tag{7.100}$$

where $R = (R_1, R_2, R_3, R_4)$ is a vector-valued function containing the components of the tensor of angular velocity. The dependence of the functions R on the components of the tensor of angular velocity is gyroscopic:

$$R = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} = -\frac{1}{v} \begin{pmatrix} 0 & -\omega_6 & \omega_5 & -\omega_3 \\ \omega_6 & 0 & -\omega_4 & \omega_2 \\ -\omega_5 & \omega_4 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}, \tag{7.101}$$

where (h_1, h_2, h_3, h_4) are some positive parameters (cf. [322, 330, 331, 345, 393]).

Since $x_{1N} \equiv 0$, we have

$$x_{2N} = Q_2 - h_1 \frac{\omega_6}{v}, \quad x_{3N} = Q_3 + h_1 \frac{\omega_5}{v}, \quad x_{4N} = Q_4 - h_1 \frac{\omega_3}{v}. \tag{7.102}$$

Similarly to the choice of the Chaplygin analytic functions (see [50, 51]),

$$\begin{aligned} Q_2 &= A \sin\alpha \cos\beta_1, \\ Q_3 &= A \sin\alpha \sin\beta_1 \cos\beta_2, \\ Q_4 &= A \sin\alpha \sin\beta_1 \sin\beta_2, \end{aligned} \tag{7.103}$$

where $A > 0$, and we take the dynamical functions s, x_{2N}, x_{3N}, and x_{4N} in the following form:

$$\begin{aligned} s(\alpha) &= B \cos\alpha, \ \ B > 0, \\ x_{2N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) &= A \sin\alpha \cos\beta_1 - h \frac{\omega_6}{v}, \quad h = h_1 > 0, \ v \neq 0, \\ x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) &= A \sin\alpha \sin\beta_1 \cos\beta_2 + h \frac{\omega_5}{v}, \quad h = h_1 > 0, \ v \neq 0, \\ x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) &= A \sin\alpha \sin\beta_1 \sin\beta_2 - h \frac{\omega_3}{v}, \quad h = h_1 > 0, \ v \neq 0. \end{aligned} \tag{7.104}$$

This shows that in the problem considered, there is an additional damping (but accelerating in certain domains of the phase space) moment of a nonconservative force (i.e., there is a dependence of the moment on the components of the tensor of angular velocity). By the dynamical symmetry of the body, $h_2 = h_3 = h_4$.

The functions $\Gamma_v(\alpha, \beta_1, \beta_2, \Omega/v)$, $\Delta_v(\alpha, \beta_1, \beta_2, \Omega/v)$, and $\Theta_v(\alpha, \beta_1, \beta_2, \Omega/v)$ in system (7.35)–(7.40) have the following form:

$$\begin{aligned} \Gamma_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) &= A \sin\alpha - \frac{h}{v} z_3, \\ \Delta_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) &= \frac{h}{v} z_2, \\ \Theta_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) &= -\frac{h}{v} z_1. \end{aligned} \tag{7.105}$$

By conditions (7.19) and (7.104), the transformed dynamical part of the equations of motion (system (7.34)–(7.40)) becomes the following analytic system:

$$v' = v\Psi(\alpha, \beta_1, \beta_2, Z), \tag{7.106}$$

$$\alpha' = -Z_3 + b\left(Z_1^2 + Z_2^2 + Z_3^2\right)\sin\alpha + b\sin\alpha\cos^2\alpha - bH_1Z_3\cos^2\alpha, \tag{7.107}$$

$$\begin{aligned} Z_3' = \sin\alpha\cos\alpha - (1 + bH_1)\left(Z_1^2 + Z_2^2\right)\frac{\cos\alpha}{\sin\alpha} + bZ_3\left(Z_1^2 + Z_2^2 + Z_3^2\right)\cos\alpha - bZ_3\sin^2\alpha\cos\alpha \\ + bH_1Z_3^2\sin\alpha\cos\alpha - H_1Z_3\cos\alpha, \end{aligned} \tag{7.108}$$

$$\begin{aligned} Z_2' = (1 + bH_1)Z_2Z_3\frac{\cos\alpha}{\sin\alpha} + (1 + bH_1)Z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} + bZ_2\left(Z_1^2 + Z_2^2 + Z_3^2\right)\cos\alpha \\ - bZ_2\sin^2\alpha\cos\alpha + bH_1Z_2Z_3\sin\alpha\cos\alpha - H_1Z_2\cos\alpha, \end{aligned} \tag{7.109}$$

$$\begin{aligned} Z_1' = (1 + bH_1)Z_1Z_3\frac{\cos\alpha}{\sin\alpha} - (1 + bH_1)Z_1Z_2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} + bZ_1\left(Z_1^2 + Z_2^2 + Z_3^2\right)\cos\alpha \\ - bZ_1\sin^2\alpha\cos\alpha + bH_1Z_1Z_3\sin\alpha\cos\alpha - H_1Z_1\cos\alpha, \end{aligned} \tag{7.110}$$

$$\beta_1' = (1 + bH_1)Z_2\frac{\cos\alpha}{\sin\alpha}, \tag{7.111}$$

$$\beta_2' = -(1 + bH_1)Z_1\frac{\cos\alpha}{\sin\alpha\sin\beta_1}, \tag{7.112}$$

where

$$\Psi(\alpha, \beta_1, \beta_2, Z) = -b\left(Z_1^2 + Z_2^2 + Z_3^2\right)\cos\alpha + b\sin^2\alpha\cos\alpha - bH_1Z_3\sin\alpha\cos\alpha;$$

as above, the dimensionless parameters b and H_1 and the constant n_1 are chosen as follows:

$$b = \sigma n_0, \quad n_0^2 = \frac{AB}{2I_2}, \quad H_1 = \frac{Bh}{2I_2n_0}, \quad n_1 = n_0. \tag{7.113}$$

Thus, system (7.106)–(7.112) can be considered on its seven-dimensional phase manifold

$$W_1 = \mathbb{R}_+^1\{v\} \times T\mathbf{S}^3\Big\{Z_1,\ Z_2,\ Z_3,\ 0 < \alpha < \pi,\ 0 < \beta_1 < \pi,\ 0 \leq \beta_2 < 2\pi\Big\}, \tag{7.114}$$

i.e., on the direct product of the number half-line and the tangent bundle of the three-dimensional sphere $\mathbf{S}^3\{0 < \alpha < \pi,\ 0 < \beta_1 < \pi,\ 0 \leq \beta_2 < 2\pi\}$.

We see that the seventh-order system (7.106)–(7.112) contains the independent sixth-order system (7.107)–(7.112) on its own six-dimensional manifold.

For the complete integration of system (7.106)–(7.112), in general, we need six independent first integrals. However, after the change of variables

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \to \begin{pmatrix} Z \\ Z_* \end{pmatrix}, \quad Z = \sqrt{Z_1^2 + Z_2^2}, \quad Z_* = Z_2/Z_1, \tag{7.115}$$

system (7.107)–(7.112) splits as follows:

$$\alpha' = -Z_3 + b\left(Z^2 + Z_3^2\right)\sin\alpha + b\sin\alpha\cos^2\alpha - bH_1Z_3\cos^2\alpha, \tag{7.116}$$

$$\begin{aligned} Z_3' = \sin\alpha\cos\alpha - (1 + bH_1)Z^2\frac{\cos\alpha}{\sin\alpha} + bZ_3\left(Z^2 + Z_3^2\right)\cos\alpha - bZ_3\sin^2\alpha\cos\alpha \\ + bH_1Z_3^2\sin\alpha\cos\alpha - H_1Z_3\cos\alpha, \end{aligned} \tag{7.117}$$

$$Z' = (1 + bH_1)\, ZZ_3 \frac{\cos\alpha}{\sin\alpha} + bZ\left(Z^2 + Z_3^2\right)\cos\alpha - bZ\sin^2\alpha\cos\alpha + bH_1 ZZ_3\sin\alpha\cos\alpha - H_1 Z\cos\alpha, \tag{7.118}$$

$$Z'_* = (\pm)\,(1 + bH_1)\, Z\sqrt{1 + Z_*^2}\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}, \tag{7.119}$$

$$\beta'_1 = (\pm)\,(1 + bH_1)\,\frac{ZZ_*}{\sqrt{1 + Z_*^2}}\frac{\cos\alpha}{\sin\alpha}, \tag{7.120}$$

$$\beta'_2 = (\mp)\,(1 + bH_1)\,\frac{Z}{\sqrt{1 + Z_*^2}}\frac{\cos\alpha}{\sin\alpha\sin\beta_1}. \tag{7.121}$$

We see that the sixth-order system splits into independent subsystems of lower orders: system (7.116)–(7.118) of order 3 and system (7.119), (7.120) (after the change of the independent variable) of order 2. This, for the complete integrability of system (7.106), (7.116)–(7.121), it suffices to specify two independent first integrals of system (7.116)–(7.118), one first integral of system (7.119), (7.120), and two additional first integrals that "attach" Eqs. (7.121) and (7.106).

Note that system (7.116)–(7.118) can be considered on the tangent bundle $T\mathbf{S}^2$ of the two-dimensional sphere $\mathbf{S}^2$.

3.2. Complete list of first integrals. System (7.116)–(7.118) has the form of a system of equations that appears in the dynamics of a three-dimensional (3D) rigid body in a nonconservative field.

Note that, by (7.19), the value of the velocity of the center of mass is a first integral of system (7.26)–(7.32) (under condition (7.21)); namely, the function of phase variables

$$\Psi_0(v, \alpha, \beta_1, \beta_2, z_1, z_2, z_3) = v^2 + \sigma^2\left(z_1^2 + z_2^2 + z_3^2\right) - 2\sigma z_3 v\sin\alpha = V_C^2 \tag{7.122}$$

is constant on phase trajectories of this system (the values of z_1, z_2, and z_3 are taken due to (7.25)).

Due to the nondegenerate change of the independent variable (for $v \neq 0$), system (7.106), (7.116)–(7.121) also possesses an analytic integral, namely, the function of phase variables

$$\Psi_1(v, \alpha, \beta_1, \beta_2, Z, Z_*, Z_3) = v^2\left(1 + b^2\left(Z^2 + Z_3^2\right) - 2bZ_3\sin\alpha\right) = V_C^2 \tag{7.123}$$

is constant on phase trajectories of this system.

Equality (7.123) allows one to find the dependence of the velocity of the characteristic point of the rigid body (the cemter D of the disk) on other phase variable without solving system (7.106), (7.116)–(7.121); namely, for $V_C \neq 0$ we have

$$v^2 = \frac{V_C^2}{1 + b^2\left(Z^2 + Z_3^2\right) - 2bZ_3\sin\alpha}. \tag{7.124}$$

Since the phase space

$$W_2 = \mathbb{R}_+^1\{v\} \times T\mathbf{S}^3\Big\{Z,\ Z_*,\ Z_3,\ 0 < \alpha < \pi,\ 0 < \beta_1 < \pi,\ 0 \leq \beta_2 < 2\pi\Big\} \tag{7.125}$$

of system (7.106), (7.116)–(7.121) has dimension 7 and contains asymptotic limit sets, we see that Eq. (7.123) determines the unique analytic (even continuous) first integral of system (7.106), (7.116)–(7.121) on the whole phase space (cf. [23, 24, 43, 53, 103, 105, 137, 152–154, 156, 213, 427, 438]).

We examine the existence of other (additional) first integrals of system (7.106), (7.116)–(7.121). Its phase space is stratified into surfaces

$$\Big\{(v, \alpha, \beta_1, \beta_2, Z, Z_*, Z_3) \in W_2 : V_C = \text{const}\Big\}; \tag{7.126}$$

the dynamics on these surfaces is determined by the first integrals of system (7.106), (7.116)–(7.121).

First, to the independent third-order subsystem (7.116)–(7.118), we put in correspondence the nonautonomous second-order system

$$\begin{aligned}
\frac{dZ_3}{d\alpha} &= \frac{R_2(\alpha, Z, Z_3)}{-Z_3 + b\left(Z^2 + Z_3^2\right)\sin\alpha + b\sin\alpha\cos^2\alpha - bH_1 Z_3\cos^2\alpha},\\
\frac{dZ}{d\alpha} &= \frac{R_1(\alpha, Z, Z_3)}{-Z_3 + b\left(Z^2 + Z_3^2\right)\sin\alpha + b\sin\alpha\cos^2\alpha - bH_1 Z_3\cos^2\alpha},\\
R_2(\alpha, Z, Z_3) &= \sin\alpha\cos\alpha + bZ_3\left(Z^2 + Z_3^2\right)\cos\alpha - bZ_3\sin^2\alpha\cos\alpha\\
&\quad - (1 + bH_1)Z^2\frac{\cos\alpha}{\sin\alpha} + bH_1 Z_3^2\sin\alpha\cos\alpha - H_1 Z_3\cos\alpha,\\
R_1(\alpha, Z, Z_3) &= bZ\left(Z^2 + Z_3^2\right)\cos\alpha - bZ\sin^2\alpha\cos\alpha\\
&\quad + (1 + bH_1)ZZ_3\frac{\cos\alpha}{\sin\alpha} + bH_1 ZZ_3\sin\alpha\cos\alpha - H_1 Z\cos\alpha.
\end{aligned} \tag{7.127}$$

Using the substitution $\tau = \sin\alpha$, we rewrite system (7.127) in the algebraic form

$$\begin{aligned}
\frac{dZ_3}{d\tau} &= \frac{\tau + bZ_3\left(Z^2 + Z_3^2\right) - bZ_3\tau^2 - (1 + bH_1)Z^2/\tau + bH_1 Z_3^2\tau - H_1 Z_3}{-Z_3 + b\tau\left(1 - \tau^2\right) + b\tau\left(Z^2 + Z_3^2\right) - bH_1 Z_3\left(1 - \tau^2\right)},\\
\frac{dZ}{d\tau} &= \frac{bZ\left(Z^2 + Z_3^2\right) - bZ_1\tau^2 + (1 + bH_1)ZZ_3/\tau + bH_1 ZZ_3\tau - H_1 Z}{-Z_3 + b\tau\left(1 - \tau^2\right) + b\tau\left(Z^2 + Z_3^2\right) - bH_1 Z_3\left(1 - \tau^2\right)}.
\end{aligned} \tag{7.128}$$

Further, introducing the homogeneous variables by the formulas

$$Z = u_1\tau, \quad Z_3 = u_2\tau, \tag{7.129}$$

we transform system (7.128) to the following form:

$$\begin{aligned}
\tau\frac{du_2}{d\tau} + u_2 &= \frac{1 - bu_2\tau^2 + bu_2\left(u_1^2 + u_2^2\right)\tau^2 - (1 + bH_1)u_1^2 - H_1 u_2 + bH_1 u_2^2\tau^2}{-u_2 + b\tau^2\left(u_1^2 + u_2^2\right) + b\left(1 - \tau^2\right) - bH_1 u_2\left(1 - \tau^2\right)},\\
\tau\frac{du_1}{d\tau} + u_1 &= \frac{bu_1\left(u_1^2 + u_2^2\right)\tau^2 - bu_1\tau^2 + (1 + bH_1)u_1u_2 - H_1 u_1 + bH_1 u_1u_2}{-u_2 + b\tau^2\left(u_1^2 + u_2^2\right) + b\left(1 - \tau^2\right) - bH_1 u_2\left(1 - \tau^2\right)},
\end{aligned} \tag{7.130}$$

which is equivalent to

$$\begin{aligned}
\tau\frac{du_2}{d\tau} &= \frac{1 - (b + H_1)u_2 + (1 + bH_1)u_2^2 - (1 + bH_1)u_1^2}{-u_2 + b\tau^2\left(u_1^2 + u_2^2\right) + b\left(1 - \tau^2\right) - bH_1 u_2\left(1 - \tau^2\right)},\\
\tau\frac{du_1}{d\tau} &= \frac{2(1 + bH_1)u_1u_2 - (b + H_1)u_1}{-u_2 + b\tau^2\left(u_1^2 + u_2^2\right) + b\left(1 - \tau^2\right) - bH_1 u_2\left(1 - \tau^2\right)}.
\end{aligned} \tag{7.131}$$

To the second-order system (7.131), we put in correspondence the first-order nonautonomous equation

$$\frac{du_2}{du_1} = \frac{1 - (b + H_1)u_2 + (1 + bH_1)u_2^2 - (1 + bH_1)u_1^2}{2(1 + bH_1)u_1u_2 - (b + H_1)u_1}, \tag{7.132}$$

which can be easily transformed to the complete differential

$$d\left(\frac{(1 + bH_1)u_2^2 + (1 + bH_1)u_1^2 - (b + H_1)u_2 + 1}{u_1}\right) = 0. \tag{7.133}$$

Therefore, Eq. (7.132) has the first integral

$$\frac{(1 + bH_1)u_2^2 + (1 + bH_1)u_1^2 - (b + H_1)u_2 + 1}{u_1} = C_1 = \text{const}, \tag{7.134}$$

which in the previous variables has the form

$$\frac{(1+bH_1)Z_3^2+(1+bH_1)Z^2-(b+H_1)Z_3\sin\alpha+\sin^2\alpha}{Z\sin\alpha}=C_1=\text{const}\,. \tag{7.135}$$

Remark 7.3. Consider system (7.116)–(7.118) with variable dissipation with zero mean (see [67, 68, 243, 260, 262, 265, 282–286, 291, 295, 309, 324, 333, 340, 384, 391, 404–408, 412, 413, 421, 437, 438]), which becomes conservative for $b=H_1$:

$$\begin{aligned}
\alpha' &= -Z_3+b\left(Z^2+Z_3^2\right)\sin\alpha+b\sin\alpha\cos^2\alpha-b^2Z_3\cos^2\alpha,\\
Z_3' &= \sin\alpha\cos\alpha-\left(1+b^2\right)Z^2\frac{\cos\alpha}{\sin\alpha}+bZ_3\left(Z^2+Z_3^2\right)\cos\alpha-bZ_3\sin^2\alpha\cos\alpha\\
&\quad +b^2Z_3^2\sin\alpha\cos\alpha-bZ_3\cos\alpha,\\
Z' &= \left(1+b^2\right)ZZ_2\frac{\cos\alpha}{\sin\alpha}+bZ\left(Z^2+Z_3^2\right)\cos\alpha-bZ\sin^2\alpha\cos\alpha+b^2ZZ_3\sin\alpha\cos\alpha-bZ\cos\alpha.
\end{aligned} \tag{7.136}$$

It possesses two analytic first integrals

$$\left(1+b^2\right)\left(Z_3^2+Z^2\right)-2bZ_3\sin\alpha+\sin^2\alpha=C_1^*=\text{const}, \tag{7.137}$$

$$Z\sin\alpha=C_2^*=\text{const}\,. \tag{7.138}$$

Obviously, the ratio of two first integrals (7.137) and (7.138) is also a first integral of system (7.136). However, for $b\neq H_1$, none of the functions

$$(1+bH_1)\left(Z_3^2+Z^2\right)-(b+H_1)Z_3\sin\alpha+\sin^2\alpha \tag{7.139}$$

and (7.138) is a first integral of system (7.116)–(7.118), but their ratio is a first integral of system (7.116)–(7.118) for all b and H_1.

We find the explicit form of an additional first integral of the third-order system (7.116)–(7.118). For this, we transform the invariant relation (7.134) for $u_1\neq 0$ as follows:

$$\left(u_2-\frac{b+H_1}{2(1+bH_1)}\right)^2+\left(u_1-\frac{C_1}{2(1+bH_1)}\right)^2=\frac{(b-H_1)^2+C_1^2-4}{4(1+bH_1)^2}. \tag{7.140}$$

We see that the parameters of this invariant relation must satisfy the condition

$$(b-H_1)^2+C_1^2-4\geq 0, \tag{7.141}$$

and the phase space of system (7.116)–(7.118) is stratified into the family of surfaces determined by Eq. (7.140).

Thus, by relation (7.134), the first equation of system (7.131) has the form

$$\tau\frac{du_2}{d\tau}=\frac{1-(b+H_1)u_2+(1+bH_1)u_2^2-(1+bH_1)U_1^2(C_1,u_2)}{-u_2+b\left(1-\tau^2\right)+b\tau^2\left(U_1^2(C_1,u_2)+u_2^2\right)-bH_1u_2\left(1-\tau^2\right)}, \tag{7.142}$$

where

$$U_1(C_1,u_2)=\frac{1}{2}\left\{C_1\pm\sqrt{C_1^2-4(1+bH_1)\left(1-(b+H_1)u_2+(1+bH_1)u_2^2\right)}\right\} \tag{7.143}$$

and the integration constant C_1 is defined by condition (7.141), or the form of the Bernoulli equation:

$$\frac{d\tau}{du_2}=\frac{(b-(1+bH_1)u_2)\tau-b\tau^3\left(1-U_1^2(C_1,u_2)-u_2^2-H_1u_2\right)}{1-(b+H_1)u_2+(1+bH_1)u_2^2-(1+bH_1)U_1^2(C_1,u_2)}. \tag{7.144}$$

Using (7.143), we can easily transform Eq. (7.144) to the nonhomogeneous linear equation

$$\frac{dp}{du_2}=\frac{2((1+bH_1)u_2-b)p+2b\left(1-H_1u_2-u_2^2-U_1^2(C_1,u_2)\right)}{1-(b+H_1)u_2+(1+bH_1)u_2^2-(1+bH_1)U_1^2(C_1,u_2)},\quad p=\frac{1}{\tau^2}. \tag{7.145}$$

This means that there exists another transcendental first integral in explicit form (i.e., through a finite combination of quadratures). Herewith, the general solution of Eq. (7.145) depends on an arbitrary constant C_2. We omit complete calculations but note that the general solution of the homogeneous linear equation obtained from (7.145), in the particular case

$$|b - H_1| = 2, \quad C_1 = \frac{1 - A_1^4}{1 + A_1^4}, \quad A_1 = \frac{1}{2}(b + H_1),$$

has the following solution:

$$p = p_0(u_2) = C[1 - A_1 u_2]^{2/(1+A_1^4)} \left| \frac{\sqrt{C_1^2 - 4A_1^2(1 - A_1 u_2)^2} \pm C_1}{\sqrt{C_1^2 - 4A_1^2(1 - A_1 u_2)^2} \mp C_1} \right|^{\pm A_1^4/(1+A_1^4)}$$
$$\times \exp \frac{2(A_1 - b)}{\left(1 + A_1^4\right) A_1(A_1 u_2 - 1)}, \quad C = \text{const}. \quad (7.146)$$

Remark 7.4. Formally, in the expression of the found first integral, we must substitute instead of C_1 the left-hand side of the first integral (7.134).

Then the additional first integral obtained has the following structure (similar to the transcendental first integral from the planar dynamics):

$$\ln|\sin\alpha| + G_2\left(\sin\alpha, \frac{Z_3}{\sin\alpha}, \frac{Z}{\sin\alpha}\right) = C_2 = \text{const}. \quad (7.147)$$

Thus, for integration of the sixth-order system (7.116)–(7.121) we already have two independent first integrals. For the complete integrability, as was noted above, it suffices to find one first integral for the (potentially separated) system (7.119), (7.120) and an additional first integral that "attaches" Eq. (7.121).

To find the first integral of the (potentially separated) system (7.119), (7.120), we put in correspondence the following nonautonomous first-order equation:

$$\frac{dZ_*}{d\beta_1} = \frac{1 + Z_*^2}{Z_*} \frac{\cos\beta_1}{\sin\beta_1}. \quad (7.148)$$

After integration, this leads to the required invariant relation

$$\frac{\sqrt{1 + Z_*^2}}{\sin\beta_1} = C_3 = \text{const}, \quad (7.149)$$

which in the variables Z_1 and Z_2 has the form

$$\frac{\sqrt{Z_1^2 + Z_2^2}}{Z_1 \sin\beta_1} = C_3 = \text{const}. \quad (7.150)$$

Further, to find an additional first integral that "attaches" Eq. (7.121), to Eqs. (7.121) and (7.119) we put in correspondence the following nonautonomous equation:

$$\frac{dZ_*}{d\beta_2} = -\left(1 + Z_*^2\right)\cos\beta_1. \quad (7.151)$$

Since, by (7.149),

$$C_3 \cos\beta_1 = \pm\sqrt{C_3^2 - 1 - Z_*^2}, \quad (7.152)$$

we have

$$\frac{dZ_*}{d\beta_2} = \mp\frac{1}{C_3}\left(1 + Z_*^2\right)\sqrt{C_3^2 - 1 - Z_*^2}. \quad (7.153)$$

Integrating this relation, we obtain the following quadrature:

$$\mp(\beta_2 + C_4) = \int \frac{C_3 dZ_*}{(1 + Z_*^2)\sqrt{C_3^2 - 1 - Z_*^2}}, \ C_4 = \text{const}\,. \tag{7.154}$$

Another integration leads to the relation

$$\mp\tan(\beta_2 + C_4) = \frac{C_3 Z_*}{\sqrt{C_3^2 - 1 - Z_*^2}}, \quad C_4 = \text{const}\,. \tag{7.155}$$

In the variables Z_1 and Z_2, this invariant relation becomes

$$\mp\tan(\beta_2 + C_4) = \frac{C_3 Z_2}{\sqrt{\left(C_3^2 - 1\right) Z_1^2 - Z_2^2}}, \quad C_4 = \text{const}\,. \tag{7.156}$$

Finally, we have the additional first integral that “attaches” Eq. (7.121):

$$\arctan \frac{C_3 Z_*}{\sqrt{C_3^2 - 1 - Z_*^2}} \pm \beta_2 = C_4, \quad C_4 = \text{const} \tag{7.157}$$

or

$$\arctan \frac{C_3 Z_2}{\sqrt{\left(C_3^2 - 1\right) Z_1^2 - Z_2^2}} \pm \beta_2 = C_4, \quad C_4 = \text{const}\,. \tag{7.158}$$

Thus, in the case considered, the system of dynamical equations (7.3)–(7.6), (7.9)–(7.14) under condition (7.104) has 8 invariant relations: the analytic nonintegrable constraint of the form (7.19) corresponding to the analytic first integral (7.122), the cyclic first integrals of the form (7.17) and (7.18), the first integral of the form (7.135); moreover, there is a first integral that can be found from Eq. (7.145) (it is a transcendental function of phase variables in the sense of complex analysis), and, finally, transcendental first integrals of the form (7.149) (or (7.150)) and (7.157) (or (7.158)).

Theorem 7.3. *System* (7.3)–(7.6), (7.9)–(7.14) *under conditions* (7.19), (7.104), (7.18), *and* (7.17) *possesses* 8 *invariant relations (complete set), four of which are transcendental functions (from the point of view of complex analysis). Herewith, at least seven of these eight relations are expressed through finite combinations of elementary functions.*

3.3. Topological analogies. We show that there exists another mechanical and topological analogy.

Theorem 7.4. *The first integral* (7.135) *of system* (7.3)–*7.6,* (7.9)–(7.14) *under conditions* (7.19), (7.104), (7.18), *and* (7.17) *is constant on phase trajectories of system* (6.145)–(6.150).

Proof. Indeed, the first integral (7.135) can be obtained by the change of coordinates by relation (7.134), whereas the first integral (6.166) can be obtained by the change of coordinates by relation (6.165). But relations (7.134) and (6.165) coincides. The theorem is proved. □

Thus, we have the following topological and mechanical analogies in the sense explained above:

(1) a motion of a free rigid body in a nonconservative field with a tracing force (under a nonintegrable constraint);
(2) a motion of a fixed physical pendulum in a flowing medium (nonconservative field);
(3) a rotation of a rigid body about the center of mass that moves rectilinearly and uniformly in a nonconservative field.

On more general topological analogies, see also [253, 265, 267, 309, 340, 342].

Chapter 8

CASES OF INTEGRABILITY CORRESPONDING TO THE MOTION OF A RIGID BODY IN THE FOUR-DIMENSIONAL SPACE, III

In this chapter, we systematize results, both new and obtained earlier, concerning the study of equations of motion of a dynamically symmetric four-dimensional (4D) rigid body in a field of nonconservative forces in the case of a special dynamical symmetry. These equations are taken from the dynamics of realistic rigid bodies of lesser dimension that interact with a resisting medium by laws of jet flow when the body is subjected to a nonconservative tracing force such that the magnitude of the velocity of a certain typical point of the body and another phase variable remain constant throughout the motion; this means that the system possesses a a nonintegrable servo constraint (see [1, 64, 70, 72, 119–121, 157, 164–167, 182, 184, 191, 198, 237, 242, 346, 351, 352, 390]).

Earlier (see [164–167]), the author has already proved the complete integrability of the equations of a plane-parallel motion of a body in a resisting medium under the jet flow conditions when the system of dynamical equations possesses a first integral, which is a transcendental (in the sense of the theory of functions of a complex variable) function of quasi-velocities having essential singularities. It was assumed that the interaction of the medium with the body is concentrated on a part of the surface of the body that has the form of a (one-dimensional) plate.

In the sequel (see [193, 196, 202, 204, 208, 209, 218, 232, 241]), the planar problem was generalized to the spatial (three-dimensional) case, where the system of dynamical equations possesses a complete set of transcendental first integrals. In this case, it was assumed that the interaction of the medium with the body is concentrated on the part of the surface of the body that has the form of a planar (two-dimensional) disk.

In this chapter, we discuss results, both new and obtained earlier, concerning the case where the interaction of the medium with the body is concentrated on the part of the surface of the body that has the form of a three-dimensional disk and the force acts in the direction perpendicular to the disk. We systematize these results and formulate them in the invariant form. We also introduce the additional dependence of the moment of a nonconservative force on the angular velocity; this dependence can be generalized to the motion in higher-dimensional spaces.

1. General Problem on the Motion under a Tracing Force

Consider the motion of a homogeneous, dynamically symmetric (case (6.1)), rigid body with front end face (a two-dimensional disk interacting with a medium that fills the four-dimensional space) in the field of a resistance force $\mathbf{S}$ under the quasi-stationarity conditions (see [28, 50, 51, 62–66, 98, 112, 119–121, 160–169, 171, 431, 432].

Let $(v, \alpha, \beta_2, \beta_1)$ be the coordinates of the velocity vector of a certain typical point D of a rigid body (namely, D be the center of two-dimensional disk) such that α is the angle between the vector $\mathbf{v}_D$ and the plane Dx_1x_2, β_2 be the angle measured in the plane Dx_1x_2 between the velocity vector $\mathbf{v}_D$ and its projection on the plane Dx_1x_2, and let β_1 be the angle measured in the plane Dx_3x_4 between the velocity vector $\mathbf{v}_D$ and its projection on the plane Dx_3x_4. Let

$$\Omega = \begin{pmatrix} 0 & -\omega_6 & \omega_5 & -\omega_3 \\ \omega_6 & 0 & -\omega_4 & \omega_2 \\ -\omega_5 & \omega_4 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

be the tensor of angular velocity of the body, let $Dx_1x_2x_3x_4$ be the coordinate system attached to the body such that the straight line CD lies in the plane Dx_1x_2 (here C is the center of mass of the body), and the axes Dx_3 and Dx_4 lie in the plane of the disk; moreover, let I_1, $I_2 = I_1$, I_3, $I_4 = I_3$, and m be the characteristics of inertia and mass.

We adopt the following expansions in the projections to the axes of the coordinate system $Dx_1x_2x_3x_4$:

$$\begin{gathered}\mathbf{DC} = \{\sigma\sin\gamma,\ -\sigma\cos\gamma,\ 0,\ 0\},\\ \mathbf{v}_D = \Big\{v\cos\alpha\sin\beta_2,\ v\cos\alpha\cos\beta_2,\ v\sin\alpha\cos\beta_1,\ v\sin\alpha\sin\beta_1\Big\}.\end{gathered} \tag{8.1}$$

In the case (6.2), the expansion is also valid for the function of the action of the interaction to the four-dimensional body:

$$\mathbf{S} = \{S_1, S_2, 0, 0\} \tag{8.2}$$

and

$$S_1 = S\sin\gamma, \quad S_2 = -S\cos\gamma, \quad \gamma = \text{const}, \tag{8.3}$$

i.e., in this case $\mathbf{F} = \mathbf{S}$, and the angle γ is measured in the plane Dx_1x_2.

Then the part of dynamical equations of the body motion (including also the case of Chaplygin analytical functions [50, 51], see below) that describe the motion of the center of mass and correspond to the space $\mathbf{R}^4$ where tangent forces to the three-dimensional disk vanish, has the form

$$\begin{aligned}&\dot v\cos\alpha\sin\beta_2 - \dot\alpha v\sin\alpha\sin\beta_2 + \dot\beta_2 v\cos\alpha\cos\beta_2\\ &\qquad - \omega_6 v\cos\alpha\cos\beta_2 + \omega_5 v\sin\alpha\cos\beta_1 - \omega_3 v\sin\alpha\sin\beta_1\\ &\qquad - \sigma(\omega_6^2+\omega_5^2+\omega_3^2)\sin\gamma - \sigma(\omega_4\omega_5+\omega_2\omega_3)\cos\gamma + \sigma\dot\omega_6\cos\gamma = \frac{S_1}{m},\end{aligned} \tag{8.4}$$

$$\begin{aligned}&\dot v\cos\alpha\cos\beta_2 - \dot\alpha v\sin\alpha\cos\beta_2 - \dot\beta_2 v\cos\alpha\sin\beta_2\\ &\qquad + \omega_6 v\cos\alpha\sin\beta_2 - \omega_4 v\sin\alpha\cos\beta_1 + \omega_2 v\sin\alpha\sin\beta_1\\ &\qquad + \sigma(\omega_6^2+\omega_4^2+\omega_2^2)\cos\gamma + \sigma(\omega_4\omega_5+\omega_2\omega_3)\sin\gamma + \sigma\dot\omega_6\sin\gamma = \frac{S_2}{m},\end{aligned} \tag{8.5}$$

$$\begin{aligned}&\dot v\sin\alpha\cos\beta_1 + \dot\alpha v\cos\alpha\cos\beta_1 - \dot\beta_1 v\sin\alpha\sin\beta_1\\ &\qquad - \omega_5 v\cos\alpha\sin\beta_2 + \omega_4 v\cos\alpha\cos\beta_2 - \omega_1 v\sin\alpha\sin\beta_1\\ &\qquad + \sigma(\omega_4\omega_6-\omega_1\omega_3)\sin\gamma - \sigma(\omega_5\omega_6+\omega_1\omega_2)\cos\gamma - \sigma\dot\omega_5\sin\gamma - \sigma\dot\omega_4\cos\gamma = 0,\end{aligned} \tag{8.6}$$

$$\begin{aligned}&\dot v\sin\alpha\sin\beta_1 + \dot\alpha v\cos\alpha\sin\beta_1 + \dot\beta_1 v\sin\alpha\cos\beta_1\\ &\qquad + \omega_3 v\cos\alpha\sin\beta_2 - \omega_2 v\cos\alpha\cos\beta_2 + \omega_1 v\sin\alpha\cos\beta_1\\ &\qquad - \sigma(\omega_2\omega_6+\omega_1\omega_5)\sin\gamma + \sigma(\omega_3\omega_6-\omega_1\omega_4)\cos\gamma + \sigma\dot\omega_3\sin\gamma + \sigma\dot\omega_2\cos\gamma = 0,\end{aligned} \tag{8.7}$$

where

$$S = s(\alpha)v^2, \quad \sigma = CD, \quad v > 0. \tag{8.8}$$

Further, the auxiliary matrix (6.11) for the calculation of the moment of the resisting force has the form

$$\begin{pmatrix} 0 & 0 & x_{3N} & x_{4N}\\ S_1 & S_2 & 0 & 0\end{pmatrix}, \tag{8.9}$$

and, therefore, the part of dynamical equations that describe the motion of the body around the center of mass and correspond to the Lie algebra so(4) has the form

$$(\lambda_4+\lambda_3)\dot\omega_1 + (\lambda_3-\lambda_4)(\omega_3\omega_5+\omega_2\omega_4) = 0, \tag{8.10}$$

$$(\lambda_2+\lambda_4)\dot\omega_2+(\lambda_2-\lambda_4)(\omega_3\omega_6-\omega_1\omega_4)=-x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)v^2\cos\gamma, \tag{8.11}$$

$$(\lambda_4+\lambda_1)\dot\omega_3+(\lambda_4-\lambda_1)(\omega_2\omega_6+\omega_1\omega_5)=-x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)v^2\sin\gamma, \tag{8.12}$$

$$(\lambda_3+\lambda_2)\dot\omega_4+(\lambda_2-\lambda_3)(\omega_5\omega_6+\omega_1\omega_2)=x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)v^2\cos\gamma, \tag{8.13}$$

$$(\lambda_1+\lambda_3)\dot\omega_5+(\lambda_3-\lambda_1)(\omega_4\omega_6-\omega_1\omega_3)=x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)v^2\sin\gamma, \tag{8.14}$$

$$(\lambda_1+\lambda_2)\dot\omega_6+(\lambda_1-\lambda_2)(\omega_4\omega_5+\omega_2\omega_3)=0. \tag{8.15}$$

Thus, the direct product

$$\mathbf{R}^1\times\mathbf{S}^3\times\mathrm{so}(4) \tag{8.16}$$

of the four-dimensional manifold by the Lie algebra so(4) is the phase space of the tenth-order system (8.4)–(8.7), (8.10)–(8.15).

Note that system (8.4)–(8.7), (8.10)–(8.15), by the dynamical symmetry

$$I_1=I_2,\quad I_3=I_4, \tag{8.17}$$

possesses the cyclic first integrals

$$\omega_1\equiv\omega_1^0=\mathrm{const},\quad \omega_6\equiv\omega_6^0=\mathrm{const}\,. \tag{8.18}$$

Therefore, in what follows, we consider the dynamics of the system on the zero level:

$$\omega_1^0=\omega_6^0=0. \tag{8.19}$$

If we consider a more general problem on the motion of the body with a tracing force $\mathbf{T}$ that acts in the plane Dx_1x_2 and obeys the fulfillment of the conditions (see also [164–167])

$$v\equiv\mathrm{const},\quad \beta_2\equiv\mathrm{const}, \tag{8.20}$$

then in system (8.4)–(8.7), (8.10)–(8.15), we must replace F_1 and F_2 by

$$T_1+S_1,\quad T_2+S_2, \tag{8.21}$$

respectively.

As a result of the corresponding choice of the magnitude T of the tracing force, it is possible to satisfy Eqs. (8.20) formally during all time of the motion. Indeed, if we express T using (8.4)–(8.7) and (8.10)–(8.15), we obtain for $\cos\alpha\neq 0$

$$\begin{aligned}T_1=T_{1,v,\beta2}(\alpha,\beta_1,\Omega)&=-m\sigma(\omega_5^2+\omega_3^2)\sin\gamma-m\sigma(\omega_4\omega_5+\omega_2\omega_3)\cos\gamma\\&\quad+m\omega_5v\sin\alpha\cos\beta_1\cos^2\beta_2-m\omega_3v\sin\alpha\sin\beta_1\cos^2\beta_2\\&\quad+m\omega_4v\sin\alpha\cos\beta_1\sin\beta_2\cos\beta_2-m\omega_2v\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_2\\&\quad-s(\alpha)v^2\left[\sin\gamma-\frac{m\sigma}{I_1+I_3}\frac{\sin\alpha}{\cos\alpha}\sin\beta_2\cdot\Lambda_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\right],\end{aligned} \tag{8.22}$$

$$\begin{aligned}T_2=T_{2,v,\beta2}(\alpha,\beta_1,\Omega)&=m\sigma(\omega_4^2+\omega_2^2)\cos\gamma+m\sigma(\omega_4\omega_5+\omega_2\omega_3)\sin\gamma\\&\quad-m\omega_4v\sin\alpha\cos\beta_1\sin^2\beta_2+m\omega_2v\sin\alpha\sin\beta_1\sin^2\beta_2\\&\quad-m\omega_5v\sin\alpha\cos\beta_1\sin\beta_2\cos\beta_2+m\omega_3v\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_2\\&\quad+s(\alpha)v^2\left[\cos\gamma-\frac{m\sigma}{I_1+I_3}\frac{\sin\alpha}{\cos\alpha}\cos\beta_2\cdot\Lambda_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\right],\end{aligned} \tag{8.23}$$

where

$$\Lambda_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)=x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\cos\beta_1+x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\sin\beta_1. \tag{8.24}$$

Conditions (8.18)–(8.20) are used for obtaining Eqs. (8.22) and (8.23).

This procedure can be interpreted in two ways. First, we have transformed the system using the tracing force (control) that provides the consideration of interesting classes of motion (8.20). Second, we can treat this as an order-reduction procedure. Indeed, system (8.4)–(8.7), (8.10)–(8.15) generates the following independent system of sixth order:

$$\dot\alpha v\cos\alpha\cos\beta_1-\dot\beta_1 v\sin\alpha\sin\beta_1-\omega_5 v\cos\alpha\sin\beta_2 \\ +\omega_4 v\cos\alpha\cos\beta_2-\sigma\dot\omega_5\sin\gamma-\sigma\dot\omega_4\cos\gamma=0, \tag{8.25}$$

$$\dot\alpha v\cos\alpha\sin\beta_1+\dot\beta_1 v\sin\alpha\cos\beta_1+\omega_3 v\cos\alpha\sin\beta_2 \\ -\omega_2 v\cos\alpha\cos\beta_2+\sigma\dot\omega_3\sin\gamma+\sigma\dot\omega_2\cos\gamma=0, \tag{8.26}$$

$$(I_1+I_3)\dot\omega_2=-x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)v^2\cos\gamma, \tag{8.27}$$

$$(I_1+I_3)\dot\omega_3=-x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)v^2\sin\gamma, \tag{8.28}$$

$$(I_1+I_3)\dot\omega_4=x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)v^2\cos\gamma, \tag{8.29}$$

$$(I_1+I_3)\dot\omega_5=x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)v^2\sin\gamma, \tag{8.30}$$

which, in addition to the permanent parameters specified above, contains the parameters v and β_2.

1.1. Two approaches to the integrability. First, we make a remark on analytic first integrals. Obviously, system (8.25)–(8.30) possesses two analytical first integrals expressed as finite combinations of elementary functions:

$$\omega_2\sin\gamma-\omega_3\cos\gamma=W_1'=\text{const}, \tag{8.31}$$

$$\omega_4\sin\gamma-\omega_5\cos\gamma=W_2'=\text{const}. \tag{8.32}$$

This means that system (8.25)–(8.30) can be reduced to a fourth-order system on its own four-dimensional phase manifold.

Further, we can study system (8.25)–(8.30) in two ways.

I. We may ignore the first integrals (8.31) and (8.32) of the system. Then, after transformations, we reduce system (8.25)–(8.30) to an equivalent system in which a lower-dimensional system is separated. For complete integration, it suffices to find several independent first integrals (owing to (8.31) and (8.32), the number of these integrals is two less than the total number of integrals).

II. Using the first integrals (8.31) and (8.32), we can express two phase variables from the list ω_2, ω_3, ω_4, and ω_5. In this case, we obtain a fourth-order system on its own four-dimensional phase manifold as a reduction of system (8.25)–(8.30).

Consider the first way.

Indeed, system (8.25)–(8.30) is equivalent to the following system:

$$\dot\alpha v\cos\alpha-\omega_5 v\cos\alpha\cos\beta_1\sin\beta_2+\omega_4 v\cos\alpha\cos\beta_1\cos\beta_2 \\ +\omega_3 v\cos\alpha\sin\beta_1\sin\beta_2-\omega_2 v\cos\alpha\sin\beta_1\cos\beta_2-\sigma\dot\omega_5\sin\gamma\cos\beta_1 \\ -\sigma\dot\omega_4\cos\gamma\cos\beta_1+\sigma\dot\omega_3\sin\gamma\sin\beta_1+\sigma\dot\omega_2\cos\gamma\sin\beta_1=0, \tag{8.33}$$

$$\dot{\beta}_1 v\sin\alpha + \omega_3 v\cos\alpha\cos\beta_1\sin\beta_2 - \omega_2 v\cos\alpha\cos\beta_1\cos\beta_2 + \omega_5 v\cos\alpha\sin\beta_1\sin\beta_2 - \omega_4 v\cos\alpha\sin\beta_1\cos\beta_2 + \sigma\dot{\omega}_3\sin\gamma\cos\beta_1 + \sigma\dot{\omega}_2\cos\gamma\cos\beta_1 + \sigma\dot{\omega}_5\sin\gamma\sin\beta_1 + \sigma\dot{\omega}_4\cos\gamma\sin\beta_1 = 0, \quad (8.34)$$

$$\dot{\omega}_2 = -\frac{v^2}{I_1+I_3} x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)\cos\gamma, \quad (8.35)$$

$$\dot{\omega}_3 = -\frac{v^2}{I_1+I_3} x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)\sin\gamma, \quad (8.36)$$

$$\dot{\omega}_4 = \frac{v^2}{I_1+I_3} x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)\cos\gamma, \quad (8.37)$$

$$\dot{\omega}_5 = \frac{v^2}{I_1+I_3} x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)\sin\gamma, \quad (8.38)$$

We introduce the new quasi-velocities in the system. For this, we transform ω_2, ω_3, ω_4, and ω_5 by the following rotations:

$$\begin{pmatrix} z_1 \\ -z_2 \end{pmatrix} = T_*(-\beta_1)\begin{pmatrix} \omega_3 \\ \omega_5 \end{pmatrix}, \quad \begin{pmatrix} z_3 \\ -z_4 \end{pmatrix} = T_*(-\beta_1)\begin{pmatrix} \omega_2 \\ \omega_4 \end{pmatrix}, \quad (8.39)$$

where

$$T_*(\beta_1) = \begin{pmatrix} \cos\beta_1 & -\sin\beta_1 \\ \sin\beta_1 & \cos\beta_1 \end{pmatrix}, \quad (8.40)$$

and also

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = T_*(\beta_2)\begin{pmatrix} z_3 \\ z_1 \end{pmatrix}, \quad \begin{pmatrix} w_3 \\ w_4 \end{pmatrix} = T_*(-\beta_2)\begin{pmatrix} -z_4 \\ z_2 \end{pmatrix}. \quad (8.41)$$

Thus, the following relations hold:

$$\begin{aligned} z_1 &= \omega_3\cos\beta_1 + \omega_5\sin\beta_1, & z_2 &= \omega_3\sin\beta_1 - \omega_5\cos\beta_1, \\ z_3 &= \omega_2\cos\beta_1 + \omega_4\sin\beta_1, & z_4 &= \omega_2\sin\beta_1 - \omega_4\cos\beta_1, \\ w_1 &= -z_1\sin\beta_2 + z_3\cos\beta_2, & w_2 &= z_3\sin\beta_2 + z_1\cos\beta_2, \\ w_3 &= z_2\sin\beta_2 - z_4\cos\beta_2, & w_4 &= z_4\sin\beta_2 + z_2\cos\beta_2. \end{aligned} \quad (8.42)$$

We see from (8.33)–(8.38) that on the manifold

$$O_2 = \left\{(\alpha,\beta_1,\omega_2,\omega_3,\omega_4,\omega_5) \in \mathbf{R}^6 : \ \alpha = \pi k/2, \ k \in \mathbf{Z}\right\}, \quad (8.43)$$

the system is not uniquely solvable with respect to $\dot{\alpha}$ and $\dot{\beta}_1$. Thus, on the manifold (8.43), the uniqueness theorem formally is violated. Moreover, first, the uncertainty appears for even or odd k due to the degeneration of the coordinates $(v,\alpha,\beta_1,\beta_2)$, which parameterize the three-dimensional sphere (note that they are not the conventional spherical coordinates), and, second, an obvious violation of the uniqueness theorem for odd k occurs since the first equation in (8.33) is degenerate in this case.

Indeed, the Jacobian of the transformation $(x_1,x_2,x_3,x_4) \to (v,\alpha,\beta_1,\beta_2)$ defined by the formulas

$$\begin{aligned} x_1 &= v\cos\alpha\sin\beta_2, \\ x_2 &= v\cos\alpha\cos\beta_2, \\ x_3 &= v\sin\alpha\cos\beta_1, \\ x_4 &= v\sin\alpha\sin\beta_1 \end{aligned} \quad (8.44)$$

is equal to

$$v^3\cos\alpha\sin\alpha;$$

it differs from the Jacobian of the generalized spherical coordinates v, α, β_1, β_2, which, in turn, is equal to

$$v^3 \sin\alpha \sin\beta_1.$$

It follows that system (8.33)–(8.38) outside the manifold (8.43) (and only outside it) is equivalent to the system

$$\dot\alpha = -w_3 + \frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\cos\alpha} \cdot \Lambda_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right), \tag{8.45}$$

$$\begin{aligned}\dot z_4 = -\frac{v^2}{I_1+I_3} s(\alpha)\cos\gamma \cdot \Lambda_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right) \\ + z_3\left[w_1\frac{\cos\alpha}{\sin\alpha} - \frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\sin\alpha} \cdot \Pi_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right)\right],\end{aligned} \tag{8.46}$$

$$\begin{aligned}\dot z_3 = \frac{v^2}{I_1+I_3} s(\alpha)\cos\gamma \cdot \Pi_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right) \\ - z_4\left[w_1\frac{\cos\alpha}{\sin\alpha} - \frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\sin\alpha} \cdot \Pi_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right)\right],\end{aligned} \tag{8.47}$$

$$\begin{aligned}\dot z_2 = -\frac{v^2}{I_1+I_3} s(\alpha)\sin\gamma \cdot \Lambda_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right) \\ + z_1\left[w_1\frac{\cos\alpha}{\sin\alpha} - \frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\sin\alpha} \cdot \Pi_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right)\right],\end{aligned} \tag{8.48}$$

$$\begin{aligned}\dot z_1 = \frac{v^2}{I_1+I_3} s(\alpha)\sin\gamma \cdot \Pi_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right) \\ - z_2\left[w_1\frac{\cos\alpha}{\sin\alpha} - \frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\sin\alpha} \cdot \Pi_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right)\right],\end{aligned} \tag{8.49}$$

$$\dot\beta_1 = w_1\frac{\cos\alpha}{\sin\alpha} - \frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\sin\alpha} \cdot \Pi_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right), \tag{8.50}$$

or, finally,

$$\dot\alpha = -w_3 + \frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\cos\alpha} \cdot \Lambda_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right), \tag{8.51}$$

$$\begin{aligned}\dot w_4 = -\frac{v^2}{I_1+I_3} s(\alpha)\sin(\beta_2+\gamma) \cdot \Lambda_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right) \\ + w_2\left[w_1\frac{\cos\alpha}{\sin\alpha} - \frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\sin\alpha} \cdot \Pi_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right)\right],\end{aligned} \tag{8.52}$$

$$\begin{aligned}\dot w_3 = \frac{v^2}{I_1+I_3} s(\alpha)\cos(\beta_2+\gamma) \cdot \Lambda_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right) \\ - w_1\left[w_1\frac{\cos\alpha}{\sin\alpha} - \frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\sin\alpha} \cdot \Pi_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right)\right],\end{aligned} \tag{8.53}$$

$$\dot{w}_2 = \frac{v^2}{I_1+I_3} s(\alpha)\sin(\beta_2+\gamma)\cdot \Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right) - w_4\left[w_1\frac{\cos\alpha}{\sin\alpha} - \frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\sin\alpha}\cdot \Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\right], \tag{8.54}$$

$$\dot{w}_1 = \frac{v^2}{I_1+I_3} s(\alpha)\cos(\beta_2+\gamma)\cdot \Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right) + w_3\left[w_1\frac{\cos\alpha}{\sin\alpha} - \frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\sin\alpha}\cdot \Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\right], \tag{8.55}$$

$$\dot{\beta}_1 = w_1\frac{\cos\alpha}{\sin\alpha} - \frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\sin\alpha}\cdot \Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right), \tag{8.56}$$

where

$$\Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right) = -x_{4N}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1 + x_{3N}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1, \tag{8.57}$$

and the function $\Lambda_{v,\beta_2}(\alpha,\beta_1,\Omega/v)$ is represented in the form (8.24).

In what follows, the dependence on the variables $(\alpha,\beta_1,\beta_2,\Omega/v)$ must be treated as the composite dependence on $(\alpha,\beta_1,\beta_2,z_1/v,z_2/v,z_3/v,z_4/v)$ (or $(\alpha,\beta_1,\beta_2,w_1/v,w_2/v,w_3/v,w_4/v)$) by virtue of (8.42).

The violation of the uniqueness theorem happens for system (8.33)–(8.38) on the manifold (8.43) for odd k in the following sense:

regular phase trajectories of system (8.33)–(8.38) pass through almost all points of the manifold (8.43) and intersect this manifold at a right angle, and also there exists a phase trajectory that completely coincides with the specified point at all time instants. However, these trajectories are different since they correspond to different values of the tracing force. Let us prove this.

As was shown above, to fulfill constraints (8.20), one must choose the value of T_1 and T_2 for $\cos\alpha \neq 0$ in the form (8.22) and (8.23).

Let

$$\lim_{\alpha\to\pi/2}\frac{s(\alpha)}{\cos\alpha}\Lambda_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right) = L\left(\beta_1,\beta_2,\frac{\Omega}{v}\right). \tag{8.58}$$

Note that $|L| < +\infty$ if and only if

$$\lim_{\alpha\to\pi/2}\left|\frac{\partial}{\partial\alpha}\left(\Lambda_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)s(\alpha)\right)\right| < +\infty. \tag{8.59}$$

The required values of the tracing force for $\alpha = \pi/2$ can be found from the relations

$$T_1 = T_{1,v,\beta 2}\left(\frac{\pi}{2},\beta_1,\Omega\right) = -m\sigma(\omega_5^2+\omega_3^2)\sin\gamma - m\sigma(\omega_4\omega_5+\omega_2\omega_3)\cos\gamma + m\omega_5 v\cos\beta_1\cos^2\beta_2 - m\omega_3 v\sin\beta_1\cos^2\beta_2 + m\omega_4 v\cos\beta_1\sin\beta_2\cos\beta_2 - m\omega_2 v\sin\beta_1\sin\beta_2\cos\beta_2 + v^2\frac{m\sigma}{I_1+I_3}\sin\beta_2\cdot L \tag{8.60}$$

and

$$T_2 = T_{2,v,\beta 2}\left(\frac{\pi}{2},\beta_1,\Omega\right) = m\sigma(\omega_4^2+\omega_2^2)\cos\gamma + m\sigma(\omega_4\omega_5+\omega_2\omega_3)\sin\gamma - m\omega_4 v\cos\beta_1\sin^2\beta_2 + m\omega_2 v\sin\beta_1\sin^2\beta_2 - m\omega_5 v\cos\beta_1\sin\beta_2\cos\beta_2 + m\omega_3 v\sin\beta_1\sin\beta_2\cos\beta_2 - v^2\frac{m\sigma}{I_1+I_3}\cos\beta_2\cdot L, \tag{8.61}$$

where the values of ω_2, ω_3, ω_4, and ω_5 are arbitrary.

On the other hand, if we support the rotation about a certain point W by the tracing force, we must choose the projections of the tracing force as follows:

$$T = T_1\left(\frac{\pi}{2}, \beta_1, \beta_2, \Omega\right) = \frac{mv^2}{R_{01}}, \tag{8.62}$$

$$T = T_2\left(\frac{\pi}{2}, \beta_1, \beta_2, \Omega\right) = \frac{mv^2}{R_{02}}, \tag{8.63}$$

where R_{01} and R_{02} are the projections of the segment CW on the corresponding coordinate axes.

Equalities (8.22)–(8.23) and (8.62)–(8.63) define, generally speaking, different values of the tracing force T for almost all points of manifold (8.43), and the remark is proved.

2. Case where the Moment of Nonconservative Forces Is Independent of the Angular Velocity

2.1. Reduced system. Similarly to the choice of Chaplygin analytical functions (see [50, 51]), we choose the dynamical functions s, x_{3N}, and x_{4N} as follows:

$$\begin{gathered} s(\alpha) = B\cos\alpha, \\ x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = x_{3N0}(\alpha, \beta_1) = A\sin\alpha\cos\beta_1, \\ x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = x_{4N0}(\alpha, \beta_1) = A\sin\alpha\sin\beta_1, \quad A, B > 0, \quad v \neq 0, \end{gathered} \tag{8.64}$$

which shows that for the considered system, the moment of the nonconservative forces is independent of the angular velocity (it depends only on the angles α, β_1, and β_2).

The functions $\Lambda_{v,\beta_2}(\alpha, \beta_1, \Omega/v)$, $\Pi_{v,\beta_2}(\alpha, \beta_1, \Omega/v)$ in system (8.51)–(8.56) have the following form:

$$\Lambda_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right) = A\sin\alpha, \quad \Pi_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right) \equiv 0. \tag{8.65}$$

Then, owing to the nonintegrable constraint (8.20), outside the manifold (8.43) (and only outside it) the dynamical part of the equations of motion (system (8.51)–(8.56)) has the form of the following analytical system:

$$\dot\alpha = -w_3 + \frac{\sigma ABv}{I_1 + I_3}\sin\alpha, \tag{8.66}$$

$$\dot w_4 = -\frac{ABv^2}{I_1 + I_3}\sin(\beta_2 + \gamma)\sin\alpha\cos\alpha + w_1 w_2\frac{\cos\alpha}{\sin\alpha}, \tag{8.67}$$

$$\dot w_3 = \frac{ABv^2}{I_1 + I_3}\cos(\beta_2 + \gamma)\sin\alpha\cos\alpha - w_1^2\frac{\cos\alpha}{\sin\alpha}, \tag{8.68}$$

$$\dot w_2 = -w_1 w_4\frac{\cos\alpha}{\sin\alpha}, \tag{8.69}$$

$$\dot w_1 = w_1 w_3\frac{\cos\alpha}{\sin\alpha}, \tag{8.70}$$

$$\dot\beta_1 = w_1\frac{\cos\alpha}{\sin\alpha}. \tag{8.71}$$

Introducing the dimensionless variables, the parameters, and the differentiation as follows:

$$w_k \mapsto n_0 v w_k, \quad k = 1, 2, 3, 4, \quad n_0^2 = \frac{AB}{I_1 + I_3}, \quad b = \sigma n_0, \quad \langle\cdot\rangle = n_0 v\langle'\rangle, \tag{8.72}$$

we reduce system (8.66)–(8.71) to the form

$$\alpha' = -w_3 + b\sin\alpha, \tag{8.73}$$

$$w_4' = -\sin(\beta_2+\gamma)\sin\alpha\cos\alpha + w_1w_2\frac{\cos\alpha}{\sin\alpha}, \tag{8.74}$$

$$w_3' = \cos(\beta_2+\gamma)\sin\alpha\cos\alpha - w_1^2\frac{\cos\alpha}{\sin\alpha}, \tag{8.75}$$

$$w_2' = -w_1w_4\frac{\cos\alpha}{\sin\alpha}, \tag{8.76}$$

$$w_1' = w_1w_3\frac{\cos\alpha}{\sin\alpha}, \tag{8.77}$$

$$\beta_1' = w_1\frac{\cos\alpha}{\sin\alpha}. \tag{8.78}$$

We see that in the sixth-order system (8.73)–(8.78), which can be considered on the six-dimensional manifold

$$T\mathbf{S}^2 \times \mathbf{R}^2, \tag{8.79}$$

the direct product of the tangent bundle $T\mathbf{S}^2$ of a two-dimensional sphere $\mathbf{S}^2$ and a two-dimensional plane, an independent fifth-order system (8.73)–(8.77) has been formed; this fifth-order system can be considered on its own five-dimensional manifold.

Moreover, the sixth order system (8.73)–(8.78) has the independent third-order subsystem

$$\alpha' = -w_3 + b\sin\alpha, \tag{8.80}$$

$$w_3' = \cos(\beta_2+\gamma)\sin\alpha\cos\alpha - w_1^2\frac{\cos\alpha}{\sin\alpha}, \tag{8.81}$$

$$w_1' = w_1w_3\frac{\cos\alpha}{\sin\alpha}, \tag{8.82}$$

the second-order system

$$w_4' = -\sin(\beta_2+\gamma)\sin\alpha\cos\alpha + w_1w_2\frac{\cos\alpha}{\sin\alpha}, \tag{8.83}$$

$$w_2' = -w_1w_4\frac{\cos\alpha}{\sin\alpha}, \tag{8.84}$$

(this system is still dependent), and the equation

$$\beta_1' = w_1\frac{\cos\alpha}{\sin\alpha}. \tag{8.85}$$

In general, for the complete integrability of system (8.73)–(8.78), it suffices to know five independent first integrals. However, after splitting the system into three parts (system (8.80)–(8.82), system (8.83)–(8.84), and Eq. (8.85)), for the complete integrability it suffices to know two independent first integrals of system (8.80)–(8.82), one first integral of system (8.83)–(8.84) (after the reduction of this system to an independent subsystem), and one first integral that "attaches" Eq. (8.85).

Note that these arguments are typical for the first way of studying the system (see p. 484). Indeed, we now ignore the two analytical first integrals (8.31) and (8.32). Therefore, when we obtain two independent first integrals of the independent third-order system (8.80)–(8.82) and also the first integral that "attaches" Eq. (8.85), we obtain a complete set of independent first integrals of the fourth-order system (8.80)–(8.82), (8.85). The obtained complete set (three integrals) together with the analytical first integrals (8.31) and (8.32) forms a complete set of five first integrals of the sixth-order system (8.80)–(8.85).

In what follows, in particular, we will see that the composition of analytical first integrals (8.31) and (8.32) yields a first integral of system (8.83), (8.84).

2.2. Complete list of invariant relations. System (8.80)–(8.82) has the form of system (4.22) that appears in the dynamics of a three-dimensional rigid body in a field of nonconservative forces (see Chap. 4). In this case, the phase variables z_1 and z_2 in system (4.22) correspond to the phase variables z and z_3 of system (8.80)–(8.82). We recall certain facts discussed in Chap. 4 in the new notation.

First, we compare the third-order system (8.80)–(8.82) with the nonautonomous second-order system

$$\begin{aligned} \frac{dw_3}{d\alpha} &= \frac{\cos(\beta_2+\gamma)\sin\alpha\cos\alpha - w_1^2\cos\alpha/\sin\alpha}{-w_3+b\sin\alpha}, \\ \frac{dw_1}{d\alpha} &= \frac{w_1w_3\cos\alpha/\sin\alpha}{-w_3+b\sin\alpha}. \end{aligned} \tag{8.86}$$

Using the substitution $\tau=\sin\alpha$, we rewrite system (8.86) in the algebraic form

$$\begin{aligned} \frac{dw_3}{d\tau} &= \frac{\cos(\beta_2+\gamma)\tau - w_1^2/\tau}{-w_3+b\tau}, \\ \frac{dw_1}{d\tau} &= \frac{w_1w_3/\tau}{-w_3+b\tau}. \end{aligned} \tag{8.87}$$

Further, introducing the homogeneous variables by the formulas

$$w_1=u_1\tau, \quad w_3=u_2\tau, \tag{8.88}$$

we reduce system (8.87) to the following form:

$$\begin{aligned} \tau\frac{du_2}{d\tau}+u_2 &= \frac{\cos(\beta_2+\gamma)-u_1^2}{-u_2+b}, \\ \tau\frac{du_1}{d\tau}+u_1 &= \frac{u_1u_2}{-u_2+b}, \end{aligned} \tag{8.89}$$

which is equivalent to

$$\begin{aligned} \tau\frac{du_2}{d\tau} &= \frac{\cos(\beta_2+\gamma)-u_1^2+u_2^2-bu_2}{-u_2+b}, \\ \tau\frac{du_1}{d\tau} &= \frac{2u_1u_2-bu_1}{-u_2+b}. \end{aligned} \tag{8.90}$$

We compare the second-order system (8.90) with the nonautonomous first-order equation

$$\frac{du_2}{du_1} = \frac{\cos(\beta_2+\gamma)-u_1^2+u_2^2-bu_2}{2u_1u_2-bu_1}, \tag{8.91}$$

which can be easily reduced to the exact differential equation

$$d\left(\frac{u_2^2+u_1^2-bu_2+\cos(\beta_2+\gamma)}{u_1}\right)=0. \tag{8.92}$$

Therefore, Eq. (8.91) has the following first integral:

$$\frac{u_2^2+u_1^2-bu_2+\cos(\beta_2+\gamma)}{u_1}=C_1=\text{const}, \tag{8.93}$$

which in the old variables has the form

$$\frac{w_3^2+w_1^2-bw_3\sin\alpha+\cos(\beta_2+\gamma)\sin^2\alpha}{w_1\sin\alpha}=C_1=\text{const}\,. \tag{8.94}$$

Remark 8.1. We consider the system (8.80)–(8.82) with variable dissipation with zero mean (see [67, 68, 243, 260, 262, 265, 282–286, 291, 295, 309, 324, 333, 340, 384, 391, 404–408, 412, 413, 421, 437, 438]), which becomes conservative for $b = 0$:

$$\begin{aligned} \alpha' &= -w_3, \\ w_3' &= \cos(\beta_2+\gamma)\sin\alpha\cos\alpha - w_1^2\frac{\cos\alpha}{\sin\alpha}, \\ w_1' &= w_1 w_3\frac{\cos\alpha}{\sin\alpha}. \end{aligned} \tag{8.95}$$

It has two analytical first integrals of the form

$$w_3^2 + w_1^2 + \cos(\beta_2+\gamma)\sin^2\alpha = C_1^* = \text{const}, \tag{8.96}$$

$$w_1\sin\alpha = C_2^* = \text{const}\,. \tag{8.97}$$

Obviously, the ratio of the first integrals (8.96) and (8.97) is also a first integral of system (8.95). However, for $b \neq 0$, both functions

$$w_3^2 + w_1^2 - bw_3\sin\alpha + \cos(\beta_2+\gamma)\sin^2\alpha \tag{8.98}$$

and (8.97) are not first integrals of system (8.80)–(8.82), but their ratio is a first integral of the system for any b.

Further, we find an explicit form of an additional first integral of the third-order system (8.80)–(8.82). First, we transform the invariant relation (8.93) for $u_1 \neq 0$ as follows:

$$\left(u_2 - \frac{b}{2}\right)^2 + \left(u_1 - \frac{C_1}{2}\right)^2 = \frac{b^2 + C_1^2}{4} - \cos(\beta_2+\gamma). \tag{8.99}$$

We see that the parameters of this invariant relation must satisfy the condition

$$b^2 + C_1^2 - 4\cos(\beta_2+\gamma) \geq 0, \tag{8.100}$$

and the phase space of system (8.80)–(8.82) is stratified into a family of surfaces defined by Eq. (8.99).

Thus, by (8.93), the first equation of system (8.90) has the form

$$\tau\frac{du_2}{d\tau} = \frac{2(\cos(\beta_2+\gamma) - bu_2 + u_2^2) - C_1U_1(C_1,u_2)}{-u_2+b}, \tag{8.101}$$

where

$$U_1(C_1,u_2) = \frac{1}{2}\left\{C_1 \pm \sqrt{C_1^2 - 4(u_2^2 - bu_2 + \cos(\beta_2+\gamma))}\right\} \tag{8.102}$$

and the integration constant C_1 is chosen from condition (8.100).

Therefore, the quadrature for the search for an additional first integral of system (8.80)–(8.82) has the form

$$\int\frac{d\tau}{\tau} = \int\frac{(b-u_2)du_2}{2(\cos(\beta_2+\gamma) - bu_2 + u_2^2) - C_1\left\{C_1 \pm \sqrt{C_1^2 - 4(u_2^2 - bu_2 + \cos(\beta_2+\gamma))}\right\}/2}. \tag{8.103}$$

Obviously, the left-hand side up to an additive constant is equal to

$$\ln|\sin\alpha|. \tag{8.104}$$

If

$$u_2 - \frac{b}{2} = p_1, \quad b_1^2 = b^2 + C_1^2 - 4\cos(\beta_2+\gamma), \tag{8.105}$$

then the right-hand side of Eq. (8.103) has the form

$$-\frac{1}{4}\int\frac{d(b_1^2-4p_1^2)}{(b_1^2-4p_1^2)\pm C_1\sqrt{b_1^2-4p_1^2}}-b\int\frac{dp_1}{(b_1^2-4p_1^2)\pm C_1\sqrt{b_1^2-4p_1^2}}$$
$$=-\frac{1}{2}\ln\left|\frac{\sqrt{b_1^2-4p_1^2}}{C_1}\pm 1\right|\pm\frac{b}{2}I_1, \quad (8.106)$$

where

$$I_1=\int\frac{dp_3}{\sqrt{b_1^2-p_3^2}(p_3\pm C_1)}, \quad p_3=\sqrt{b_1^2-4p_1^2}. \quad (8.107)$$

In the calculation of integral (8.107), the following three cases are possible.

I. $b>2$:

$$I_1=-\frac{1}{2\sqrt{b^2-4}}\ln\left|\frac{\sqrt{b^2-4}+\sqrt{b_1^2-p_3^2}}{p_3\pm C_1}\pm\frac{C_1}{\sqrt{b^2-4}}\right|$$
$$+\frac{1}{2\sqrt{b^2-4}}\ln\left|\frac{\sqrt{b^2-4}-\sqrt{b_1^2-p_3^2}}{p_3\pm C_1}\mp\frac{C_1}{\sqrt{b^2-4}}\right|+\text{const}\,. \quad (8.108)$$

II. $b<2$:

$$I_1=\frac{1}{\sqrt{4-b^2}}\arcsin\frac{\pm C_1p_3+b_1^2}{b_1(p_3\pm C_1)}+\text{const}\,. \quad (8.109)$$

III. $b=2$:

$$I_1=\mp\frac{\sqrt{b_1^2-p_3^2}}{C_1(p_3\pm C_1)}+\text{const}\,. \quad (8.110)$$

Returning to the variable

$$p_1=\frac{w_3}{\sin\alpha}-\frac{b}{2}, \quad (8.111)$$

we obtain the final form of the integral I_1.

I. $b>2$:

$$I_1=-\frac{1}{2\sqrt{b^2-4}}\ln\left|\frac{\sqrt{b^2-4}\pm 2p_1}{\sqrt{b_1^2-4p_1^2}\pm C_1}\pm\frac{C_1}{\sqrt{b^2-4}}\right|$$
$$+\frac{1}{2\sqrt{b^2-4}}\ln\left|\frac{\sqrt{b^2-4}\mp 2p_1}{\sqrt{b_1^2-4p_1^2}\pm C_1}\mp\frac{C_1}{\sqrt{b^2-4}}\right|+\text{const}\,. \quad (8.112)$$

II. $b<2$:

$$I_1=\frac{1}{\sqrt{4-b^2}}\arcsin\frac{\pm C_1\sqrt{b_1^2-4p_1^2}+b_1^2}{b_1\Big(\sqrt{b_1^2-4p_1^2}\pm C_1\Big)}+\text{const}\,. \quad (8.113)$$

III. $b=2$:

$$I_1=\mp\frac{2p_1}{C_1\Big(\sqrt{b_1^2-4p_1^2}\pm C_1\Big)}+\text{const}\,. \quad (8.114)$$

Thus, we have found an additional first integral for the third-order system (8.80)–(8.82), i.e., we have a complete set of first integrals that are transcendental functions of their phase variables.

Remark 8.2. In the expression of the found first integral, we must formally substitute the left-hand side of the first integral (8.93) instead of C_1. Then the obtained additional first integral has the following structure similar to the transcendental first integral from the planar dynamics):

$$\ln|\sin\alpha| + G_2\left(\sin\alpha, \frac{w_3}{\sin\alpha}, \frac{w_1}{\sin\alpha}\right) = C_2 = \text{const}\,. \tag{8.115}$$

Thus, we have already found two independent first integrals for the integration of the sixth-order system (8.80)–(8.85). Now, using the first way of studying the system (where we ignore two analytical first integrals (8.31) and (8.32)), and the complete integrability, it suffices to find one first integral for system (8.83), (8.84) (which is potentially separated) and one additional first integral that "attaches" Eq. (8.85).

After the change of the variables

$$\begin{aligned} w_* &= w_3\sin(\gamma+\beta_2) + w_4\cos(\gamma+\beta_2), \\ w_{**} &= w_1\sin(\gamma+\beta_2) - w_2\cos(\gamma+\beta_2), \end{aligned} \tag{8.116}$$

system (8.83), (8.84) can be reduced to the form

$$\frac{dw_*}{d\beta_1} = -w_{**}, \quad \frac{dw_{**}}{d\beta_1} = w_*, \tag{8.117}$$

which implies the existence of the analytical first integral

$$w_*^2 + w_{**}^2 = C_3 = \text{const}\,. \tag{8.118}$$

The following question arises: How is the first integral (8.118) related to the analytical first integrals of the form (8.31) and (8.32)?

Indeed, two ways of the study (**I** and **II**, see p. 484) correspond to the following alternatives. For the complete integration of the sixth-order system (8.25)–(8.30):

(1) either we find five independent first integrals of the sixth-order system (8.25)–(8.30),
(2) or we transform the sixth-order system (8.25)–(8.30) such that independent subsystems of lower orders are extracted from it.

Thus, since after the introduction of such coordinates as w_* and w_{**}, the vector field of the system is stratified so that the independent second-order subsystem (8.117) is formed, we must find only four independent first integrals instead of five (three first integrals for integration of the fourth-order system (8.80)–(8.82), (8.85) and one first integral for integration of the separated second-order system (8.117)).

Finally, we rewrite the analytical first integrals (8.31) and (8.32) in the new variables as follows:

$$w_{**}\cos\beta_1 - w_*\sin\beta_1 = W_1'' = \text{const}, \tag{8.119}$$

$$w_{**}\sin\beta_1 + w_*\cos\beta_1 = W_2'' = \text{const}\,. \tag{8.120}$$

Obviously, the analytical first integrals (8.119) and (8.120) yield the analytical first integral (8.118) (to see this, it suffices to add the squares of the left-hand sides of Eqs. (8.119) and (8.120)).

Further, for integration of the fourth-order system (8.80)–(8.82), (8.85), we have found two independent first integrals. For its complete integrability, it suffices to find another additional first integral that "attaches" Eq. (8.85).

Since

$$\frac{du_1}{d\tau} = \frac{u_1(2u_2-b)}{(b-u_2)\tau}, \quad \frac{d\beta_1}{d\tau} = \frac{u_1}{(b-u_2)\tau}, \tag{8.121}$$

we have

$$\frac{du_1}{d\beta_1} = 2u_2 - b. \tag{8.122}$$

Obviously, for $u_1 \neq 0$, the following equality holds:

$$u_2 = \frac{1}{2}\left(b \pm \sqrt{b_1^2 - 4\left(u_1 - \frac{C_1}{2}\right)^2}\right), \quad b_1^2 = b^2 + C_1^2 - 4\cos(\beta_2 + \gamma); \tag{8.123}$$

therefore, integration of the quadrature

$$\beta_1 + \text{const} = \pm \int \frac{du_1}{\sqrt{b_1^2 - 4\left(u_1 - \frac{C_1}{2}\right)^2}} \tag{8.124}$$

leads to the invariant relation

$$2(\beta_1 + C_4) = \pm \arcsin \frac{2u_1 - C_1}{\sqrt{b^2 + C_1^2 - 4\cos(\beta_2 + \gamma)}}, \quad C_4 = \text{const}. \tag{8.125}$$

In other words, the relation

$$\sin\left[2(\beta_1 + C_4)\right] = \pm \frac{2u_1 - C_1}{\sqrt{b^2 + C_1^2 - 4\cos(\beta_2 + \gamma)}} \tag{8.126}$$

holds. After the transition to the old variables, we obtain

$$\sin\left[2(\beta_1 + C_4)\right] = \pm \frac{2w_1 - C_1 \sin\alpha}{\sqrt{b^2 + C_1^2 - 4\cos(\beta_2 + \gamma)}\sin\alpha}. \tag{8.127}$$

Thus, we have obtained an additional invariant relation that “attaches” Eq. (8.85). However, we must formally substitute the left-hand side of (8.93) into the last expression instead of C_1.

We perform certain transformations which lead us to the following form of the additional first integral:

$$\tan^2\left[2(\beta_1 + C_4)\right] = \frac{(u_1^2 - u_2^2 + bu_2 - \cos(\beta_2 + \gamma))^2}{u_1^2(4u_2^2 - 4bu_2 + b^2)}; \tag{8.128}$$

here Eq. (8.93) is used.

Returning to the old coordinates, we obtain the additional invariant relation in the form

$$\tan^2\left[2(\beta_1 + C_4)\right] = \frac{(w_1^2 - w_3^2 + bw_3 \sin\alpha - \cos(\beta_2 + \gamma)\sin^2\alpha)^2}{w_1^2(4w_3^2 - 4bw_3\sin\alpha + b^2\sin^2\alpha)}, \tag{8.129}$$

or, finally,

$$-\beta_1 \pm \frac{1}{2}\arctan \frac{w_1^2 - w_3^2 + bw_3\sin\alpha - \cos(\beta_2 + \gamma)\sin^2\alpha}{w_1(2w_3 - b\sin\alpha)} = C_4 = \text{const}. \tag{8.130}$$

Therefore, in the considered case, the system of dynamical equations (8.4)–(8.7), (8.10)–(8.15) under the condition (8.64) possesses eight invariant relations: the analytical nonintegrable constraints (8.20), the cyclic first integrals (8.18) and (8.19), the first integral (8.94), the first integral expressed by the relations (8.108)–(8.115), which is a transcendental function of its phase variables (in the sense of the complex analysis) expressed as a finite combination of elementary functions, the transcendental first integral (8.130) (see also (8.129)), and, finally, the analytical first integral (8.118).

Theorem 8.1. *System* (8.4)–(8.7), (8.10)–(8.15) *under the conditions* (8.20), (8.64), *and* (8.19) *possesses eight invariant relations* (*a complete set*), *three of which are transcendental functions* (*in the sense of the complex analysis*). *All these relations are expressed as finite combinations of elementary functions.*

2.3. Topological analogies. We consider the following third-order system:

$$\ddot{\xi} + b_* \dot{\xi} \cos \xi + R_3 \sin \xi \cos \xi - \dot{\eta}_1^2 \frac{\sin \xi}{\cos \xi} = 0,$$
$$\ddot{\eta}_1 + b_* \dot{\eta}_1 \cos \xi + \dot{\xi} \dot{\eta}_1 \frac{1 + \cos^2 \xi}{\cos \xi \sin \xi} = 0, \quad b_* > 0, \tag{8.131}$$

that describes a fixed spherical pendulum in a flowing medium in the case where the moment of forces is independent of the angular velocity, i.e., a mechanical system in a nonconservative field of forces (see [120, 162, 188, 201, 203, 235, 238, 276, 316, 317, 319, 320, 338, 359, 360, 376, 377, 386, 392, 429, 442]). In general, the order of such a system must be equal to 4, but the phase variable η_1 is cyclic, which leads to the stratification of the phase space and reduction of order.

The phase space is the tangent bundle

$$T\mathbf{S}^2\{\dot{\xi}, \dot{\eta}_1, \xi, \eta_1\} \tag{8.132}$$

of the two-dimensional sphere $\mathbf{S}^2\{\xi, \eta_1\}$, where the equation of the large circles

$$\dot{\eta}_1 \equiv 0 \tag{8.133}$$

defines the family of integral manifolds.

It is easy to verify that system (8.131) is equivalent to the dynamical system with variable dissipation with zero mean on the tangent bundle (8.132) of the two-dimensional sphere. Moreover, the following theorem holds.

Theorem 8.2. *System* (8.4)–(8.7)*,* (8.10)–(8.15) *under the conditions* (8.20)*,* (8.138)*, and* (8.19) *is equivalent to dynamical system* (8.131)*.*

Indeed, it suffices to take $\alpha = \xi$, $\beta_1 = \eta_1$, $b = -b_*$, and $R_3 = \cos(\gamma + \beta_2)$.

On more general topological analogies, see also [253, 265, 267, 309, 340, 342].

3. Case where the Moment of Nonconservative Forces Depends on the Angular Velocity

3.1. Dependence of the moment of nonconservative forces on the angular velocity. We continue to study the dynamics of a three-dimensional rigid body in the four-dimensional space. This section is devoted to the study of the case of the motion where the moment of forces depends on the angular velocity. We introduce this dependence in the general case; this will allow us to generalize this dependence to higher-dimensional bodies.

Let $x = (x_{1N}, x_{2N}, x_{3N}, x_{4N})$ be the coordinates of the point N of application of a nonconservative force (interaction with a medium) on a two-dimensional disk and $Q = (Q_1, Q_2, Q_3, Q_4)$ be the components independent of the angular velocity. We introduce only the linear dependence of the functions $(x_{1N}, x_{2N}, x_{3N}, x_{4N})$ on the angular velocity since the introduction of this dependence itself is not a priori obvious (see [33, 34, 48, 49, 57–66, 120, 121, 169, 182, 203, 205, 249, 250, 261, 274–276, 440]).

Thus, we accept the following dependence:

$$x = Q + R, \tag{8.134}$$

where $R = (R_1, R_2, R_3, R_4)$ is the vector-valued function containing the components of the tensor of angular velocity. Here the dependence of the function R on the angular velocity tensor is gyroscopic:

$$R = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} = -\frac{1}{v} \begin{pmatrix} 0 & -\omega_6 & \omega_5 & -\omega_3 \\ \omega_6 & 0 & -\omega_4 & \omega_2 \\ -\omega_5 & \omega_4 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}, \tag{8.135}$$

where (h_1, h_2, h_3, h_4) are some positive parameters (cf. [322, 330, 331, 345, 393]).

Now, for our problem, since $x_{1N} \equiv x_{2N} \equiv 0$, we have

$$x_{3N} = Q_3 - \frac{h_1}{v}(\omega_4 - \omega_5), \quad x_{4N} = Q_4 - \frac{h_1}{v}(\omega_3 - \omega_2). \tag{8.136}$$

3.2. Reduced system. Similarly to the choice of Chaplygin analytical functions (see [50, 51])

$$Q_3 = A \sin\alpha \cos\beta_1, \quad Q_4 = A \sin\alpha \sin\beta_1, \quad A > 0, \tag{8.137}$$

we take the dynamical functions s, x_{3N}, and x_{4N} of the following form:

$$\begin{aligned} &s(\alpha) = B\cos\alpha, && B > 0, \\ &x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = A\sin\alpha\cos\beta_1 - \frac{h}{v}(\omega_4 - \omega_5), && h = h_1 > 0, \quad v \neq 0, \\ &x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = A\sin\alpha\sin\beta_1 - \frac{h}{v}(\omega_3 - \omega_2), && h = h_2 > 0, \quad v \neq 0, \end{aligned} \tag{8.138}$$

which shows that an additional dependence of the damping (or accelerating in some domains of the phase space) moment of the nonconservative forces is also present in the system considered (i.e., the moment depends on the angular velocity). Moreover, by virtue of dynamical symmetry (8.17) of the body we have $h_1 = h_2$ and $h_3 = h_4$.

Further, we use the first way of studying the system, but also take into account the second way (see p. 484).

We introduce the following phase variables:

$$\begin{aligned} u_1 &= \omega_2 - \omega_3, \\ u_2 &= \omega_4 - \omega_5, \\ u_3 &= \omega_2\cos\beta_2 - \omega_3\sin\beta_2, \\ u_4 &= \omega_4\cos\beta_2 - \omega_5\sin\beta_2. \end{aligned} \tag{8.139}$$

These coordinates are well defined for

$$\cos\beta_2 \neq \sin\beta_2, \tag{8.140}$$

and the Jacobian of the mapping is equal to

$$-\frac{1}{(\cos\beta_2 - \sin\beta_2)^2}; \tag{8.141}$$

the inverse transformation is defined as follows:

$$\begin{aligned} \omega_2 &= \frac{u_3 - u_1\sin\beta_2}{\cos\beta_2 - \sin\beta_2}, & \omega_3 &= \frac{u_3 - u_1\cos\beta_2}{\cos\beta_2 - \sin\beta_2}, \\ \omega_4 &= \frac{u_4 - u_2\sin\beta_2}{\cos\beta_2 - \sin\beta_2}, & \omega_5 &= \frac{u_4 - u_2\cos\beta_2}{\cos\beta_2 - \sin\beta_2}. \end{aligned} \tag{8.142}$$

The particular case where

$$\cos\beta_2 = \sin\beta_2, \tag{8.143}$$

which simplifies the dynamical equations, can be considered separately.

Under the condition (8.138), Eqs. (8.33)–(8.38) outside the manifold

$$O_3 = \left\{(\alpha, \beta_1, \omega_2, \omega_3, \omega_4, \omega_5) \in \mathbf{R}^6 : \ \alpha = \frac{\pi}{2} + \pi k, \ k \in \mathbf{Z}\right\} \tag{8.144}$$

(and only outside it) are transformed to the following equations:

$$\dot\alpha - u_3\sin\beta_1 + u_4\cos\beta_1 - \sigma n_0^2 v\sin\alpha + \sigma H_1'\big[-u_1\sin\beta_1 + u_2\cos\beta_1\big] = 0, \tag{8.145}$$

$$\dot\beta_1\sin\alpha - \cos\alpha\big[u_3\cos\beta_1 + u_4\sin\beta_1\big] - \sigma H_1'\cos\alpha\big[u_1\cos\beta_1 + u_2\sin\beta_1\big] = 0, \tag{8.146}$$

$$\dot{u}_1 = -n_0^2 v^2 r_1 \sin\alpha\cos\alpha\sin\beta_1 - \frac{Bvh}{I_1+I_3} r_1 u_1 \cos\alpha, \tag{8.147}$$
$$\dot{u}_2 = n_0^2 v^2 r_1 \sin\alpha\cos\alpha\cos\beta_1 - \frac{Bvh}{I_1+I_3} r_1 u_2 \cos\alpha, \tag{8.148}$$
$$\dot{u}_3 = -n_0^2 v^2 \sin\alpha\cos\alpha\sin\beta_1\cos(\gamma+\beta_2) - \frac{Bvh}{I_1+I_3} u_1 \cos\alpha\cos(\gamma+\beta_2), \tag{8.149}$$
$$\dot{u}_4 = n_0^2 v^2 \sin\alpha\cos\alpha\cos\beta_1\cos(\gamma+\beta_2) - \frac{Bvh}{I_1+I_3} u_2 \cos\alpha\cos(\gamma+\beta_2), \tag{8.150}$$

where

$$r_1 = \cos\gamma - \sin\gamma \neq 0, \quad n_0^2 = \frac{AB}{I_1+I_3}, \quad H_1' = \frac{Bh}{I_1+I_3}. \tag{8.151}$$

We note that the particular case

$$\cos\gamma = \sin\gamma \quad (r_1 = 0), \tag{8.152}$$

which simplifies the dynamical equations, can also be considered separately (similarly to the case (8.143)).

Let us introduce the following phase variables:

$$\begin{aligned} v_1 &= -u_1\sin\beta_1 + u_2\cos\beta_1, \\ v_2 &= u_1\cos\beta_1 + u_2\sin\beta_1, \\ v_3 &= -u_3\sin\beta_1 + u_4\cos\beta_1, \\ v_4 &= u_3\cos\beta_1 + u_4\sin\beta_1. \end{aligned} \tag{8.153}$$

Then, outside the manifold

$$O_4 = \left\{(\alpha, \beta_1, u_1, u_2, u_3, u_4) \in \mathbf{R}^6 : \ \beta_1 = \pi k, \ k \in \mathbf{Z}\right\} \tag{8.154}$$

(and only outside it), system (8.145)–(8.150) has the form

$$\dot{\alpha} = -v_3 - bH_1 v_1 + b\sin\alpha, \tag{8.155}$$
$$\dot{\beta}_1 = [v_4 + bH_1 v_2]\frac{\cos\alpha}{\sin\alpha}, \tag{8.156}$$
$$\dot{v}_1 = n_0^2 v^2 r_1 \sin\alpha\cos\alpha - H_1' v r_1 v_1 \cos\alpha - v_2\cdot[v_4 + bH_1 v_2]\frac{\cos\alpha}{\sin\alpha}, \tag{8.157}$$
$$\dot{v}_2 = -H_1' v r_1 v_2 \cos\alpha + v_1\cdot[v_4 + bH_1 v_2]\frac{\cos\alpha}{\sin\alpha}, \tag{8.158}$$
$$\dot{v}_3 = n_0^2 v^2 \sin\alpha\cos\alpha\cos(\gamma+\beta_2) - H_1' v v_1 \cos\alpha\cos(\gamma+\beta_2) - v_4\cdot[v_4 + bH_1 v_2]\frac{\cos\alpha}{\sin\alpha}, \tag{8.159}$$
$$\dot{v}_4 = -H_1' v v_2 \cos\alpha\cos(\gamma+\beta_2) + v_3\cdot[v_4 + bH_1 v_2]\frac{\cos\alpha}{\sin\alpha}, \tag{8.160}$$

where, as before, we introduce the dimensionless parameters as follows:

$$n_0^2 = \frac{AB}{I_1+I_3}, \quad b = \sigma n_0, \quad [b] = 1, \quad H_1 = \frac{H_1'}{n_0} = \frac{Bh}{(I_1+I_3)n_0}, \quad [H_1] = 1. \tag{8.161}$$

We also introduce the following auxiliary change of a part of the phase variables as follows:

$$s_1 = v_3 + bH_1 v_1, \quad s_2 = v_4 + bH_1 v_2. \tag{8.162}$$

Then system (8.155)–(8.160) after the introduction of the dimensionless variables and differentiation

$$v_k \mapsto n_0 v v_k, \quad k = 1, \ldots, 4, \quad \langle\cdot\rangle = n_0 v \langle'\rangle \tag{8.163}$$

takes the form

$$\alpha' = -s_1 + b\sin\alpha, \tag{8.164}$$
$$\beta_1' = s_2\frac{\cos\alpha}{\sin\alpha}, \tag{8.165}$$

$$s_1' = R_1 \sin\alpha\cos\alpha - s_2^2\frac{\cos\alpha}{\sin\alpha} - R_1 H_1 v_1 \cos\alpha, \tag{8.166}$$

$$s_2' = s_1 s_2 \frac{\cos\alpha}{\sin\alpha} - R_1 H_1 v_2 \cos\alpha, \tag{8.167}$$

$$v_1' = R_2 \sin\alpha\cos\alpha - s_2 v_2 \frac{\cos\alpha}{\sin\alpha} - H_1 R_2 v_1 \cos\alpha, \tag{8.168}$$

$$v_2' = s_2 v_1 \frac{\cos\alpha}{\sin\alpha} - H_1 R_2 v_2 \cos\alpha, \tag{8.169}$$

where

$$R_1 = bH_1(\cos\gamma - \sin\gamma) + \cos(\gamma + \beta_2), \quad R_2 = r_1 = \cos\gamma - \sin\gamma. \tag{8.170}$$

We see that for $H_1 = 0$, the independent fourth-order subsystem (8.164)–(8.167) on the tangent bundle $T\mathbf{S}^2$ of the two-dimensional sphere $\mathbf{S}^2\{0 < \alpha < \pi,\ 0 \le \beta_1 < 2\pi\}$ can be extracted from system (8.164)–(8.169). Further, the independent third-order subsystem (8.164), (8.166), (8.167) can be extracted from fourth-order subsystem (8.164)–(8.167); this third-order subsystem can be considered on its own three-dimensional phase manifold.

This fact is quite obvious since, for $H_1 = 0$, the moment of forces is independent of the angular velocity tensor (see the previous section and system (8.80)–(8.82), (8.85)). This allows one to integrate completely the fourth-order system (8.164)–(8.167) and, therefore, the sixth-order system (8.164)–(8.169), since there exist two independent analytical first integrals (8.31) and (8.32) or (8.119) and (8.120) (see above about the discussion on the two ways of the study, p. 484).

In this case, the inequality $H_1 \neq 0$ is important. Therefore, we transform the analytical first integrals (8.31) and (8.32) or (8.119) and (8.120). Their explicit form is as follows:

$$\frac{u_3 - u_1 \sin\beta_2}{\cos\beta - 2 - \sin\beta_2}\sin\gamma - \frac{u_3 - u_1\cos\beta_2}{\cos\beta - 2 - \sin\beta_2}\cos\gamma = W_1' = \text{const}, \tag{8.171}$$

$$\frac{u_4 - u_2 \sin\beta_2}{\cos\beta - 2 - \sin\beta_2}\sin\gamma - \frac{u_4 - u_2\cos\beta_2}{\cos\beta - 2 - \sin\beta_2}\cos\gamma = W_2' = \text{const}\,. \tag{8.172}$$

If we consider the case (8.20) (i.e., in particular, the case where the value β_2 is identically constant along phase trajectories), then the following analytical functions are constant on phase trajectories of the considered system:

$$u_3(\sin\gamma - \cos\gamma) + u_1\cos(\gamma + \beta_2) = W_1^0 = \text{const}, \tag{8.173}$$

$$u_4(\sin\gamma - \cos\gamma) + u_2\cos(\gamma + \beta_2) = W_2^0 = \text{const}\,. \tag{8.174}$$

In other variables, these two invariant relations have the form

$$(v_2\cos\beta_1 - v_1\sin\beta_1)\cos(\gamma+\beta_2) + (v_4\cos\beta_1 - v_3\sin\beta_1)(\sin\gamma - \cos\gamma) = W_1^0 = \text{const}, \tag{8.175}$$

$$(v_2\sin\beta_1 + v_1\cos\beta_1)\cos(\gamma+\beta_2) + (v_4\sin\beta_1 + v_3\cos\beta_1)(\sin\gamma - \cos\gamma) = W_2^0 = \text{const} \tag{8.176}$$

or

$$R_1 v_2 \cos\beta_1 - R_1 v_1 \sin\beta_1 + R_2[s_1\sin\beta_1 - s_2\cos\beta_1] = W_1^0 = \text{const}, \tag{8.177}$$

$$R_1 v_2 \sin\beta_1 + R_1 v_1 \cos\beta_1 - R_2[s_1\cos\beta_1 + s_2\sin\beta_1] = W_2^0 = \text{const}, \tag{8.178}$$

where

$$R_1 = \cos(\gamma+\beta_2) + bH_1(\cos\gamma - \sin\gamma), \quad R_2 = \cos\gamma - \sin\gamma. \tag{8.179}$$

We express the values v_1 and v_2 from the relations (8.177) and (8.178). We have

$$v_2 R_1 = R_2 s_2 + \psi_1(\beta_1, W_1^0, W_2^0), \tag{8.180}$$

$$v_1 R_1 = R_2 s_1 + \psi_2(\beta_1, W_1^0, W_2^0), \tag{8.181}$$

where

$$\psi_1(\beta_1, W_1^0, W_2^0) = W_1^0 \cos\beta_1 + W_2^0 \sin\beta_1,$$
$$\psi_2(\beta_1, W_1^0, W_2^0) = W_2^0 \cos\beta_1 - W_1^0 \sin\beta_1. \tag{8.182}$$

Then system (8.164)–(8.167) becomes the following independent fourth-order system:

$$\alpha' = -s_1 + b\sin\alpha, \tag{8.183}$$

$$s_1' = R_1 \sin\alpha\cos\alpha - s_2^2 \frac{\cos\alpha}{\sin\alpha} - R_2 H_1 s_1 \cos\alpha - H_1 \psi_2(\beta_1, W_1^0, W_2^0)\cos\alpha, \tag{8.184}$$

$$s_2' = s_1 s_2 \frac{\cos\alpha}{\sin\alpha} - R_2 H_1 s_2 \cos\alpha - H_1 \psi_1(\beta_1, W_1^0, W_2^0)\cos\alpha, \tag{8.185}$$

$$\beta_1' = s_2 \frac{\cos\alpha}{\sin\alpha}. \tag{8.186}$$

System (8.183)–(8.186) can be considered as system (8.164)–(8.167) reduced to the levels (W_1^0, W_2^0) of the analytical first integrals (8.177) and (8.178).

Obviously,

$$\psi_1(\beta_1, 0, 0) \equiv \psi_2(\beta_1, 0, 0) \equiv 0. \tag{8.187}$$

Therefore, we consider system (8.183)–(8.186) on the zero levels of the analytical first integrals (8.177) and (8.178):

$$W_1^0 = W_2^0 = 0. \tag{8.188}$$

Then it takes the form

$$\alpha' = -s_1 + b\sin\alpha, \tag{8.189}$$

$$s_1' = R_1 \sin\alpha\cos\alpha - s_2^2 \frac{\cos\alpha}{\sin\alpha} - R_2 H_1 s_1 \cos\alpha, \tag{8.190}$$

$$s_2' = s_1 s_2 \frac{\cos\alpha}{\sin\alpha} - R_2 H_1 s_2 \cos\alpha, \tag{8.191}$$

$$\beta_1' = s_2 \frac{\cos\alpha}{\sin\alpha}. \tag{8.192}$$

This system can be considered on the tangent bundle $T\mathbf{S}^2$ of the two-dimensional sphere $\mathbf{S}^2\{0 < \alpha < \pi,\ 0 \le \beta_1 < 2\pi\}$. Moreover, we can extract the independent third-order subsystem (8.189)–(8.191) on its own three-dimensional phase manifold from system (8.189)–(8.192).

Thus, for integration of the sixth-order system, we have first used the first way of studying (see p. 484) and have not taken into account the existence of two independent analytical first integrals (8.31) and (8.32). Subsequently, we have reduced the considered sixth-order system to levels (in particular, to the zero level) of the first integrals, i.e., we have used the second way of studying.

3.3. Complete list of invariant relations. System (8.189)–(8.191) is similar to system (4.74) that arises in the dynamics of a three-dimensional rigid body in a nonconservative field (see Chap. 4). The phase variables z_1 and z_2 in system (4.74) correspond to the phase variables s_2 and s_1 of system (8.189)–(8.191).

Compare the third-order system (8.189)–(8.191) with the nonautonomous second-order system

$$\frac{ds_1}{d\alpha} = \frac{R_1 \sin\alpha\cos\alpha - s_2^2 \cos\alpha/\sin\alpha - R_2 H_1 s_1 \cos\alpha}{-s_1 + b\sin\alpha},$$
$$\frac{ds_2}{d\alpha} = \frac{s_1 s_2 \cos\alpha/\sin\alpha - R_2 H_1 s_2 \cos\alpha}{-s_1 + b\sin\alpha}. \tag{8.193}$$

We rewrite system (8.193) in the algebraic form using the substitution $\tau = \sin\alpha$:

$$\begin{aligned} \frac{ds_1}{d\tau} &= \frac{R_1\tau - s_2^2/\tau - R_2H_1s_1}{-s_1 + b\tau}, \\ \frac{ds_2}{d\tau} &= \frac{s_1s_2/\tau - R_2H_1s_2}{-s_1 + b\tau}. \end{aligned} \tag{8.194}$$

If we introduce the homogeneous variables by the formulas

$$s_1 = t_1\tau, \quad s_2 = t_2\tau, \tag{8.195}$$

we reduce system (8.194) to the following form:

$$\begin{aligned} \tau\frac{dt_1}{d\tau} + t_1 &= \frac{R_1 - t_2^2 - R_2H_1t_1}{-t_1 + b}, \\ \tau\frac{dt_2}{d\tau} + t_2 &= \frac{t_1t_2 - R_2H_1t_2}{-t_1 + b}, \end{aligned} \tag{8.196}$$

which is equivalent to

$$\begin{aligned} \tau\frac{dt_1}{d\tau} &= \frac{t_1^2 - t_2^2 - (b + R_2H_1)t_1 + R_1}{-t_1 + b}, \\ \tau\frac{dt_2}{d\tau} &= \frac{2t_1t_2 - (b + R_2H_1)t_2}{-t_1 + b}. \end{aligned} \tag{8.197}$$

We compare the second-order system (8.197) with the nonautonomous first-order equation

$$\frac{dt_1}{dt_2} = \frac{t_1^2 - t_2^2 - (b + R_2H_1)t_1 + R_1}{2t_1t_2 - (b + R_2H_1)t_2}, \tag{8.198}$$

which can be easily reduced to the exact differential equation

$$d\left(\frac{t_1^2 + t_2^2 - (b + R_2H_1)t_1 + R_1}{t_2}\right) = 0. \tag{8.199}$$

Therefore, Eq. (8.198) possesses the first integral

$$\frac{t_1^2 + t_2^2 - (b + R_2H_1)t_1 + R_1}{t_2} = C_1 = \text{const}, \tag{8.200}$$

which in the old variables has the form

$$\frac{s_1^2 + s_2^2 - (b + R_2H_1)s_1\sin\alpha + R_1\sin^2\alpha}{s_2\sin\alpha} = C_1 = \text{const}\,. \tag{8.201}$$

Remark 8.3. We consider system (8.189)–(8.191) with variable dissipation with zero mean (see [67, 68, 243, 260, 262, 265, 282–284, 286, 286, 291, 295, 309, 324, 333, 340, 384, 391, 404–408, 412, 413, 421, 437, 438], which becomes conservative for $b = R_2H_1$:

$$\begin{aligned} \alpha' &= -s_1 + b\sin\alpha, \\ s_1' &= R_1\sin\alpha\cos\alpha - s_2^2\frac{\cos\alpha}{\sin\alpha} - bs_1\cos\alpha, \\ s_2' &= s_1s_2\frac{\cos\alpha}{\sin\alpha} - bs_2\cos\alpha. \end{aligned} \tag{8.202}$$

It has two analytical first integrals of the form

$$s_1^2 + s_2^2 - 2bs_1\sin\alpha + R_1\sin^2\alpha = C_1^* = \text{const}, \tag{8.203}$$

$$s_2\sin\alpha = C_2^* = \text{const}\,. \tag{8.204}$$

Obviously, the ratio of the first integrals (8.203) and (8.204) is also a first integral of system (8.202). However, for $b \neq R_2H_1$, both functions

$$s_1^2 + s_2^2 - (b + R_2H_1)s_1 \sin\alpha + R_1 \sin^2\alpha \tag{8.205}$$

and (8.204) are not first integrals of system (8.189)–(8.191), but their ratio is a first integral of system (8.189)–(8.191) for any b and R_2H_1.

Further, we find the explicit form of an additional first integral of the third-order system (8.189)–(8.191). For this, we transform the invariant relation (8.200) for $u_1 \neq 0$ as follows:

$$\left(t_1 - \frac{b + R_2H_1}{2}\right)^2 + \left(t_2 - \frac{C_1}{2}\right)^2 = \frac{(b + R_2H_1)^2 + C_1^2 - 4R_1}{4}. \tag{8.206}$$

We see that the parameters of this invariant relation must satisfy the condition

$$(b + R_2H_1)^2 + C_1^2 - 4R_1 \geq 0, \tag{8.207}$$

and the phase space of system (8.189)–(8.191) is stratified into a family of surfaces defined by Eq. (8.206).

Thus, by relation (8.200), the first equation of system (8.197) has the form

$$\tau\frac{dt_1}{d\tau} = \frac{2t_1^2 - 2(b + R_2H_1)t_1 + 2R_1 - C_1U_1(C_1, t_1)}{b - t_1}, \tag{8.208}$$

where

$$\begin{gathered} U_1(C_1, t_1) = \frac{1}{2}\{C_1 \pm U_2(C_1, t_1)\}, \\ U_2(C_1, t_1) = \sqrt{C_1^2 - 4(R_1 - (b + R_2H_1)t_1 + t_1^2)}, \end{gathered} \tag{8.209}$$

and the integration constant C_1 is chosen from condition (8.207).

Therefore, the quadrature for the search for an additional first integral of system (8.189)–(8.191) has the form

$$\int \frac{d\tau}{\tau} = \int \frac{(b - t_1)dt_1}{2(R_1 - (b + R_2H_1)t_1 + t_1^2) - C_1\{C_1 \pm U_2(C_1, t_1)\}/2}. \tag{8.210}$$

Obviously, the left-hand side (up to an additive constant) is equal to

$$\ln|\sin\alpha|. \tag{8.211}$$

If

$$t_1 - \frac{b + R_2H_1}{2} = w_1, \quad b_1^2 = (b + R_2H_1)^2 + C_1^2 - 4R_1, \tag{8.212}$$

then the right-hand side of Eq. (8.210) has the form

$$\begin{aligned} -\frac{1}{4}\int &\frac{d(b_1^2 - 4w_1^2)}{(b_1^2 - 4w_1^2) \pm C_1\sqrt{b_1^2 - 4w_1^2}} \\ &- (b + R_2H_1)\int \frac{dw_1}{(b_1^2 - 4w_1^2) \pm C_1\sqrt{b_1^2 - 4w_1^2}} \\ &= -\frac{1}{2}\ln\left|\frac{\sqrt{b_1^2 - 4w_1^2}}{C_1} \pm 1\right| \pm \frac{b + R_2H_1}{2}I_1, \end{aligned} \tag{8.213}$$

where

$$I_1 = \int \frac{dw_3}{\sqrt{b_1^2 - w_3^2}(w_3 \pm C_1)}, \quad w_3 = \sqrt{b_1^2 - 4w_1^2}. \tag{8.214}$$

In the calculation of integral (8.214), the following three cases are possible.

I. $(b + R_2H_1)^2 - 4R_1 > 0$:

$$I_1 = -\frac{1}{2\sqrt{(b+R_2H_1)^2-4R_1}} \ln\left|\frac{\sqrt{(b+R_2H_1)^2-4R_1}+\sqrt{b_1^2-w_3^2}}{w_3 \pm C_1} \pm \frac{C_1}{\sqrt{(b+R_2H_1)^2-4R_1}}\right|$$
$$+ \frac{1}{2\sqrt{(b+R_2H_1)^2-4R_1}} \ln\left|\frac{\sqrt{(b+R_2H_1)^2-4R_1}-\sqrt{b_1^2-w_3^2}}{w_3 \pm C_1} \mp \frac{C_1}{\sqrt{(b+R_2H_1)^2-4R_1}}\right| + \text{const}\,. \tag{8.215}$$

II. $(b+R_2H_1)^2-4R_1<0$:

$$I_1 = \frac{1}{\sqrt{4R_1-(b+R_2H_1)^2}} \arcsin\frac{\pm C_1w_3+b_1^2}{b_1(w_3\pm C_1)} + \text{const}\,. \tag{8.216}$$

III. $(b+R_2H_1)^2-4R_1=0$:

$$I_1 = \mp\frac{\sqrt{b_1^2-w_3^2}}{C_1(w_3\pm C_1)} + \text{const}\,. \tag{8.217}$$

When we return to the variable

$$w_1 = \frac{s_1}{\sin\alpha} - \frac{b+R_2H_1}{2}, \tag{8.218}$$

we obtain the final form for the value I_1:

I. $(b+R_2H_1)^2-4R_1>0$:

$$I_1 = -\frac{1}{2\sqrt{(b+R_2H_1)^2-4R_1>0}} \ln\left|\frac{\sqrt{(b+R_2H_1)^2-4R_1}\pm 2w_1}{\sqrt{b_1^2-4w_1^2}\pm C_1} \pm \frac{C_1}{\sqrt{(b+R_2H_1)^2-4R_1}}\right|$$
$$+ \frac{1}{2\sqrt{(b+R_2H_1)^2-4R_1}} \ln\left|\frac{\sqrt{(b+R_2H_1)^2-4R_1}\mp 2w_1}{\sqrt{b_1^2-4w_1^2}\pm C_1} \mp \frac{C_1}{\sqrt{(b+R_2H_1)^2-4R_1}}\right| + \text{const}\,. \tag{8.219}$$

II. $(b+R_2H_1)^2-4R_1<0$:

$$I_1 = \frac{1}{\sqrt{4R_1-(b+R_2H_1)^2}} \arcsin\frac{\pm C_1\sqrt{b_1^2-4w_1^2}+b_1^2}{b_1(\sqrt{b_1^2-4w_1^2}\pm C_1)} + \text{const}\,. \tag{8.220}$$

III. $(b+R_2H_1)^2-4R_1=0$:

$$I_1 = \mp\frac{2w_1}{C_1(\sqrt{b_1^2-4w_1^2}\pm C_1)} + \text{const}\,. \tag{8.221}$$

Thus, we have found an additional first integral of the third-order system (8.189)–(8.191), i.e., we have a complete set of first integrals that are transcendental functions of their phase variables.

Remark 8.4. We must formally substitute the left-hand side of the first integral (8.200) in the expression of the found first integral instead of C_1.

Then the obtained additional first integral has the following structure (which is similar to the form of the transcendental first integral in the planar dynamics):

$$\ln|\sin\alpha| + G_2\left(\sin\alpha, \frac{s_1}{\sin\alpha}, \frac{s_2}{\sin\alpha}\right) = C_2 = \text{const}\,. \tag{8.222}$$

Thus, we have already found two independent first integrals for integration of the fourth-order system (8.189)–(8.192). To complete integration, it suffices to find an additional first integral that “attaches” Eq. (8.192).

Since

$$\frac{dt_2}{d\tau} = \frac{2t_1t_2-(b+R_2H_1)t_2}{(b-t_1)\tau}, \quad \frac{d\beta_1}{d\tau} = \frac{t_2}{(b-t_1)\tau}, \tag{8.223}$$

we have

$$\frac{dt_2}{d\beta_1} = 2t_1 - (b + R_2H_1). \tag{8.224}$$

It is obvious that for $t_2 \neq 0$ the following equality holds:

$$t_1 = \frac{1}{2}\left((b + R_2H_1) \pm \sqrt{b_1^2 - (2t_2 - C_1)^2}\right), \tag{8.225}$$

$$b_1^2 = (b + R_2H_1)^2 + C_1^2 - 4R_1,$$

and, therefore, integration of the quadrature

$$\beta_1 + \text{const} = \pm \int \frac{dt_2}{\sqrt{b_1^2 - (2t_2 - C_1)^2}} \tag{8.226}$$

leads to the invariant relation

$$2(\beta_1 + C_3) = \pm \arcsin \frac{2t_1 - C_1}{\sqrt{(b + R_2H_1)^2 + C_1^2 - 4R_1}}, \quad C_3 = \text{const}\,. \tag{8.227}$$

In other words, the equality

$$\sin\left[2(\beta_1 + C_3)\right] = \pm \frac{2t_2 - C_1}{\sqrt{(b + R_2H_1)^2 + C_1^2 - 4R_1}} \tag{8.228}$$

holds and, after the transition to the old variables,

$$\sin\left[2(\beta_1 + C_3)\right] = \pm \frac{2s_2 - C_1 \sin\alpha}{\sqrt{(b + R_2H_1)^2 + C_1^2 - 4R_1} \sin\alpha}. \tag{8.229}$$

Thus, we have obtained an additional invariant relation that "attaches" Eq. (8.192). However, we must formally substitute the left-hand side of (8.200) into the last expression instead of C_1.

We perform some transformations that lead to the following explicit form of the additional first integral (here Eq. (8.200) is used):

$$\tan^2\left[2(\beta_1 + C_3)\right] = \frac{(t_2^2 - t_1^2 + (b + R_2H_1)t_1 - R_1)^2}{t_2^2(2t_1 - (b + R_2H_1))^2}. \tag{8.230}$$

Returning to the old coordinates, we obtain an additional invariant relation in the form

$$\tan^2\left[2(\beta_1 + C_3)\right] = \frac{(s_2^2 - s_1^2 + (b + R_2H_1)s_1 \sin\alpha - R_1 \sin^2\alpha)^2}{s_2^2(2s_1 - (b + R_2H_1)\sin\alpha)^2}, \tag{8.231}$$

or, finally,

$$\beta_1 \pm \frac{1}{2} \arctan \frac{s_2^2 - s_1^2 + (b + R_2H_1)s_1 \sin\alpha - R_1 \sin^2\alpha}{s_2(2s_1 - (b + R_2H_1)\sin\alpha)} = C_3 = \text{const}\,. \tag{8.232}$$

Therefore, the system of dynamical equations (8.4)–(8.7), (8.10)–(8.15) under the condition (8.138) has nine invariant relations in the considered case: the analytical nonintegrable constraints (8.20), the cyclic first integrals (8.18) and (8.19), the analytical first integrals (8.31) and (8.32), the first integral (8.201), the first integral expressed by relations (8.215)–(8.222), which is a transcendental function of its phase variables (in the sense of the complex analysis) and is expressed as a finite combination of elementary functions, and, finally, the transcendental first integral (8.232).

Theorem 8.3. *System* (8.4)–(8.7), (8.10)–(8.15) *under the conditions* (8.20), (8.138), (8.19), *and* (8.188) *possesses nine invariant relations* (*a complete set*), *three of which are transcendental functions from the point of view of the complex analysis. All the relations are expressed as finite combinations of elementary functions.*

We also note that, in a similar theorem (see Theorem 8.1 of this chapter), the question is on a complete set of first integrals consisting of eight first integrals, although there exist nine first integrals. But in the proof of Theorem 8.1, we use the first way of studying (see p. 484), which implies the introduction of phase coordinates (in particular, w_k, $k = 1, \ldots, 4$) in which the vector field of the system admits additional stratifications. Here, the analytical first integrals (8.31) and (8.32) are not used directly; this allows one to use a fewer first integrals.

And, in the proof of Theorem 8.3, we have used the second way of studying, which implies the reduction of the system examined to (zero) levels of the analytical first integrals (8.31) and (8.32). In this case, we need a complete set of first integrals.

3.4. Topological analogies. We consider the following third-order system:

$$\ddot{\xi} + (b_* - H_1^*)\dot{\xi}\cos\xi + R_3\sin\xi\cos\xi - \dot{\eta}_1^2\frac{\sin\xi}{\cos\xi} + H_1^{**}[W_1^0\sin\eta_1 - W_2^0\cos\eta_1] = 0,$$
$$\ddot{\eta}_1 + (b_* - H_1^*)\dot{\eta}_1\cos\xi + \dot{\xi}\dot{\eta}_1\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + H_1^{**}[W_1^0\cos\eta_1 + W_2^0\sin\eta_1] = 0, \quad b_* > 0, \quad H_1^{**} > 0, \tag{8.233}$$

which describes a fixed spherical pendulum in a flowing medium in the case where the moment of forces depends on the angular velocity, i.e., a mechanical system in a nonconservative force field (see [120, 162, 188, 201, 203, 235, 238, 276, 316, 317, 319, 320, 338, 359, 360, 376, 377, 386, 392, 429, 442]). In contrast to previous chapters, the order of such a system is equal to 4 (but not 3) since the phase variable η_1 is not cyclic and hence the phase space is not stratified and the order reduction does not occur.

The phase space of this system is the tangent bundle

$$T\mathbf{S}^2\{\dot{\xi}, \dot{\eta}_1, \xi, \eta_1\} \tag{8.234}$$

of the two-dimensional sphere $\mathbf{S}^2\{\xi, \eta_1\}$, where the equation of the large circles

$$\dot{\eta}_1 \equiv 0 \tag{8.235}$$

defined a family of integral manifolds only for $W_1^0 = W_2^0 = 0$.

It is easy to verify that system (8.233) is equivalent to the dynamical system with variable dissipation with zero mean on the tangent bundle (8.234) of the two-dimensional sphere. Moreover, the following theorem holds.

Theorem 8.4. *System* (8.4)–(8.7), (8.10)–(8.15) *under conditions* (8.20), (8.138), *and* (8.19) *is equivalent to dynamical system* (8.233).

Indeed, it suffices to take $\alpha = \xi$, $\beta_1 = \eta_1$, $b = -b_*$, $H_1 = H_1^{**}$, $R_2H_1 = -H_1^*$, and $R_1 - bR_2H_1 = R_3$.

On more general topological analogies, see also [253, 265, 267, 309, 340, 342].

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