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INDEPENDENT SETS AND CHROMATIC NUMBERS OF CIRCLE GRAPHS

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Let the vertices of a circle graph be divided into several groups. This paper contains lower bounds on the size of an independent set that can be contained in one group of this subdivision. Bibliography: 7 titles.

For a graph $G = (V,\ E)$ we use the following notation:
$V(G)$ and $E(G)$ are the sets of vertices and edges of the graph G, respectively;
$d_G(x)$ is the *degree* of a vertex $x \in V(G)$, i.e., the number of edges incident to x;
$\Delta(G)$ is the *maximal* vertex degree of the graph G;
$\delta(G)$ is the *minimal* vertex degree of the graph G;
$\chi(G)$ is the *chromatic number* of the graph G, i.e., the minimal number of colors in a proper vertex coloring of G;
$\omega(G)$ is the *clique number* of the graph G, i.e., the size of a maximal clique as an induced subgraph of G;
$\bar{G}$ is the *additional* graph to $G = (V, E)$, i.e., the graph on the vertex set V the edge set of which consists of all edges that do not belong to E.

1. Introduction

A *circle graph* is a graph of intersections of a finite set of chords of a circle. The vertices of such a graph are chords of a circle and two vertices are adjacent if and only if the corresponding chords have a common inner point. Denote by CIR the set of all circle graphs.

Properties of circle graphs are being studied from the midtwentieth century. The problem of how to verify whether a graph belongs to CIR or not is NP-hard. Mathematicians have studied different properties of circle graphs. In paper [1], one can find algorithms for searching cliques and independent sets in circle graphs. In [2], circle graphs are characterized in terms of intersections of cocyclic paths.

Set the notation $f(k) = \max(\chi(G) \mid G \in CIR;\ \omega(G) \leq k)$. The question of finding tight bounds on $f(k)$ was considered in several papers. For arbitrary k, the best known bound is $f(k) \leq 2^{k+6}$ (see [3]). For small k, more tight bounds are known: $f(2) = 5$ (papers [4] and [5]) and $f(3) \leq 30$ (see [6]).

Consider a slightly different approach to circle graphs. Denote by $CIR(n)$ the complete circle graph on n points, i.e. the graph of intersections of all chords that connect pairs of given n points of a circle. This graph has $\frac{n(n-1)}{2}$ vertices and C_n^4 edges. Let the vertices of this graph be divided into m sets such that no chords with a common end belong to the same set. Clearly, $m \geq n$. The following question arises: Is it true that there is a rather large independent set in one of these sets? The question of how to estimate the size of such an independent set if we fix the value $\frac{m}{n} = k$ is rather nontrivial. We answer it in the following lemma.

Lemma 1. *Let all vertices of the graph $CIR(n)$ be divided into m sets such that no chords with a common end belong to the same set. Then one of these sets contains an independent set of size at least $\lceil \frac{n}{m} \ln[\frac{n-1}{2}] \rceil$.*

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That means that for sufficiently large n and fixed k, a rather large independent set can be found in some set. The following theorem is a consequence of this lemma.

Theorem 1. *Let $k \geq 1$. Consider a graph additional to $CIR(n)$. Let its vertices be divided into kn sets such that no chords with a common end belong to the same set. Then the chromatic number of the subgraph induced on one of these sets is at least $\lceil \frac{1}{k} \ln[\frac{n-1}{2}]\rceil$.*

We also prove the following corollary.

Corollary 1. *Let $n > 2e^{2t-2} + 3$ for some positive integer t and integers from 1 to n are written at n points of a circle in an arbitrary order. Then one can choose t pairwise disjoint chords such that the sums of numbers at the ends of all chosen chords are equal.*

If we drop the condition that no chords in one set have a common end, we obtain the following results.

Theorem 2. *Let k be a positive number and all vertices of the graph $CIR(n)$ be divided into kn sets. Then the subgraph induced by one of these sets contains an independent set of size at least $\sqrt{\frac{\ln(n/2)}{k}}$.*

Corollary 2. *Assume that n, k, and t be positive integers such that $n > 2e^{2kt^2}$. Let positive integers that do not exceed kn be written at n points of a circle in an arbitrary order. Then one can choose more than t pairwise disjoint chords such that the sums of numbers at the ends of all chosen chords are equal.*

2. Maximal independent sets

We begin with an important lemma.

Lemma 2. *Denote by a_i the number of vertices of degree i in a graph G. Then G has an independent set of size $a_0 + a_1/2 + a_2/3 + \dots$.*

This lemma is known (in a slightly different formulation it can be found in [7, Lemma 25.1]). For completeness, we give here a proof of this fact.

Proof. We prove the statement by induction on the number of vertices. Denote by $S(G)$ the sum $a_0 + a_1/2 + a_2/3 + \dots$. It is enough to prove that $S(G) \leq \omega(\bar{G})$.

The base of induction. If G is a complete graph on n vertices, then $a_0 = a_1 = \dots = a_{n-2} = 0$, $a_{n-1} = n$ and $S(G) = 1$.

The induction step. Let the graph be not complete. We delete from G a vertex A of minimal degree $\delta(G)$ and all vertices adjacent to A. Since G is not complete the obtained graph G_1 has at least one vertex. The induction assumption is true for this graph. Since the vertex A is not adjacent to any vertex of a maximal independent set of the graph G_1, we have $\omega(\bar{G}) \geq \omega(\bar{G}_1) + 1$.

Since we have deleted $\delta(G) + 1$ vertices of degree at least $\delta(G)$ from G, the contribution of deleted vertices to $S(G)$ is at least 1. Since the degrees of remaining vertices have not increased, their contribution to $S(G)$ has not decreased. Thus, $S(G)$ can decrease by at most 1. By induction assumption the resulting sum is at least $\omega(\bar{G}_1) \leq \omega(\bar{G}) - 1$. Hence, $S(G) \leq \omega(\bar{G})$. ☐

We pass to the proof of Lemma 1.

Proof. Assume the contrary. Consider an arbitrary chord A. It divides the circle into two arcs. Let us count the numbers of points distinct from the ends of A in these arcs. We denote the least of these numbers by $\ell(A)$ and call it the *length* of the chord A. Any chord that intersects A has ends in distinct half-planes with respect to A. Hence, A can be intersected by at most $\ell(A)$ other chords of the set that contains it. Therefore, the degree of A in the subgraph induced by the set that contains A does not exceed $\ell(A)$.

Denote the number $[\frac{n-3}{2}]$ by s. Clearly, the graph $CIR(n)$ has exactly n chords of each of the lengths $1, 2, \dots, s$. By Lemma 2, for a set with exactly a_i chords of length i the size of a maximal independent set in the corresponding circle graph is at least $a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \dots + \frac{a_s}{s+1}$. Hence,

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \dots + \frac{a_s}{s+1} < \lceil \frac{n}{m} \ln(s+1) \rceil.$$

Since the left-hand side does not exceed the size of the maximal independent set and the right-hand side is greater than this size, the difference between the right and the left sides is at least 1. Therefore, $a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \dots + \frac{a_s}{s+1} < \frac{n}{m} \ln(s+1)$. We sum such inequalities over all sets of chords. Since the number of chords of each length $i \le s$ is equal to n and each of these chords occurs in exactly one sum, we obtain $n(1 + \frac{1}{2} + \dots + \frac{1}{s+1}) < n \ln(s+1)$. This contradicts the inequality $1 + \frac{1}{2} + \dots + \frac{1}{s+1} > \ln(s+1)$. □

We proceed with the proof of Corollary 1.

Proof. Divide all chords into $2n$ sets: let the ith set consist of all chords with the sum of numbers exactly $i + 2$ at the ends (some sets may be empty). By Lemma 1, one can find $\lceil \frac{1}{2} \ln[\frac{n-1}{2}]\rceil$ chords in some set without common inner point. Since $n > 2e^{2t-2} + 3$, we have $\frac{1}{2} \ln[e^{2t-2} + 1] \ge t - 1$ chords, i.e., at least t chords. □

We pass to the proof of Theorem 2.

Proof. Denote by t the size of the maximal independent set of the graph $CIR(n)$ the chords of which belong to one set of the division. Each point on the circle can be incident to at most t chords of one set, since all these chords form an independent set. We prove that a chord A can be intersected by at most $t\ell(A)$ other chords of the set that contains A. Since any chord of this set that intersects A has ends in distinct half-planes with respect to A and one of these half-planes contains $\ell(A)$ points, the number of such chords does not exceed $t\ell(A)$.

Denote the number $[\frac{n-3}{2}]$ by s. Let a set have exactly a_i chords of length i (for each $i \le s$). By Lemma 2, we have $t \ge a_0 + \frac{a_1}{t+1} + \frac{a_2}{2t+1} + \frac{a_3}{3t+1} + \dots + \frac{a_s}{st+1}$. Summing such inequalities over all sets, we obtain

$$knt \ge n\Big(1 + \frac{1}{t+1} + \frac{1}{2t+1} + \frac{1}{3t+1} \dots + \frac{1}{st+1}\Big). \tag{1}$$

Since the function $f(x) = \frac{1}{x+1}$ decreases for $x > 0$, by the integral estimation we obtain

$$\begin{aligned} 1 + \frac{1}{t+1} + \frac{1}{2t+1} + \dots + \frac{1}{st+1} &\ge \int_0^{s+1} \frac{1}{tx+1}\, dx \\ &= \frac{\ln(t(s+1)+1)}{t} \ge \frac{\ln(s+2)}{t} \ge \frac{\ln(n/2)}{t}. \end{aligned} \tag{2}$$

Substituting this inequality in (1), we deduce that $knt > n\frac{\ln(n/2)}{t}$, whence $t > \sqrt{\frac{\ln(n/2)}{k}}$. Therefore, the subgraph induced by some set of chords has an independent set of size at least $\sqrt{\frac{\ln(n/2)}{k}}$. □

We proceed to the proof of Corollary 2.

Proof. Divide all chords into $2kn$ sets: let the ith set consist of all chords with the sum exactly $i + 1$ at the ends (some sets may be empty). By Theorem 2, one can find at least $\sqrt{\frac{\ln(n/2)}{2k}}$ pairwise disjoint chords in some set. Since $n > 2e^{2kt^2}$, this number is at least $\sqrt{\frac{\ln e^{2kt^2}}{2k}} = t$. □

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REFERENCES

1. F. Gavril, "Algorithms for a maximum clique and a maximum independent set of a circle graph," *Networks*, **3**, 261–273 (1975).
2. H. de Fraysseix, "A characterization of circle graphs," *Europ. J. Combinatorics*, **5**, 223–238 (1984).
3. A. V. Kostochka and J. Kratochvil, "Covering and coloring polygon-circle graphs," *Disc. Math.*, **163**, 299–305 (1997).
4. A. V. Kostochka, "On upper bounds for the chromatic numbers of graphs," *Trudy Inst. Math.*, **10**, 204–226 (1988).
5. A. A. Ageev. "A triangle-free circle graph with chromatic number 5," *Disc. Math.*, **152**, Nos. 1–3, 295–298 (1996).
6. G. V. Nenashev, "An upper bound on the chromatic number of circle graphs without K_4," *Zap. Nauchn. Semin. POMI*, **391**, 149–156 (2011).
7. V. A. Emelichev, O. I. Melnikov, V. I. Sarvanov, and R. I. Tyshkevich, *Lectures in Graph Theory* [in Russian], Nauka, Moscow (1990).