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SMALL DEVIATION PROBABILITIES FOR SUMS OF INDEPENDENT POSITIVE RANDOM VARIABLES WITH A DISTRIBUTION THAT SLOWLY VARIES AT ZERO

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In the note, we study small deviation probabilities for sums of independent, identically distributed positive random variables whose distribution function is slowly varying at zero. Bibliography: 5 titles.

1. Introduction and results

Consider independent copies $\{X_i\}_{i\geq 1}$ of a positive random variable X. Set $S_n = X_1 + \ldots + X_n,\ n \geq 1$.

The present paper is a continuation of the research from [1] and [2], where, in particular, it was proved that if the distribution $V(x) = \mathbf{P}(X < x)$ decreases as "a power" at zero, then, with the notation

$$L(h) = \mathbf{E}e^{-hX}, \quad m(h) = -\big(\log L(h)\big)', \quad \text{and} \quad \sigma^2(h) = \big(\log L(h)\big)'', \quad h \geq 0, \tag{1.1}$$

the small deviation probability of S_n has the classical asymptotics as $n \to \infty$:

$$\mathbf{P}(S_n < x) \sim \frac{1}{h\sigma(h)\sqrt{2\pi n}}\, L^n(h)\, e^{h\,x}, \tag{1.2}$$

uniformly in $0 < x \leq \mu\, n$, where the constant μ is such that $\mu < \mathbf{E}X \leq \infty$ and $h = h(x/n)$ is the unique solution of the equation

$$m(h) = \frac{x}{n}. \tag{1.3}$$

We treat the case where $V(x)$ is a slowly varying function at zero. This case has not been examined in the literature. The results obtained show, in particular, that the asymptotics of type (1.2) in this situation holds for "not too small" $x = x(n)$ only.

Let us introduce the key condition of the present work:

$$\frac{1}{y}\int_0^y u\, dV(u) \sim l(y), \quad y \to +0, \tag{1.4}$$

where the function $l(y)$ is slowly varying at zero (and, without loss of generality, it can be assumed positive and continuous for $0 < y \leq y_0$).

Notice that (1.4) implies that $l(+0) = 0$ and

$$V(y) \sim \widetilde{l}(y) = \int_0^y l(u)/u\; du, \quad y \to +0, \tag{1.5}$$

where the function $\widetilde{l}(y)$ is slowly varying at zero.

Conditions (1.4) and (1.5) are obviously satisfied if the distribution V is absolutely continuous in a vicinity of zero with a density $p(x)$ such that $p(y) \sim l(y)/y,\ y \to +0$.

Now we formulate the results.

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Theorem 1. *Let condition* (1.4) *hold. Then*

$$\mathbf{P}(S_n < x) \sim \frac{\sqrt{hx/n}}{h\sigma(h)}\, L^n(h)\,(hx)^{hx}/\Gamma(1+hx), \quad n \to \infty, \tag{1.6}$$

uniformly in $0 < x \le \mu\, n$, *where the function* $h = h(x/n)$ *is the unique solution of Eq.* (1.3) *and the constant* μ *is such that* $\mu < \mathbf{E}X$.

In particular, if $x > 0$ *is such that* $\lim x/n = 0$, *then*

$$\mathbf{P}(S_n < x) \sim L^n(h)\,(hx)^{hx}/\Gamma(1+hx), \quad n \to \infty. \tag{1.6$'$}$$

Remark 1. If the random variable X is unlattice, then asymptotics (1.2) (or (1.6)) hold uniformly in $\varepsilon\, n \le x \le \mu\, n$ for any $\varepsilon > 0$. In this case, (1.4) is not required.

The end of the section includes an example of application of formula (1.6′).

In the following result, the small deviations of the sum S_n are studied in more detail.

Let (see (1.4) and (1.5))

$$\kappa(y) = l(y)/\widetilde{l}(y) = y\,\big(\log \widetilde{l}(y)\big)', \quad 0 < y \le y_0, \tag{1.7}$$

and let $\kappa(y) = \kappa(y_0)$ on (y_0, ∞).

Note that the function $\kappa(y)$ is slowly varying at zero and $\kappa(y) \to 0$ as $y \to +0$.

Denote $\tau = n\,\kappa(1/h)$.

Theorem 2. *Let* (1.4) *hold. Then the following relations hold as* $n \to \infty$*:*
(1)

$$\mathbf{P}(x - y < S_n \le x) = L^n(h)\, e^{h\,x}\, \frac{1 - e^{-h\,y}}{h\sigma(h)\sqrt{2\pi n}}\, \big(e^{-\beta^2/2} + O\,(\tau^{-1/2} + (\tau^{1/\varepsilon}\, h\, y)^{-1}\big) \tag{1.8}$$

uniformly in $x > 0$, $y > 0$, *and* $h > \varepsilon$ *for any* $\varepsilon > 0$, *where* $\beta = (x - n\,m(h))/(\sigma(h)\sqrt{n})$;
(2)

$$\mathbf{P}(x - y \le S_n < x) = L^n(h)\,\left(h^\tau\, \frac{x^\tau - (x-y)^\tau}{\Gamma(1+\tau)} + o\,(1)\right) \tag{1.9}$$

uniformly in $\{h:\ h > n,\ \varepsilon_n < \tau < 1/\varepsilon_n\}$, $\{x : 0 < hx = O\,(1+\tau)\}$, *and* $\{y : 0 < y \le x\}$, *where* ε_n *is a positive sequence running to zero*;
(3)

$$\mathbf{P}(S_n < x) \sim L^n(h) \tag{1.10}$$

uniformly in $h \to \infty$ *and* $\{x : 0 < xh \to 0,\ \tau/(hx) = O\,(1)\}$.

Theorem 1 is a direct corollary of Theorem 2 for h that satisfy Eq. (1.3) in view of Lemma 1 of the present work and the Stirling formula $\Gamma(1+\lambda) = \sqrt{2\pi\lambda}\ (\lambda/e)^\lambda\ e^{\theta/\lambda},\ 0 < \theta < 1/12$.

Remark 2. Let

$$\liminf_{y \to +0}\ \min_{y \le u \le 1}\ \kappa(u)/\kappa(y) > 0 \tag{1.11}$$

(for example, $\kappa(y)$ does not decrease at $(0,1)$).

Then, if $h_n \to \infty$ so that $n\,\kappa(1/h_n) \to \infty$, it is possible to assume in statement (1) of Theorem 2 that $\varepsilon < h < h_n$, and if $h_n \to \infty$ so that $n\,\kappa(1/h_n) \to 0$, one can replace the condition $h \to \infty$ in statement (3) of Theorem 2 by $h > h_n$.

Consider an example.

Example. Let $a > 0$, $\delta > 0$, $a_1, a_2, \ldots$ be constants. Assume that

$$V(e^{-t}) = a\,t^{-\delta}\,\Big(1 + \sum_{j=1}^{k} a_j\, t^{-j} + O(t^{-k-1})\Big), \quad t \to +\infty, \tag{1.12}$$

for any integer $k \ge 1$. Then

$$h = \frac{\delta}{\varepsilon\xi}\Big(1 + \sum_{j=1}^{k} \pi_j(s)\, \xi^{-j} + O(s/\xi)^{k+1}\Big) \tag{1.13}$$

and

$$\begin{aligned} \log L(h) = \log a - \delta s - \frac{\delta}{\xi}\Big(1 + \sum_{j=1}^{k-1} \pi_j(s)\, \xi^{-j}\Big) \\ + \delta \sum_{j=1}^{k} \xi^{-j}\Big(\sum_{m=0}^{j} \pi_j^{(m)}(s)/j^{m+1}\Big) + O(s/\xi)^{k+1} \end{aligned} \tag{1.14}$$

in (1.6′), where $k \ge 1$, $\varepsilon = x/n$, $\xi = -\log\varepsilon$, $s = \log\xi$, and $\pi_j(\cdot)$ are polynomials of degree j with explicitly determined coefficients depending on δ and $a_1, \dots, a_j$ (for example, $\pi_1(s) = \delta^{-1} a_1 - C - \log\delta + s$, where C is the Euler constant).

In particular, if $x > 0$ is such that $\xi = \log(n/x) \to \infty$ and $\delta\, n/\xi \to \rho$ as $n \to \infty$ with a constant $\rho \in [0, \infty)$, then

$$\mathbf{P}(S_n < x) \sim A\, a^n\, e^{-\delta n(1-1/\xi)\,\log\xi}, \quad n \to \infty,$$

where $A = e^{\rho(\log\rho + \delta^{-1} a_1 - C - \log\delta)}/\Gamma(1+\rho)$, and if

$$n/\xi \to \infty \quad \text{and} \quad n(\log\xi/\xi)^2 \to 0,$$

then

$$\mathbf{P}(S_n < x) \sim a^n \sqrt{\frac{\xi}{2\pi\delta n}}\, e^{-\delta n(\log\xi - (\delta^{-1} a_1 - C - \log\delta + \log\xi)/\xi)}, \quad n \to \infty.$$

2. Proof of statement (1) of Theorem 2

Introduce an auxiliary random variable $X(h)$, $h \ge 0$, with distribution $e^{-hr}\, V(dr)/L(h)$. Recall (see (1.1)) that $m(h) = \mathbf{E}X(h)$ and $\sigma^2(h) = \mathbf{Var}X(h)$.

Lemma 1. *Let* (1.4) *hold. Then* (*see* (1.5) *and* (1.7))

$$V(1/h) \sim L(h) \sim \widetilde{l}(1/h), \tag{2.1}$$

$$(-1)^k\, L^{(k)}(h) \sim (k-1)!\; l(1/h)/h^k, \quad k = 1, 2, \dots, \tag{2.2}$$

$$h\, m(h) \sim h^2\, \sigma^2(h) \sim \kappa(1/h), \quad \text{and} \quad h\mathbf{E}X^3(h)/\sigma^2(h) = O(1) \tag{2.3}$$

as $h \to \infty$.

Proof of Lemma 1. For $y > 0$ set

$$\mu(y) = \int_0^y u\, dV(u) \quad \text{and} \quad \widehat{\mu}(y) = \mu(y)/y, \quad \widetilde{\mu}(y) = \int_0^y \widehat{\mu}(u)\, du/u.$$

We have

$$\begin{aligned} (-1)^k\, L^{(k)}(h) &= \int_0^\infty e^{-hy} y^{k-1}\, d\mu(y) = -\int_0^\infty \mu(y)\, dy^{k-1}\, e^{-hy} \\ &= -\int_0^\infty \widehat{\mu}(y)\, \{(k-1)y^{k-1} - hy^k\}\, e^{-hy}\, dy \end{aligned}$$

$$\sim l(1/h)\,h^{-k}\int_0^\infty e^{-t}\{t^k-(k-1)t^{k-1}\}\,dt$$

for $h\to\infty$ whence (2.2) follows; next, by (1.5),

$$L(h)=h\int_0^\infty e^{-hu}\,V(u)\,du=\int_0^\infty e^{-y}\,V(y/h)\,dy\sim V(1/h),$$

and therefore, (2.1) holds.

Now we check (2.3). By (2.1) and (2.2),

$$h^2\,\sigma^2(h)=\frac{hL'(h)}{L(h)}\Big(\frac{hL''(h)}{L'(h)}-\frac{hL'(h)}{L(h)}\Big)\sim-\frac{hL'(h)}{L(h)}\sim\kappa(1/h)$$

as $h\to\infty$; similarly,

$$h\mathbf{E}X^3(h)/\sigma^2(h)=\big|hL'''(h)/(L(h)\,\sigma^2(h))\big|\to 2.$$

Lemma 1 is proved. □

Lemma 2. *Set* $f_h(t)=\mathbf{E}e^{it\,X(h)}$. *If* (1.4) *holds, then*

$$1-|f_h(v)|\geq\delta\,e^{-2\pi h/v}\,l(1/v)/L(h),\quad v>v_0,\quad h>0,\tag{2.4}$$

where v_0 *is large enough and a positive number* δ *does not depend on* v *and* h.

Proof of Lemma 2. Set $Y=X(h)$. We have

$$1-|f_h(v)|\geq I=\int_0^{a+(2-\tau)\pi}2\sin^2\big(\tfrac{y-a}{2}\big)\,dF_Y(y/v),$$

where $a=a(v)\in[0,2\pi)$, $\tau\in(0,1)$, and $F_Y(y)$ is the distribution function of Y.

If $s_1=[a-(2-\tau)\pi,a-\tau\pi]$ and $s_2=[a+\tau\pi,a+(2-\tau)\pi]$, then

$$2\sin^2\big(\tfrac{y-a}{2}\big)\geq c_0=2\sin^2(\tau\pi/2).$$

Let $\tau=1/2$. Then $c_0=1$, $s_1=[a-3\pi/2,a-\pi/2]$, and $s_2=[a+\pi/2,a+3\pi/2]$. Hence,

$$I\geq\int_{s_2}dF_Y(y/v)+\int_{s_1}dF_Y(y/v)=I_1+I_2.\tag{2.5}$$

It is obvious that

$$I_1\geq\int_\pi^{3\pi/2}dF_Y(y/v),\ \ a\in[0,\pi/2],\quad I_1\geq\int_{3\pi/2}^{2\pi}dF_Y(y/v),\ \ a\in[\pi/2,\pi],$$

$$I_2\geq\int_{\pi/2}^{\pi}dF_Y(y/v),\ \ a\in[3\pi/2,2\pi],\quad\text{and}\quad I_2\geq\int_0^{\pi/2}dF_Y(y/v),\ \ a\in[\pi,3\pi/2].$$

Further, if $b>a$, then

$$\int_a^b dF_Y(y/v)=\frac{1}{L(h)}\int_{a/v}^{b/v}e^{-hu}\,dV(u)\geq\frac{1}{L(h)}\,e^{-bh/v}\int_{a/v}^{b/v}dV(u).\tag{2.6}$$

Since $\int\limits_{\alpha}^{\beta} dV(u) = \int\limits_{\alpha}^{\beta} d\mu(u)/u \geq (\mu(\beta) - \mu(\alpha))/\beta,\ \alpha < \beta$, for any fixed $0 < a < b < \infty$,

$$\int\limits_{a/v}^{b/v} dV(u) \geq \frac{v}{b}\,\mu(1/v)\,\frac{\mu(b/v) - \mu(a/v)}{\mu(1/v)} \sim \frac{v}{b}\,\mu(1/v)\,(b-a) \sim \frac{b-a}{b}\,l(1/v), \quad v \to \infty. \tag{2.7}$$

Relation (2.4) follows from (2.5), (2.6), and (2.7) taking into account the lower bounds for I_1 and I_2. Lemma 2 is proved. □

Proof of statement (1) *of Theorem* 2. As in [2], denote independent copies of a random variable $X(h)$ by $X_j(h)$, $j \geq 1$, and set (see (1.1))

$$S_n(h) = X_1(h) + \cdots X_n(h), \quad g_h(t) = \mathbf{E}\exp\Big(it\,\frac{S_n(h) - n\,m(h)}{\sigma(h)\sqrt{n}}\Big),$$
$$\tau = h\sigma(h)\sqrt{n},\ \beta = \frac{x - n\,m(h)}{\sigma(h)\sqrt{n}}, \quad \text{and} \quad \delta_\gamma(h) = \int\limits_0^{1/\gamma} |g_h(t) - e^{-t^2/2}|\,dt \quad (\gamma > 0). \tag{2.8}$$

In [3, Lemma 2], it was proved that for any positive x, h, y, and γ,

$$\mathbf{P}(x - y < S_n \leq x) = L^n(h)\,e^{h\,x}\,\frac{1 - e^{-h\,y}}{h\sigma(h)\sqrt{2\pi n}}$$
$$\times\Big(e^{-\beta^2/2} + \theta\,(|\beta|\,e^{-\beta^2/2}/\tau + 1/\tau^2 + \rho_\gamma(h,y))\Big), \tag{2.9}$$

where $|\theta|$ is bounded above by an absolute constant,

$$\rho_\gamma(h,y) = \delta_\gamma(h) + (1 + \delta_\gamma(h))(1 + \tfrac{1}{h\,y})\,\tau\gamma. \tag{2.10}$$

It is known (see [4, Chap. 5, Lemma 1]) that

$$|g_h(t) - e^{-t^2/2}| \leq 16\nu\,e^{-t^2/3},\ |t| \leq 1/\nu, \tag{2.11}$$

where (see (2.3))

$$\nu = \frac{4}{\sigma^3(h)\sqrt{n}}\mathbf{E}|X(h) - \mathbf{E}X(h)|^3 \leq \frac{1}{c_1\tau} \tag{2.12}$$

(hereinafter, $c_i,\ i = 1, 2, \ldots$, are positive constants).

Assuming that τ is large enough (otherwise, (1.6) is a consequence of the estimate $\mathbf{P}(S_n < x) \leq L^n(h)\,e^{h\,x}$), we set $\gamma = \widetilde{\tau}^{-k-1}$, $k \geq 1$. From (2.8), (2.11), and (2.12) it follows that

$$\delta_\gamma(h) \leq c_2/\tau + I(h), \quad I(h) = \tau\int\limits_{c_1}^{\tau^k} |f_h(th)|^n dt. \tag{2.13}$$

Let first $h > h_0$, where h_0 is large enough (a critical case). Estimating $|f_h(th)|$ by (2.4) and (2.3) and using the fact that $l(\cdot)$ is slowly varying at zero, we obtain for any fixed k (see (1.7)) the estimate

$$|f_h(t\,h)| \leq e^{-c_3\,\kappa(1/h)/\tau}, \quad t \in (c_1, \tau^k). \tag{2.14}$$

Therefore,

$$I(h) \leq c_4\,/\tau. \tag{2.15}$$

Let now $h \in [\varepsilon, h_0]$. Then $c_5 \leq h^2\,\sigma^2(h) \leq c_6$ (and $L(h) \leq 1$), and, similarly to (2.14) we show that

$$|f_h(t\,h)| \leq e^{-c_7/\sqrt{n}}, \quad t \in (c_1, \tau^k),$$

for $t \ge t_0$. Thus, (2.15) takes place again.

Finally, if $c_1 \le t \le t_0$ and $\varepsilon \le h \le h_0$, then $|f_h(t\,h)| \le e^{-c_9}$ since the mapping between the distributions of X and $X(h)$ is continuous in h and X is unlattice due to (1.4). Therefore, (2.15) is valid again.

Equality (1.8) follows from (2.9), (2.10), (2.13), and (2.15); to check Remark 1, one can use the last part of the proof of statement (1) of Theorem 2. Remark 2, regarding to statement (1) of Theorem 2, holds since (1.11) implies that $\min\limits_{1\le h\le h_n} \kappa(1/h) \ge \delta\,\kappa(1/h_n)$ for some $\delta > 0$. □

3. Proofs of statements (2,3) of Theorem 2

It is known (see [5, Theorem 2 and (1.8)]) that

$$L^n(h)\,e^{h\,x} \ge \mathbf{P}(S_n < x) \ge L^n(u)(1 - n\,m(u)/x) \tag{3.1}$$

for $x > 0$, $h > 0$, $K > 1$ and $u = K\,h$. By Lemma 1,

$$\log L(u) - \log L(h) = -\int\limits_h^u m(t)dt \sim -\kappa(1/h)\,\log K, \quad m(u) \sim m(h)/K, \tag{3.2}$$

as $h \to \infty$. Substituting (3.2) into (3.1) and letting K tend to infinity, we arrive at (1.10).

Remark 2 to statement (3) of Theorem 2 is valid since (1.11) is equivalent to the condition $\limsup\limits_{y\to+0} \max\limits_{0<u\le y} \kappa(u)/\kappa(y) < \infty$, which implies that $n \max\limits_{h_n\le h<\infty} \kappa(1/h) \to 0$.

Now let us check relation (1.9). We have (see, for instance, [3, (2.1)]) the relation

$$\mathbf{P}(x - y \le S_n < x) = L^n(h) \int\limits_{h\,(x-y)}^{h\,x} e^u\,dF_h(u), \tag{3.3}$$

where $F_h(\cdot)$ is the distribution function of $h\,S_n(h)$ (see (2.8)).

Let $G_\tau(\cdot)$ be gamma-distribution with parameter $\tau = n\,\kappa(1/h)$ and the corresponding characteristic function $\widehat{G}_\tau(t) = (1 - it)^{-\tau}$.

Then

$$\int\limits_{h\,(x-y)}^{h\,x} e^t\,dF_h(t) = \int\limits_{h\,(x-y)}^{h\,x} e^u\,dG_\tau(u) + \int\limits_{h\,(x-y)}^{h\,x} e^u\,d\Delta_h(u) = I + J, \tag{3.4}$$

where $\Delta_h(u) = F_h(u) - G_\tau(u)$.

Obviously,

$$I = \int\limits_{h\,(x-y)}^{h\,x} e^u\,e^{-u}u^{\tau-1}/\Gamma(\tau)\,du = h^\tau\,\frac{x^\tau - (x-y)^\tau}{\Gamma(1+\tau)}. \tag{3.5}$$

To prove statement (2) of Theorem 2, it is sufficient to show that there exists a sequence $0 < \varepsilon_n \to 0$ such that

$$J = o\,(1), \quad n \to \infty, \tag{3.6}$$

uniformly in

$$h > n, \quad 0 < hx = O\,(1+\tau), \quad \text{and} \quad \varepsilon_n < \tau < 1/\varepsilon_n. \tag{3.7}$$

Let us prove this.

Denoting $\sup\limits_u |\Delta_h(u)|$ by Δ_h, we obtain the inequality

$$|J| \le 2\,e^{h\,x}\,\Delta_h. \tag{3.8}$$

Now let us estimate Δ_h. To this end, we apply Theorems 1 and 2 of [4, Chap. V]. Simple calculations show that

$$\Delta_h \le C\,(T^{-\nu} + T\,\delta_T) \tag{3.9}$$

for any $T > 1$, where $\nu = \min(1,\tau)$, $\delta_T = \sup\limits_{0<t\le T} |q_h^n(t) - \widehat{G}_\tau(t)|/t$, and $q_h(t) = \mathbf{E}e^{it\,hX(h)}$.

Let $0 < t \le T$. It follows from (1.1) and (2.3) that

$$|n(q_h(t)-1)| \le n\,t\,h\,m(h) \sim t\,\tau, \quad h \to \infty.$$

Hence, if

$$T\,\tau/\sqrt{n} = o\,(1), \tag{3.10}$$

then

$$q_h^n(t) = e^{n(q_h(t)-1)} + O\,(t^2\,\tau^2/n), \quad 0 < t \le T, \quad n \to \infty. \tag{3.11}$$

Now we evaluate $q_h(t) - 1$ more accurately. Set $u(y) = e^{-y}(e^{it\,y} - 1)/y$. Using the notation of Lemma 1, we get the relations

$$L(h)(q_h(t)-1) = \int\limits_0^\infty u(y)\,h\,d\mu(y/h) = -\int\limits_0^\infty \widehat{\mu}(y/h)\,y\,u'(y)\,dy. \tag{3.12}$$

Standard reasoning using properties of slowly varying functions shows that, under condition (1.4),

$$\int\limits_0^\infty \widehat{\mu}(y/h)\,y\,u'(y)\,dy = l(1/h)\Big(\int\limits_0^\infty y\,u'(y)\,dy + \theta\,t\,\varepsilon(h)\Big), \quad h \to \infty,$$

where $t > 0$, $\varepsilon(h) \to 0$, and $|\theta|$ is bounded from above by a constant uniformly in t and $h > 1$.

From this relation, (3.12), and Lemma 1 it follows that a similar estimate (with different θ and $\varepsilon(h)$) is valid for $n(q_h(t)-1)$:

$$n(q_h(t)-1) = \tau\Big(\int\limits_0^\infty y\,u'(y)\,dy + \theta\,t\,\varepsilon(h)\Big), \quad h \to \infty. \tag{3.13}$$

Taking into consideration the equality

$$\int\limits_0^\infty y\,u'(y)\,dy = -\int\limits_0^\infty u(y)\,dy = \log\,(1 - it)$$

and assuming that

$$\tau\varepsilon(h) = o\,(1), \quad n \to \infty, \tag{3.14}$$

we deduce from (3.13) that

$$e^{n(q_h(t)-1)} - \widehat{G}_\tau(t) = t\,\widehat{G}_\tau(t)\,O\,(\tau\varepsilon(h)), \quad n \to \infty,$$

uniformly in $t \in (0, T]$ with a suitable T which will be chosen later. Thus (see also (3.9) and (3.11)),

$$\delta_T \le C\,(\tau\varepsilon(h) + T\,\tau^2/n), \tag{3.15}$$

where the constant C does not depend on T for all n large enough.

Let $T = (\tau\varepsilon(h))^{-1/(1+\nu)}$ (see (3.7)). Assuming, without loss of generality, that $n\,\varepsilon^2(h) > 1$ (in this case, condition (3.10) is satisfied), we deduce from (3.7), (3.8), and (3.9) that

$$|J| \le c\,e^{c\tau}(\varepsilon(h))^{\nu/(1+\nu)}$$

for all n (and h) large enough with some constant c. Thus, under an appropriate choice of ε_n, (3.6) follows.

Statement (2) of Theorem 2 is proved. □

4. Verification of the statement of the example

Let condition (1.12) be fulfilled. We claim (see (1.1)) that in this case,

$$L(h) = a\,\tau^{-\delta}\Big(1 + \sum_{\nu=1}^{k} b_\nu \tau^{-\nu} + O\left(\tau^{-k-1}\right)\Big)$$

and

$$h\,L'(h) = -a\,\tau^{-\delta-1}\Big(\delta + \sum_{\nu=1}^{k} (\delta+\nu)\,b_\nu\,\tau^{-\nu} + O\left(\tau^{-k-1}\right)\Big) \tag{4.1}$$

as $h \to \infty$ for any $k \geq 1$, where $\tau = \log h$ and the coefficients b_ν are given by explicit formulas. In particular, $b_1 = a_1 - \delta C$ and $b_2 = a_2 - a_1(1+\delta)C + \delta(1+\delta)(C^2 + \pi^2/6)/2$, where C is the Euler constant.

Set $y_0 = \tau^{-k-1}/h$, $y_1 = (k+1)\log\tau/h$, and $y_2 = (\delta+k+1)\log\tau/h$.

We represent

$$L(h) = \Big(\int_0^{y_0} + \int_{y_0}^{y_1} + \int_{y_1}^{\infty}\Big) V(y)e^{-hy}\,dhy = I_1 + I_2 + I_3. \tag{4.2}$$

Thus,

$$I_1 \leq V(y_0)hy_0 \sim a\tau^{-\delta-k-1}$$

and

$$I_3 \leq \tau^{-\delta-k-1} + \int_{y_1}^{y_2} V(y)e^{-hy}\,dhy \sim (1+a)\tau^{-\delta-k-1}. \tag{4.3}$$

Let us estimate I_2. Put $g(t) = a\,t^{-\delta}\left(1 + \sum_{j=1}^{k} a_j\,t^{-j}\right)$. Then

$$I_2 = \int_{y_0}^{y_1} g(-\log y)e^{-hy}\,dhy + O\left(1\right)\int_{y_0}^{y_1} (-\log y)^{-\delta-k-1}e^{-hy}\,dhy = I_4 + O\left(\tau^{-\delta-k-1}\right), \tag{4.4}$$

where

$$I_4 = \int_{t_0}^{t_1} g(\tau+t)e^{-t}e^{-e^{-t}}\,dt, t_0 = -\log\log\tau - \log\left(k+1\right), \quad \text{and} \quad t_1 = (k+1)\log\tau. \tag{4.5}$$

Further,

$$g(\tau+t) = \sum_{l=0}^{k} g^{(l)}(\tau)\,t^l/l! + g^{(k+1)}(\tau+\theta t)\,t^{k+1}/(k+1)!, \quad 0<\theta<1,$$

where

$$g^{(l)}(u) = au^{-\delta-l}\sum_{\nu=0}^{k} a_{\nu l}u^{-\nu}, \quad a_{0l} = (-1)^l\Gamma(l+\delta)/\Gamma(\delta),$$

and

$$a_{\nu l} = a_\nu\,(-1)^l\,l!/\Gamma(\delta)\,\sum_{m=0}^{l} \Gamma(l+\delta-m)\,C_{\nu+m-1}^m/(l-m)!, \quad \nu \geq 1.$$

Therefore,

$$I_4=\sum_{l=0}^{k} g^{(l)}(\tau)/l!\int_{t_0}^{t_1} t^l\, e^{-t}e^{-e^{-t}}\,dt+O\left(\tau^{-\delta-k-1}\right)=\sum_{l=0}^{k} c_l\, g^{(l)}(\tau)+O\left(\tau^{-\delta-k-1}\right)$$

with $c_l=(-1)^l\Gamma^{(l)}(1)/l!$ ($c_0=1$, $c_1=C$, and $c_2=C^2/2+\pi^2/12$).

This equality and (4.2)–(4.5) imply the first equality in (4.1) with coefficients

$$b_\nu=\sum_{l=0}^{\nu} c_l\, a_{\nu-l,l}.$$

The second statement is checked similarly in view of the equality $h\,L'(h)=\int_0^\infty V(y)\,d(hye^{-hy})$.

From (4.1) (see also [4, Chap. VI, Lemma 1] and (1.1)) it follows that

$$h\,m(h)=\delta\,\tau^{-1}\Big(\sum_{\nu=0}^{k}\beta_\nu\,\tau^{-\nu}+O\left(\tau^{-k-1}\right)\Big) \tag{4.6}$$

with $\beta_0=1$ and

$$\beta_\nu=\frac{\nu}{\delta}\sum(-1)^{r-1}(r-1)!\prod_{l=1}^{\nu}\frac{(b_l)^{m_l}}{m_l!},\ \nu\ge 1, \tag{4.7}$$

where summation is taken over all integer nonnegative solutions $(m_1,\dots,m_\nu)$ of the equations $1\cdot m_1+\cdots+\nu\cdot m_\nu=\nu$ and $r=m_1+\cdots+m_\nu$.

In particular, $\beta_1=b_1/\delta$, $\beta_2=(2b_2-b_1^2)/\delta$, and $\beta_3=(3b_3-3b_1b_2+b_1^3)/\delta$.

Consider the equation

$$m(h)=\varepsilon; \tag{4.8}$$

we claim that the function $h=h(\varepsilon)$ from (1.13) is its approximate solution with necessary degree of accuracy.

First let us note that, in the notation of (1.13),

$$\begin{aligned}\tau=\log h&=\log\delta+\xi-s+\xi\sum_{\nu=2}^{k+1}Q_\nu(s)\,\xi^{-\nu}+O(s/\xi)^{k+1}\\&=\xi\Big(1+\sum_{\nu=1}^{k+1}Q_\nu(s)\,\xi^{-\nu}\Big)+O(s/\xi)^{k+1},\end{aligned} \tag{4.9}$$

where (see the notation in (4.7))

$$Q_1(t)=\log\delta-t\quad\text{and}\quad Q_{\nu+1}(t)=\sum(-1)^{r-1}(r-1)!\prod_{l=1}^{\nu}\pi_l^{m_l}(t)/m_l!,\ \nu\ge 1. \tag{4.10}$$

For instance,

$$Q_2(s)=\pi_1(s),\ Q_3(s)=\pi_2(s)-\pi_1^2(s)/2,\ Q_4(s)=\pi_3(s)-\pi_1(s)\pi_2(s)+\pi_1^3(s)/3.$$

In addition,

$$\tau^{-l}=\xi^{-l}\Big(1+\sum_{\nu=1}^{k+1}Q_\nu(s)\,\xi^{-\nu}+O(s^{k+1}/\xi^{k+2})\Big)^{-l}=\xi^{-l}\Big(\sum_{\nu=0}^{k}c_{l\nu}(s)\,\xi^{-\nu}+O(s/\xi)^{k+1}\Big), \tag{4.11}$$

where $l \geq 1$ is integer, $c_{l0}(s) = 1$, and (see (4.7)) the coefficients

$$c_{l\nu}(t) = \frac{1}{(l-1)!} \sum (-1)^r (l+r-1)! \prod_{l=1}^{\nu} Q_l^{m_l}(t)/m_l!, \quad \nu \geq 1, \tag{4.12}$$

are polynomials of degree ν (that depend on the functions $\pi_j(t)$ for $1 \leq j < \nu$).

Substituting (4.6), (4.11), and (1.13) into (4.8), we find the representation of the functions $\pi_\nu(\cdot)$ under which (4.8) is satisfied in the form of a recurrence relation:

$$\pi_m(t) = \sum_{l=0}^{m} \beta_l \, c_{l+1,m-l}(t), \tag{4.13}$$

which makes it possible to calculate these functions taking into account (4.7), (4.10), and (4.12).

Thus, the solution $h(\varepsilon)$ of Eq. (4.8) has the form

$$h(\varepsilon) = \frac{\delta}{\varepsilon\xi} \Big(1 + \sum_{j=1}^{k} \pi_j(s)\, \xi^{-j} + O(s/\xi)^{k+1}\Big), \quad \varepsilon \to +0, \tag{4.14}$$

where $k \geq 1$ is an arbitrary integer, $\xi = -\log \varepsilon$, and $s = \log \xi$.

Relation (1.13) follows from (4.14) for $\varepsilon = x/n \to +0$.

To check (1.14), we use (4.14) and the equality

$$\log L(h(\varepsilon)) = -\delta \, \log\log(1/\varepsilon) + \log a - \varepsilon\, h(\varepsilon) + \int_0^{\varepsilon} (h(\varepsilon) - \frac{\delta}{\varepsilon |\log \varepsilon|})\, d\varepsilon, \quad \varepsilon > 0, \tag{4.15}$$

which holds under condition (1.12).

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