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MULTIVARIATE ESTIMATES FOR THE CONCENTRATION FUNCTIONS OF WEIGHTED SUMS OF INDEPENDENT, IDENTICALLY DISTRIBUTED RANDOM VARIABLES

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In this paper, we formulate and prove multidimensional generalizations of results obtained previously by the author and A. Yu. Zaitsev. Let $X, X_1, \dots, X_n$ be independent, identically distributed random variables. We study the behavior of the concentration function of the random variable $\sum\limits_{k=1}^{n} X_k a_k$ according to the arithmetic structure of the vectors a_k. Recently, interest to this problem increased significantly due to study of distributions of eigenvalues of random matrices. In this paper, we formulate and prove some refinements of results of Rudelson–Vershinin and Friedland–Sodin. Bibliography: 29 titles.

1. Introduction

Let $X, X_1, \dots, X_n$ be independent, identically distributed (i.i.d.) random variables. The concentration function of an $\mathbf{R}^d$-valued vector Y with distribution $F = \mathcal{L}(Y)$ is defined by the equality

$$Q(F, \lambda) = \sup_{x \in \mathbf{R}^d} \mathbf{P}(Y \in x + \lambda B), \quad \lambda > 0,$$

where $B = \{x \in \mathbf{R}^n : \|x\| \le 1\}$. Let $a = (a_1, \dots, a_n) \ne 0$, where $a_k = (a_{k1}, \dots, a_{kd}) \in \mathbf{R}^d$, $k = 1, \dots, n$. We study the behavior of the concentration function of the sum $S_a = \sum\limits_{k=1}^{n} X_k a_k$ according to the arithmetic structure of the vectors a_k. This problem is called the Littlewood–Offord problem. It was considered in [1–7]. The classical one-dimensional results were obtained by Littlewood and Offord [8] and Erdös [9] for i.i.d. X_k taking values ± 1 with probabilities $1/2$ and integer coefficients $a_k \ne 0$. In this case, the concentration function is of order $O(n^{-1/2})$ (a similar estimate holds for the multidimensional Littlewood–Offord problem, see [10]). However, if we assume that all the a_k are different, then the estimate can be significantly improved up to order $O(n^{-3/2})$ (see [11,12]). Recently, the behavior of the concentration function of weighted sums S_a was actively investigated due to study of distributions of eigenvalues of random matrices.

In the sequel, let F_a be the distribution of the sum $S_a = \sum\limits_{k=1}^{n} X_k a_k$ and let G be the distribution of the symmetrized random variable $\widetilde{X} = X_1 - X_2$. Let

$$M(\tau) = \tau^{-2} \int\limits_{|x| \le \tau} x^2 \, G\{dx\} + \int\limits_{|x| > \tau} G\{dx\} = \mathbf{E} \min\left\{\widetilde{X}^2/\tau^2, 1\right\}, \quad \tau > 0. \tag{1}$$

The symbol c will be used for absolute positive constants. Note that c can be different in different (or even in the same) formulas. We write $A \ll B$ if $|A| \le cB$ and $B > 0$. Similarly, $A \ll_d B$ if $|A| \le c^d B$ and $B > 0$. Note that $\ll_d$ allows constants to be exponential with respect to d. Also we write $A \asymp B$ if $A \ll B$ and $B \ll A$. Similarly, $A \asymp_d B$ if $A \ll_d B$ and

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$B \ll_d A$. For $x = (x_1, \dots, x_d) \in \mathbf{R}^d$ we denote $\|x\|^2 = x_1^2 + \dots + x_d^2$ and $\|x\|_\infty = \max\limits_j |x_j|$. The inner product in $\mathbf{R}^d$ is denoted by $\langle \cdot, \cdot \rangle$. The product of a vector $t = (t_1, \dots, t_d) \in \mathbf{R}^d$ and a multivector a is denoted by $t \cdot a = (\langle t, a_1 \rangle, \dots, \langle t, a_n \rangle) \in \mathbf{R}^n$.

Simplest properties of one-dimensional concentration functions are well studied (see, for instance, [13–15]). It is well known that $Q(F, \mu) \ll_d (1 + \mu/\lambda)^d \, Q(F, \lambda)$ for all $\mu, \lambda > 0$. Hence,

$$Q(F, c\lambda) \asymp_d Q(F, \lambda) \tag{2}$$

and

$$\text{if } Q(F, \lambda) \ll K, \text{ then } Q(F, \mu) \ll_d K(1 + (\mu/\lambda))^d. \tag{3}$$

Recall that for any one-dimensional distribution F, the classical Esséen and Kolmogorov–Rogozin inequalities hold [16] (see as well [14] and [15]). One can find their multidimensional analogs in [17–20].

For a random vector Y with distribution $F = \mathcal{L}(Y)$ in $\mathbf{R}^d$, the following Esséen inequality holds (see [3, Lemma 3.4]):

$$Q(F, \sqrt{d}) \ll_d \int\limits_{B(\sqrt{d})} |\widehat{F}(t)| \, dt, \tag{4}$$

where $\widehat{F}(t) = \mathbf{E} \exp(i \langle t, Y \rangle)$ is the characteristic function of Y. Let $\int\limits_{\mathbf{R}^d} |\widehat{F}(u)| \, du < \infty$ (otherwise, we can achieve this by applying smoothing); we assume additionally that the distribution F is symmetric and $\widehat{F}(t) \geq 0$ for all $t \in \mathbf{R}$. Then, applying relation (4) to the measure $\dfrac{\widehat{F}(t) \, dt}{\int\limits_{\mathbf{R}^d} \widehat{F}(u) \, du}$, we obtain the inequality

$$Q(F, \sqrt{d}) \gg_d \int\limits_{B(\sqrt{d})} \widehat{F}(t) \, dt. \tag{5}$$

One can find estimates of that type but with a different dependence on the dimension d in [21]. Thereby,

$$Q(F, \sqrt{d}) \asymp_d \int\limits_{B(\sqrt{d})} \widehat{F}(t) \, dt. \tag{6}$$

Use of relation (6) allows us to simplify our arguments compared to those in [3] and [22].

Recall a multidimensional generalization of the Kolmogorov–Rogozin inequality.

Proposition 1. *Let $Y_1, \dots, Y_n$ be independent random vectors with distributions $F_k = \mathcal{L}(Y_k)$. Let $\lambda_1, \dots, \lambda_n$ be positive numbers, $\lambda_k \leq \lambda$, $k = 1, \dots, n$. Then*

$$Q\Big(\mathcal{L}\Big(\sum_{k=1}^n Y_k\Big), \lambda\Big) \ll \lambda \Big(\sum_{k=1}^n \lambda_k^2 \left(1 - Q(\widetilde{F}_k, \lambda_k)\right)\Big)^{-1/2}, \tag{7}$$

where $\widetilde{F}_k$ are the distributions of the corresponding symmetrized random vectors.

Siegel [18] improved the statement of Proposition 1. He showed that the following result holds.

Proposition 2. *Under the conditions of Proposition* 1,

$$Q\Big(\mathcal{L}\big(\sum_{k=1}^n Y_k\big), \lambda\Big) \ll \lambda \Big(\sum_{k=1}^n \lambda_k^2 M_k(\lambda_k)\Big)^{-1/2}. \tag{8}$$

One can find one-dimensional versions and refinements of these results in [13–15, 23–28]. Note that the constants in (7) and (8) do not depend on the dimension d. However, there exist estimates of Kolmogorov–Rogozin type with constants depending on d (see, for example, [23, 29]).

The Littlewood–Offord problem was considered in [1–7, 22]. In this paper, we formulate and prove multidimensional generalizations of results of [6]. They also refine results of [3] and [22].

Now we formulate results of [3] and [22] in common notation.

Friedland and Sodin [22] have simplified the reasoning of Rudelson and Vershynin [2] and obtained the following result.

Proposition 3. *Let $X, X_1, \dots, X_n$ be i.i.d. random variables, let $Q(\mathcal{L}(X), 1) \le 1 - p$, where $p > 0$, and let $a_1, \dots, a_n \in \mathbf{R}^d$. If, for some $0 < D < d$ and $\alpha > 0$,*

$$\sum_{k=1}^{n} (\langle t, a_k \rangle - m_k)^2 \ge \alpha^2 \text{ for all } m_1, \dots, m_n \in \mathbf{Z},\ t \in \mathbf{R}^d, \text{ such that } \max_k |\langle t, a_k \rangle| \ge 1/2 \text{ and } \|t\| \le D, \tag{9}$$

then

$$Q(F_a, d/D) \ll_d \exp(-cp\alpha^2) + \Big(\frac{\sqrt{d}}{\sqrt{p}D}\Big)^d \big(\det \mathbb{N}\big)^{-1/2}, \tag{10}$$

where

$$\mathbb{N} = \sum_{k=1}^{n} \mathbb{N}_k, \quad \mathbb{N}_k = \begin{pmatrix} a_{k1}^2 & a_{k1}a_{k2} & \dots & a_{k1}a_{kd} \\ a_{k2}a_{k1} & a_{k2}^2 & \dots & a_{k2}a_{kd} \\ \dots & \dots & \dots & \dots \\ a_{kd}a_{k1} & \dots & \dots & a_{kd}^2 \end{pmatrix}, \quad a_k = (a_{k1}, \dots, a_{kd}), \quad k = 1, \dots, n. \tag{11}$$

Note that the statement of Proposition 3 in [22] was formulated and proved in a weakened form. The right-hand side of inequality (10) contained p^2 instead of p. The possibility of replacing p^2 by p was noted, for example, in [3] (see Proposition 4). It easily follows from elementary properties of the concentration function.

Moreover, in [22] it was assumed that $0 < D < d$. Furthermore, the left-hand side of inequality (10) contained $Q(F_a, 1)$ instead of $Q(F_a, d/D)$. Since $d/D > 1$ for $0 < D < d$, the value $Q(F_a, 1)$ can be essentially less than $Q(F_a, d/D)$. If the authors of [22] considered their result for $D = d$, they could deduce from it the inequality for any $D > 0$ and with $Q(F_a, d/D)$ instead of $Q(F_a, 1)$ in the same simple way as we below deduce Corollary 1 from Theorem 1.

Note that

$$\big(\operatorname{dist}(t \cdot a, \mathbf{Z}^n)\big)^2 = \sum_{k=1}^{n} \min_{m_k \in \mathbf{Z}} (\langle t, a_k \rangle - m_k)^2 = \sum_{k=1}^{n} \langle t, a_k \rangle^2 \tag{12}$$

for $\max_k |\langle t, a_k \rangle| \le 1/2$, where

$$\operatorname{dist}(t \cdot a, \mathbf{Z}^n) = \min_{m \in \mathbf{Z}^n} \| t \cdot a - m \|.$$

Thus, the assumption that $\max_k |\langle t, a_k \rangle| \ge 1/2$ in condition (9) is natural.

Let us now formulate the multidimensional Theorem 3.3 of [3] in the same notation.

Proposition 4. *Let $X, \dots, X_n$ be i.i.d. random variables with zero mean and let $Q(\mathcal{L}(X), 1) \le 1 - p$, where $p > 0$.*

Consider $a = (a_1, \dots, a_n)$, $a_k \in \mathbf{R}^d$, *such that* $\sum\limits_{k=1}^{n} \langle t, a_k\rangle^2 \geq \|t\|^2$ *for any* $t \in \mathbf{R}^d$. *Let* α, $D > 0$, $\gamma \in (0,1)$, *and*

$$\Big(\sum_{k=1}^{n}(\langle t, a_k\rangle - m_k)^2\Big)^{1/2} \geq \min\{\gamma\|t\cdot a\|, \alpha\} \text{ for all } m_1, \dots, m_n \in \mathbf{Z} \text{ and } \|t\| \leq D. \tag{13}$$

Then

$$Q\Big(F_a, \frac{d}{D}\Big) \ll_d \Big(\frac{\sqrt{d}}{\gamma D\sqrt{p}}\Big)^d + \exp(-2\,p\,\alpha^2). \tag{14}$$

Note that the assumption $\mathbf{E}\,X = 0$ is unnecessary in the formulation of [3, Theorem 3.3].

It is obvious that if

$$0 < D \leq D(a) = \inf\big\{\|t\| > 0 : t \in \mathbf{R}^d, \operatorname{dist}(t\cdot a, \mathbf{Z}^n) \leq \min\{\gamma\|\,t\cdot a\|, \alpha\}\big\}, \tag{15}$$

then condition (13) holds. Rudelson and Vershynin [3] called the value $D(a)$ the essential least common denominator of a vector $a \in (\mathbf{R}^d)^n$.

Now we formulate one of the main results of this paper.

Theorem 1. *Let* $X, X_1, \dots, X_n$ *be i.i.d. random variables. Let* $a = (a_1, \dots, a_n)$, $a_k \in \mathbf{R}^d$. *Assume that for some* $\alpha > 0$, *condition* (9) *holds for* $D = \sqrt{d}$, *i.e.,*

$$\begin{gathered}\sum_{k=1}^{n}(\langle t, a_k\rangle - m_k)^2 \geq \alpha^2 \text{ for all } m_1, \dots, m_n \in \mathbf{Z},\ t \in \mathbf{R}^d \text{ such that} \\ \max_k |\,\langle t, a_k\rangle\,| \geq 1/2,\ \ \|t\| \leq \sqrt{d}.\end{gathered} \tag{16}$$

Then

$$Q(F_a, \sqrt{d}) \ll_d \Big(\frac{1}{\sqrt{M(1)}}\Big)^d \frac{1}{\sqrt{\det \mathbb{N}}} + \exp\big(-c\,\alpha^2 M(1)\big),$$

where $M(1)$ *is defined in* (1) *and the matrix* $\mathbb{N}$ *is defined in* (11).

Hence, it is easy to see what follows from Theorem 1 under the conditions of Proposition 3. Namely, the following statement holds.

Corollary 1. *Let the conditions of Theorem* 1 *be satisfied under condition* (9) *instead of* (16) *for an arbitrary* $D > 0$. *Then*

$$Q\Big(F_a, \frac{d}{D}\Big) \ll_d \Big(\frac{\sqrt{d}}{D\sqrt{M(1)}}\Big)^d \frac{1}{\sqrt{\det \mathbb{N}}} + \exp(-c\,\alpha^2 M(1)).$$

Note that the value $M(1)$ is essential in refining results of [3] and [22]. It is clear that $M(1)$ may be significantly larger than p. For example, p may equal 0 while $M(1) > 0$ for any nondegenerate distribution $F = \mathcal{L}(X)$. Therefore, Corollary 1 is an essential improvement of Proposition 3. It is obvious that Corollary 1 is related to Proposition 3 in the same way as the multidimensional variant of Esséen's inequality (8) is related to the multidimensional variant of the Kolmogorov–Rogozin inequality (7).

The proofs of Theorem 1 and Corollary 1 are somewhat easier than those in [3] and [22] because they do not include complicated decompositions of the integration set. This is achieved by using relation (6) and methods of Esséen [29] (see the proof of [15, Chap. II, Lemma 4]).

We reformulate Corollary 1 for the random variables X_k/τ, $\tau > 0$.

Corollary 2. *Let* $V_{a,\tau} = \mathcal{L}\big(\sum\limits_{k=1}^{n} a_k X_k/\tau\big)$*. Then*

$$Q\Big(V_{a,\tau}, \frac{d}{D}\Big) = Q\Big(F_a, \frac{d\,\tau}{D}\Big) \ll_d \exp\big(-c\,\alpha^2 M(\tau)\big) + \Big(\frac{\sqrt{d}}{D\,\sqrt{M(\tau)}}\Big)^d \frac{1}{\sqrt{\det \mathbb{N}}}.$$

under the conditions of Corollary 1*. For* $\tau = D/d$,

$$Q\big(F_a, 1\big) \ll_d \Big(\frac{\sqrt{d}}{D\,\sqrt{M(D/d)}}\Big)^d \frac{1}{\sqrt{\det \mathbb{N}}} + \exp\big(-c\,\alpha^2 M(D/d)\big)\,.$$

To prove Corollary 2, it suffices to use relation (1).

Note that τ can be arbitrarily small in Corollary 2. Applying this statement for τ tending to zero, we obtain the estimate

$$Q(F_a, 0) \ll_d \Bigg(\frac{\sqrt{d}}{D\,\sqrt{\mathbf{P}(\widetilde{X} \neq 0)}}\Bigg)^d \frac{1}{\sqrt{\det \mathbb{N}}} + \exp\big(-c\,\alpha^2\,\mathbf{P}(\widetilde{X} \neq 0)\big).$$

This estimate can also be deduced from results of [3] and [22].

Now we formulate refinements of Proposition 4. They are analogs of Theorem 1 and Corollaries 1 and 2.

Theorem 2. *Let* $X, X_1, \dots, X_n$ *be i.i.d. random variables. Let* $a = (a_1, \dots, a_n)$, $a_k \in \mathbf{R}^d$, $\alpha > 0$, $\gamma \in (0, 1)$, *and*

$$\Big(\sum_{k=1}^{n} (\langle t, a_k\rangle - m_k)^2\Big)^{1/2} \geq \min\{\gamma\|t \cdot a\|, \alpha\} \qquad (17)$$

for all $m_1, \dots, m_n \in \mathbf{Z}$ *and* $t \in \mathbf{R}^d$, $\|t\| \leq \sqrt{d}$.

Then

$$Q\big(F_a, \sqrt{d}\big) \ll_d \Big(\frac{1}{\gamma\,\sqrt{M(1)}}\Big)^d \frac{1}{\sqrt{\det \mathbb{N}}} + \exp\big(-c\,\alpha^2 M(1)\big)\,.$$

Note that Theorem 2 yields a more general result than the result of Proposition 4 because the condition $\sum\limits_{k=1}^{n} \langle t, a_k\rangle^2 \geq \|t\|^2$ is omitted in the formulation of Theorem 2.

Corollary 3. *Let* $X, X_1, \dots, X_n$ *be i.i.d. random variables. Let* $a = (a_1, \dots, a_n)$, $a_k \in \mathbf{R}^d$, $\alpha > 0$, $D > 0$, $\gamma \in (0, 1)$, *and*

$$\Big(\sum_{k=1}^{n} (\langle t, a_k\rangle - m_k)^2\Big)^{1/2} \geq \min\{\gamma\|t \cdot a\|, \alpha\} \qquad (18)$$

for all $m_1, \dots, m_n \in \mathbf{Z}$ *and* $t \in \mathbf{R}^d$, $\|t\| \leq D$.

Then

$$Q\Big(F_a, \frac{d}{D}\Big) \ll_d \Big(\frac{\sqrt{d}}{D\,\gamma\,\sqrt{M(1)}}\Big)^d \frac{1}{\sqrt{\det \mathbb{N}}} + \exp\big(-c\,\alpha^2 M(1)\big)\,.$$

Note that if the condition $\sum\limits_{k=1}^{n} \langle t, a_k\rangle^2 \geq \|t\|^2$ is satisfied, then $\dfrac{1}{\sqrt{\det \mathbb{N}}} \leq 1$. Hence, Corollary 3 yields a more general result than that of Proposition 4. Now we reformulate Corollary 3 for the variables X_k/τ, $\tau > 0$.

Corollary 4. *Let* $V_{a,\tau} = \mathcal{L}\big(\sum\limits_{k=1}^{n} a_k X_k/\tau\big)$. *Then*

$$Q\Big(V_{a,\tau}, \frac{d}{D}\Big) = Q\Big(F_a, \frac{d\,\tau}{D}\Big) \ll_d \Big(\frac{\sqrt{d}}{D\,\gamma\,\sqrt{M(\tau)}}\Big)^d \frac{1}{\sqrt{\det \mathbb{N}}} + \exp\big(-c\,\alpha^2 M(\tau)\big)$$

under the conditions of Corollary 3. *If* $\tau = D/d$, *then*

$$Q\big(F_a, 1\big) \ll_d \Big(\frac{\sqrt{d}}{D\,\gamma\,\sqrt{M(D/d)}}\Big)^d \frac{1}{\sqrt{\det \mathbb{N}}} + \exp\big(-c\,\alpha^2 M(D/d)\big)\,.$$

To prove Corollary 4, it suffices to use relation (1).

2. Proofs

Proof of Theorem 1. We represent the distribution $G = \mathcal{L}(\widetilde{X})$ as a mixture $G = qE + \sum\limits_{j=0}^{\infty} p_j G_j$, where $q = \mathbf{P}(\widetilde{X} = 0)$, $p_j = \mathbf{P}(\widetilde{X} \in C_j)$, $j = 0, 1, 2, \ldots$, $C_0 = \{x : |x| > 1\}$, $C_j = \{x : 2^{-j} < |x| \le 2^{-j+1}\}$, E is the probability measure concentrated at zero, and G_j are probability measures defined (for $p_j > 0$) by the equality $G_j\{X\} = \dfrac{1}{p_j} G\{X \bigcap C_j\}$ for any Borel set X. If $p_j = 0$, then we can take as G_j arbitrary measures.

For $z \in \mathbf{R}$ and $\gamma > 0$ we introduce symmetric d-dimensional infinitely divisible distributions $H_{z,\gamma}$ with characteristic functions

$$\widehat{H}_{z,\gamma}(t) = \exp\Big(-\frac{\gamma}{2}\sum_{k=1}^{n}\big(1 - \cos(2\,z\,\langle\, t, a_k\rangle)\big)\Big), \quad t \in \mathbf{R}^d. \tag{19}$$

It is clear that these functions are everywhere positive.

For any characteristic function $\widehat{W}(t)$ of a random vector Y,

$$|\widehat{W}(t)|^2 = \mathbf{E}\exp\big(i\big\langle\, t, \widetilde{Y}\big\rangle\big) = \mathbf{E}\cos\big(\big\langle\, t, \widetilde{Y}\big\rangle\big),$$

where $\widetilde{Y}$ is the corresponding symmetrized random vector. Then

$$|\widehat{W}(t)| \le \exp\Big(-\frac{1}{2}\big(1 - |\widehat{W}(t)|^2\big)\Big) = \exp\Big(-\frac{1}{2}\mathbf{E}\big(1 - \cos\big(\big\langle\, t, \widetilde{Y}\big\rangle\big)\big)\Big). \tag{20}$$

Using inequalities (4) and (20), we conclude that

$$\begin{aligned} Q(F_a, \sqrt{d}) &\ll_d \int\limits_{B(\sqrt{d})} |\widehat{F_a}(t)|\,dt \\ &\ll_d \int\limits_{B(\sqrt{d})} \exp\Big(-\frac{1}{2}\sum_{k=1}^{n}\mathbf{E}\big(1 - \cos(2\,\langle\, t, a_k\rangle\,\widetilde{X})\big)\Big)\,dt = I. \end{aligned}$$

It is clear that

$$\sum_{k=1}^{n}\mathbf{E}\big(1-\cos(2\,\langle t,a_k\rangle\,\widetilde{X})\big)=\sum_{k=1}^{n}\int\limits_{-\infty}^{\infty}\big(1-\cos(2\,\langle\,t,a_k\rangle\,x)\big)\,G\{dx\}$$
$$=\sum_{k=1}^{n}\sum_{j=0}^{\infty}\int\limits_{-\infty}^{\infty}\big(1-\cos(2\,\langle\,t,a_k\rangle\,x)\big)\,p_j\,G_j\{dx\}$$
$$=\sum_{j=0}^{\infty}\sum_{k=1}^{n}\int\limits_{-\infty}^{\infty}\big(1-\cos(2\,\langle\,t,a_k\rangle\,x)\big)\,p_j\,G_j\{dx\}.$$

We denote $\beta_j=2^{-2j}p_j$, $\beta=\sum\limits_{j=0}^{\infty}\beta_j$, and $\mu_j=\beta_j/\beta$, $j=0,1,2,\ldots$. It is obvious that $\sum\limits_{j=0}^{\infty}\mu_j=1$ and $p_j/\mu_j=2^{2j}\beta$ (for $p_j>0$).

Now we estimate the value β:

$$\beta=\sum_{j=0}^{\infty}\beta_j=\sum_{j=0}^{\infty}2^{-2j}p_j\;=\mathbf{P}\big(|\widetilde{X}|>1\big)+\sum_{j=1}^{\infty}2^{-2j}\,\mathbf{P}\big(2^{-j}<|\widetilde{X}|\le 2^{-j+1}\big)$$
$$\ge\int\limits_{|x|>1}G\{dx\}+\sum_{j=1}^{\infty}\int\limits_{2^{-j}<|x|\le 2^{-j+1}}\frac{x^2}{4}G\{dx\}$$
$$\ge\frac{1}{4}\int\limits_{|x|>1}G\{dx\}+\frac{1}{4}\int\limits_{|x|\le 1}x^2\,G\{dx\}=\frac{1}{4}M(1).$$

Thus,

$$\beta\ge\frac{1}{4}M(1). \tag{21}$$

Now we proceed like in the proof of Esséen's lemma [29] (see [15, Chap. II, Lemma 4]). Applying the Hölder inequality, it is easy to see that

$$I\le\prod_{j=0}^{\infty}I_j^{\mu_j}, \tag{22}$$

where

$$I_j=\int\limits_{B(\sqrt{d})}\exp\Big(-\frac{p_j}{2\,\mu_j}\sum_{k=1}^{n}\int\limits_{-\infty}^{\infty}\big(1-\cos(2\,\langle\,t,a_k\rangle\,x)\big)\,G_j\{dx\}\Big)\,dt$$
$$=\int\limits_{B(\sqrt{d})}\exp\Big(-\,2^{2j-1}\beta\,\sum_{k=1}^{n}\int\limits_{C_j}\big(1-\cos(2\,\langle\,t,a_k\rangle\,x)\big)\,G_j\{dx\}\Big)\,dt$$

if $p_j>0$, and $I_j=1$ for $p_j=0$.

Applying Jensen's inequality to the exponential in the integrand (see [15, p. 19]), we conclude that

$$I_j \le \int\limits_{B(\sqrt{d})} \int\limits_{C_j} \exp\Big(-2^{2j-1}\beta \sum_{k=1}^{n} \big(1-\cos(2\,\langle\, t, a_k\rangle\, x)\big)\Big)\, G_j\{dx\}\, dt$$

$$= \int\limits_{C_j} \int\limits_{B(\sqrt{d})} \exp\Big(-2^{2j-1}\beta \sum_{k=1}^{n} \big(1-\cos(2\,\langle t, a_k\rangle\, x)\big)\Big)\, dt\, G_j\{dx\}$$

$$\le \sup_{z\in C_j} \int\limits_{B(\sqrt{d})} \widehat{H}_{z,1}^{2^{2j}\beta}(t)\, dt.$$

We estimate the function $\widehat{H}_{\pi,1}(t)$ for $\max\limits_k |\,\langle\, t, a_k\rangle\,| \le 1/2$. It is clear that there exists a c such that $1-\cos x \ge cx^2$ for $|x|\le\pi$. Thus,

$$\widehat{H}_{\pi,1}(t) \le \exp\Big(-\frac{1}{2}\sum_{k=1}^{n}\big(1-\cos\left(2\pi\,\langle\, t, a_k\rangle\right)\big)\Big)$$

$$\le \exp\Big(-c\sum_{k=1}^{n} |\,\langle\, t, a_k\rangle\,|^2\Big) \le \exp\left(-c\,\langle \mathbb{N}t, t\rangle\right)$$

for $\max\limits_k |\,\langle\, t, a_k\rangle\,| \le 1/2$, where the matrix $\mathbb{N}$ is defined in (11).

It is well known that

$$\int\limits_{\mathbf{R}^d} \exp\left(-c\,\langle \mathbb{N}t, t\rangle\right)\, dt \ll_d \frac{1}{\sqrt{\det\mathbb{N}}}. \tag{23}$$

For t such that $\max\limits_k |\,\langle\, t, a_k\rangle\,| \ge 1/2$ and $\|t\|\le\sqrt{d}$, one can proceed in the same way as in [3] and [22]; namely, taking into account that

$$1-\cos x \ge c\min_{m\in\mathbf{Z}} |\, x-2\pi m|^2,$$

we see that

$$\begin{aligned}\widehat{H}_{\pi,1}(t) &\le \exp\Big(-c\sum_{k=1}^{n} \min_{m_k\in\mathbf{Z}} \big|2\pi\,\langle\, t, a_k\rangle - 2\pi m_k\big|^2\Big)\\ &= \exp\Big(-c\sum_{k=1}^{n} \min_{m_k\in\mathbf{Z}} |\,\langle\, t, a_k\rangle - m_k|^2\Big) \le \exp(-c\,\alpha^2)\end{aligned} \tag{24}$$

for $\|t\|\le\sqrt{d}$ and $\max\limits_k |\,\langle\, t, a_k\rangle\,| \ge 1/2$.

Now we use estimates (23) and (24) to estimate the integrals I_j. At first, we consider the case where $j=1,2,\ldots$. Note that the characteristic functions $\widehat{H}_{z,\gamma}(t)$ satisfy the equalities

$$\widehat{H}_{z,\gamma}(t) = \widehat{H}_{y,\gamma}\big(zt/y\big) \quad\text{and}\quad \widehat{H}_{z,\gamma}(t) = \widehat{H}_{z,1}^{\gamma}(t). \tag{25}$$

If $z \in C_j$, then $2^{-j} < |z| \le 2^{-j+1} < \pi$. Hence, if $\| t\| \le \sqrt{d}$, then $\|zt/\pi\| < \sqrt{d}$. Thus, using equalities (25) with $y = \pi$ and estimates (23) and (24), we conclude that

$$\sup_{z\in C_j} \int\limits_{B(\sqrt{d})} \widehat{H}_{z,1}^{2^{2j}\beta}(t)\, dt \le \int\limits_{B(\sqrt{d})} \exp(-c\,\beta\,\langle \mathbb{N}t, t\rangle)\, dt + \int\limits_{B(\sqrt{d})} \exp(-2^{2j} c\,\alpha^2\beta)\, dt$$
$$\ll_d \Big(\frac{1}{\sqrt{\beta}}\Big)^d \frac{1}{\sqrt{\det \mathbb{N}}} + \exp(-c\,\alpha^2\beta)$$

for $z \in C_j$.

Now we consider the case $j = 0$. Equalities (25) imply that

$$Q(H_{z,\gamma}, \sqrt{d}) = Q\big(H_{1,\gamma}, \sqrt{d}/z\big) \tag{26}$$

for $z > 0$ and $\gamma > 0$. Thus, according to relations (2), (6), (25), and (26), we obtain the relation

$$\sup_{z\in C_0} \int\limits_{B(\sqrt{d})} \widehat{H}_{z,1}^{\beta}(t)\, dt = \sup_{z\ge 1} \int\limits_{B(\sqrt{d})} \widehat{H}_{z,\beta}(t)\, dt \asymp_d \sup_{z\ge 1}\, Q(H_{z,\beta}, \sqrt{d})$$
$$= \sup_{z\ge 1}\, Q\big(H_{1,\beta}, \sqrt{d}/z\big) \le Q(H_{1,\beta}, \sqrt{d}) \ll_d Q\big(H_{1,\beta}, \sqrt{d}/\pi\big) = Q(H_{\pi,\beta}, \sqrt{d})$$
$$\asymp_d \int\limits_{B(\sqrt{d})} \widehat{H}_{\pi,\beta}(t)\, dt = \int\limits_{B(\sqrt{d})} \widehat{H}_{\pi,1}^{\beta}(t)\, dt.$$

Estimates (23) and (24) for the characteristic function $\widehat{H}_{\pi,1}(t)$ and the relation $\mathrm{Vol}(B(\sqrt{d})) \ll_d 1$ imply that

$$\int\limits_{B(\sqrt{d})} \widehat{H}_{\pi,1}^{\beta}(t)\, dt \le \int\limits_{B(\sqrt{d})} \exp(-c\beta\,\langle \mathbb{N}t, t\rangle)\, dt + \int\limits_{B(\sqrt{d})} \exp(-c\,\alpha^2\beta)\, dt$$
$$\ll_d \Big(\frac{1}{\sqrt{\beta}}\Big)^d \frac{1}{\sqrt{\det \mathbb{N}}} + \exp(-c\,\alpha^2\beta).$$

We obtained the same bound for all the integrals I_j for $p_j \neq 0$. Since $\sum\limits_{j=0}^{\infty} \mu_j = 1$,

$$I \le \prod_{j=0}^{\infty} I_j^{\mu_j} \ll_d \Big(\frac{1}{\sqrt{\beta}}\Big)^d \frac{1}{\sqrt{\det \mathbb{N}}} + \exp(-c\,\alpha^2\beta).$$

Hence,

$$Q\Big(F_a, \sqrt{d}\Big) \ll_d \Big(\frac{1}{\sqrt{\beta}}\Big)^d \frac{1}{\sqrt{\det \mathbb{N}}} + \exp(-c\,\alpha^2\beta)$$
$$\ll_d \Big(\frac{1}{\sqrt{M(1)}}\Big)^d \frac{1}{\sqrt{\det \mathbb{N}}} + \exp(-c\,\alpha^2 M(1)),$$

as was claimed. □

Now we deduce Corollary 1 from Theorem 1.

Proof of Corollary 1. We denote

$$b = (b_1, \dots, b_n) = \frac{D}{\sqrt{d}} a = \frac{D}{\sqrt{d}} (a_1, \dots, a_n) \in (\mathbf{R}^d)^n.$$

Then the equality $Q(F_a, d/D) = Q(F_b, \sqrt{d})$ holds. The conditions of Theorem 1 for the multivector a are valid for the multivector b as well. Indeed, $\sum\limits_{k=1}^{n} (\langle u, b_k\rangle - m_k)^2 \geq \alpha^2$ for all $m_1, \dots, m_n \in \mathbf{Z}$ and $u \in \mathbf{R}^d$ such that $\|u\| \leq \sqrt{d}$ and $\max\limits_k |\langle u, b_k\rangle| \geq 1/2$. This follows from condition (9) of Corollary 1 if we denote $u = \dfrac{\sqrt{d}\,t}{D}$. It remains to apply Theorem 1 to the multivector b. □

Proof of Theorem 2. We proceed similarly to the proof of Theorem 1. Using the notation of Theorem 1, we recall that

$$Q\left(F_a, \sqrt{d}\right) \ll_d \prod_{j=0}^{\infty} \sup_{z \in C_j} \int\limits_{B(\sqrt{d})} \widehat{H}_{z,1}^{2^{2j}\beta}(t)\, dt \leq \prod_{j=0}^{\infty} \sup_{z \in C_j} \int\limits_{B(\sqrt{d})} \widehat{H}_{\pi,1}^{2^{2j}\beta}\big(zt/\pi\big)\, dt.$$

The conditions of Theorem 2 imply that

$$\begin{aligned}\widehat{H}_{\pi,1}(t) &\leq \exp\Big(-c\sum_{k=1}^{n} \min_{m_k \in \mathbf{Z}} \big|2\pi\langle t, a_k\rangle - 2\pi m_k\big|^2\Big)\\ &\leq \exp(-c\,\alpha^2) + \exp\big(-C\,\gamma^2 \langle \mathbb{N}t, t\rangle\big)\end{aligned}$$

for all $\|t\| \leq \sqrt{d}$, where $\mathbb{N}$ is defined in (11). Hence,

$$\begin{aligned}Q\left(F_a, \sqrt{d}\right) &\ll_d \int\limits_{B(\sqrt{d})} \exp(-c\,\gamma^2\beta\, \langle \mathbb{N}t, t\rangle)\, dt + \int\limits_{B(\sqrt{d})} \exp(-c\,\alpha^2\beta)\, dt\\ &\ll_d \Big(\frac{1}{\gamma\sqrt{\beta}}\Big)^d \frac{1}{\sqrt{\det \mathbb{N}}} + \exp(-c\,\alpha^2\beta).\end{aligned}$$

According to (21), $\beta \geq M(1)/4$. Then

$$Q\left(F_a, \sqrt{d}\right) \ll_d \Big(\frac{1}{\gamma\sqrt{M(1)}}\Big)^d \frac{1}{\sqrt{\det \mathbb{N}}} + \exp(-c\,\alpha^2 M(1)),$$

as was claimed. □

Proof of Corollary 3. This proof is similar to the proof of Corollary 1. We denote $b = \dfrac{D}{\sqrt{d}} a \in (\mathbf{R}^d)^n$ and $u = \dfrac{\sqrt{d}\,t}{D}$. Then $\Big(\sum\limits_{k=1}^{n} (\langle u, b_k\rangle - m_k)^2\Big)^{1/2} \geq \min\{\gamma\|t\cdot a\|, \alpha\}$ for all $m_1, \dots, m_n \in \mathbf{Z}$ and $\|u\| \leq \sqrt{d}$. Thus, the conditions of Theorem 2 for the multivector a are valid for the multivector b as well. It remains to note that $Q\big(F_a, d/D\big) = Q(F_b, \sqrt{d})$ and to apply Theorem 2 to the multivector b. □

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