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DIFFRACTION OF ELECTRIC WAVES ON A CONE FORMED OF PERFECTLY MAGNETICALLY AND ELECTRICALLY CONDUCTING SURFACES

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We solve the problem of diffraction of the field of radial electric dipole on a cone whose surface is formed by finite perfectly magnetically conducting and truncated semiinfinite perfectly electrically conducting conical surfaces. The problem is solved by the Wiener–Hopf technique with the use of the Kontorovich–Lebedev integral transformation. We obtain the exact solution of the problem in the static limit and its approximate solution in the low-frequency case. We deduce an expression for the directional pattern of a cone with perfectly absorbing vertex (within the framework of the Macdonald model). We also clarify the effect of the edge of absorbing fragment of the cone surface on its scattering properties.

The laws of diffraction of electromagnetic waves on the boundary of the joint of perfectly magnetically and electrically conducting surfaces are used to model the field scattered by the edges of perfectly absorbing ("black") coatings with an aim to decrease the radar scattering widths. Plane scatterers with perfectly magnetically conducting properties were studied in [11]. A more general case in which a part of the plane surface is an impedance surface was analyzed in [7]. The results of the experimental investigations of "black" surfaces can be found in [10]. The solutions of diffraction problems for perfectly magnetically conducting and "black" bodies of canonical shapes are presented in [4].

The main aim of the present work is to obtain a mathematically rigorous solution of the following electrodynamic problem: In a spherical coordinate system (r,θ,φ), we consider a semiinfinite circular conical surface $\boldsymbol{Q}$ with a vertex at the origin. The cone $\boldsymbol{Q}$ consists of two parts: $\boldsymbol{Q}=\boldsymbol{Q}_1\cup\boldsymbol{Q}_2$, where $\boldsymbol{Q}_1:\ \{r\in[0,c),\ \theta=\gamma,\ \varphi\in[0,2\pi)\}$ is a perfectly magnetically conducting finite cone on the surface of which we impose a boundary condition of equality of the tangential component of the magnetic field to zero, i.e., $\boldsymbol{H}^{(t)}_{\tan}=0$, and $\boldsymbol{Q}_2:\ \{r\in(c,\infty),\ \theta=\gamma,\ \varphi\in[0,2\pi)\}$ is a semiinfinite perfectly electrically conducting cone, $\boldsymbol{E}^{(t)}_{\tan}=0$, with truncated vertex (see Fig. 1). This problem plays the key role in studying the field scattered by the cone for which a part of the surface in the form of a finite cone with vertex is perfectly absorbing (within the framework of the Macdonald model) [9]. For the solution of this problem, we use the Wiener–Hopf technique. The investigations of the diffraction of waves on perfectly electrically conducting conical surfaces with edges can be found in [8].

Assume that the surface $\boldsymbol{Q}$ is excited by an axisymmetrically radial electric dipole located on the axis of symmetry of the cone that coincides with the z-axis of the corresponding Cartesian coordinate system (x,y,z). In a homogeneous medium, a radial electric dipole emits the field with nonzero components E_r, E_θ, and H_φ independent of the azimuthal coordinate φ. The time dependence of the field is described by the factor $e^{-i\omega t}$. In what follows, it is omitted.

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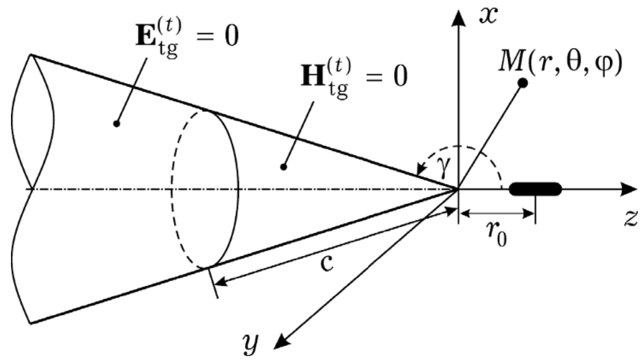

Fig. 1

The total field formed as a result of the interaction of the dipole field with the conical surface $\boldsymbol{Q}$ also has the axial symmetry and its components can be expressed via the Debye scalar potential U as follows:

$$E_r = -\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right), \qquad E_\theta = \frac{1}{r}\frac{\partial^2}{\partial r\,\partial\theta}(rU),$$
$$H_\varphi = ikZ^{-1}\frac{\partial U}{\partial\theta}, \tag{1}$$

where the function $U = U(r,\theta)$ satisfies the Helmholtz equation and Z is the resistance of the medium.

In the free space, we represent the magnetic component of the dipole field in the form [5]

$$H^{\mathrm{i}}_\varphi = iI^e_r hk^3\sqrt{\pi/2}\,\frac{H^{(1)}_{3/2}(kR)}{(kR)^{3/2}}\,r\sin\theta. \tag{2}$$

Here, $I^e_r h$ is the dipole moment, I^e_r is the amplitude of electric current, h is the dipole length, $H^{(1)}_{3/2}(\cdot)$ is the Hankel function of the first kind, $R = \sqrt{r^2 + r_0^2 - 2rr_0\cos\theta}$, r_0 is the radial coordinate of the dipole on the axis of symmetry of the cone, and k is the wave vector $(k = k' + ik''$, k', $k'' > 0)$.

We find the Debye potential of the dipole field in a homogeneous medium from relation (2). For this purpose, in view of the theorem on summation for cylindrical functions [3], we write

$$\frac{H^{(1)}_{3/2}(kR)}{(kR)^{3/2}} = -\frac{\sqrt{2\pi}}{(kr)^{3/2}(kr_0)^{3/2}\sin\theta}\sum_{n=1}^{\infty}\lambda_n P^1_{\lambda_n-1/2}(\cos\theta)\begin{cases} J_{\lambda_n}(kr_0)H^{(1)}_{\lambda_n}(kr), & r \ge r_0, \\ J_{\lambda_n}(kr)H^{(1)}_{\lambda_n}(kr_0), & r \le r_0, \end{cases}$$

where $P^1_{\lambda_n-1/2}(\cos\theta)$ is the associated Legendre function, $J_{\lambda_n}(kr_0)$ is the Bessel function, and $\lambda_n = n + 1/2$, $n = 1, 2, \ldots$.

Further, in view of relations (1) and (2), we get an expression for the Debye potential of the dipole field in the free space. Here, we present this expression in the integral form:

$$U^{\mathrm{i}}(r,\theta) = \frac{A_0}{\pi i\sqrt{sr}} \int\limits_{\Gamma^*} \frac{\nu P_{\nu-1/2}(-\cos\theta)}{\cos(\pi\nu)} K_\nu(sr_0) I_\nu(sr)\, d\nu\,, \tag{3}$$

where $A_0 = \pi I_r^e hZ/\left(r_0\sqrt{sr_0}\right)$, Γ^* is the contour of integration parallel to the imaginary axis $\operatorname{Im}\nu$ in the strip Π: $\{|\operatorname{Re}\nu| < 1/2\}$ of regularity of the integrand, $I_\nu(sr)$ and $K_\nu(sr_0)$ are the modified Bessel and Macdonald functions, respectively, $s = -ik$, and $P_{\nu-1/2}(\cos\theta)$ is the Legendre function. Here and in what follows, we use [3] the definition

$$P^1_{\nu-1/2}(\pm\cos\theta) = \pm\frac{\partial}{\partial\theta}[P_{\nu-1/2}(\pm\cos\theta)]\,.$$

The integral in (3) is absolutely and uniformly convergent for $0 \le \theta < \pi$, $0 < r/r_0 < \infty$, and determines the Debye potential of the dipole field in the free space to within a quantity proportional to the residue of the integrand at the point $\nu = 1/2$. This residue is independent of the variable θ and does not contribute to the components of the field. We can deform the contour of integration Γ^* in (3) within the domain of regularity Π and change the order of arguments of the modified Bessel and Macdonald functions without changing the value of the integral.

The electrodynamic problem is now reduced to the following mixed boundary-value problem for the Helmholtz equation:

$$\Delta U + k^2 U = 0\,, \tag{4}$$

where $U = U(r,\theta)$ is the unknown scalar potential of the diffracted field and Δ is the Laplace operator, which can be represented for the axisymmetric case in a spherical coordinate system as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right).$$

The function U must satisfy the following boundary conditions:

– on the perfectly magnetically conducting surface $\left(r,\theta \in \boldsymbol{Q}_1\right)$

$$\frac{\partial U^{\mathrm{t}}}{\partial\theta} = 0\,; \tag{5$'$}$$

– on the perfectly electrically conducting surface $\left(r,\theta \in \boldsymbol{Q}_2\right)$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U^{\mathrm{t}}}{\partial\theta}\right) = 0\,. \tag{5$''$}$$

Here, U^{t} $(\equiv U + U^{\mathrm{i}})$ is the scalar potential of the total field. In addition, for the uniqueness of solution, the required function must satisfy the condition of boundary absorption at infinity and the condition of boundedness of energy of the electromagnetic field in any bounded volume.

We represent the solution of boundary-value problem (4), (5) in the form of the Kontorovich–Lebedev integral, which is written as

$$U(r,\theta) = \frac{1}{\pi i\sqrt{sr}} \int\limits_{\Gamma^*} \nu F(\nu) P_{\nu-1/2}(\cos\theta) I_\nu(sr)\, d\nu . \tag{6}$$

Here, $F(\nu)$ is the unknown transform, which is an even function regular in the strip Π.

With regard for representation (6), we reduce the boundary-value problem (4), (5) to a system of dual integral equations:

$$\frac{1}{\pi i\sqrt{sr}} \int\limits_{\Gamma^*} \nu(\nu^2 - 1/4)\left[F(\nu)P_{\nu-1/2}(\cos\gamma) + \frac{A_0 P_{\nu-1/2}(-\cos\gamma) K_\nu(sr_0)}{\cos(\pi\nu)}\right] I_\nu(sr)\, d\nu = 0, \quad c < r < \infty , \tag{7$'$}$$

$$\frac{1}{\pi i\sqrt{sr}} \int\limits_{\Gamma^*} \nu\left[F(\nu)P^1_{\nu-1/2}(\cos\gamma) - A_0 \frac{P^1_{\nu-1/2}(-\cos\gamma) K_\nu(sr_0)}{\cos(\pi\nu)}\right] I_\nu(sr)\, d\nu = 0 , \quad 0 \le r < c , \tag{7$''$}$$

where $\operatorname{Re} s > 0$ and we take $\operatorname{Im} s = 0$ to guarantee the convergence of integrals (7). In the final relations, we pass to complex values of this parameter, including imaginary.

We now supplement the right-hand sides of Eqs. (7) to the complete interval and apply the inverse Kontorovich–Lebedev transformation. We exclude the unknown function $F(\nu)$ from the obtained relations and arrive at the following functional equation:

$$\Phi_1(\nu) M(\nu) - \frac{2A_0 K_\nu(sr_0)}{\pi \sin\gamma P_{\nu-1/2}(\cos\gamma)} = \Phi_2(\nu) . \tag{8}$$

Here,

$$\Phi_1(\nu) = \sqrt{s} \int\limits_0^c g_1(r) K_\nu(sr) \frac{dr}{\sqrt{r}} , \tag{9$'$}$$

$$\Phi_2(\nu) = \sqrt{s} \int\limits_c^\infty g_2(r) K_\nu(sr) \frac{dr}{\sqrt{r}} , \tag{9$''$}$$

where $g_1(r) = -rE^{\mathrm{t}}_r(r,\gamma)$ and $g_2(r) = H^{\mathrm{t}}_\varphi(r,\gamma)/i\omega\varepsilon$ are new unknown functions, different from zero in the domains $0 \le r < c$ and $c < r < \infty$, respectively, and

$$M(\nu) = \frac{P^1_{\nu-1/2}(\cos\gamma)}{(\nu^2-1/4)P_{\nu-1/2}(\cos\gamma)}. \tag{10}$$

The function $M(\nu)$ is even and regular and has no zeros in the domain Π. As $|\nu|\to\infty$, the function $M(\nu)$ tends to zero as ν^{-1} in Π. Outside the domain Π, the function $M(\nu)$ has simple real zeros and poles at the points $\nu=\pm z_n$ and $\nu=\pm\nu_n$, $n=1,2,\dots$, respectively, obtained as solutions of the equations

$$P^1_{z_n-1/2}(\cos\gamma)/(\nu_n^2-1/4) = 0, \tag{11$'$}$$

$$P_{\nu_n-1/2}(\cos\gamma) = 0. \tag{11$''$}$$

The asymptotics of the roots of the equations $P^1_{\eta_n-1/2}(\cos\gamma)=0$ and $P_{\nu_n-1/2}(\cos\gamma)=0$ as $n\to\infty$ are as follows [2]:

$$\eta_n = \pi(n-3/4)/\gamma+O(1/n), \qquad \nu_n = \pi(n-1/4)/\gamma+O(1/n). \tag{12}$$

Consider new unknown functions (9) appearing in Eq. (8). As $r\to 0$, the function $g_1(r) = O(r^{z_1-1/2})$, where z_1 is the first positive root of Eq. (11$'$). Since the minimal value of z_1 is reached as $\gamma\to\pi$ and, in addition, $z_1\to 3/2+2(\pi-\gamma)^2/9$ [3], the function $g_1(r)$ is bounded for all $0<\gamma<\pi$ at the vertex of the cone $\boldsymbol{Q}_1$. On the boundary of the joint of the surfaces $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$, the function $g_1(r)$ has an integrable singularity $g_1(r)=O\left((r-c)^{-1/2}\right)$ as $r\to c-0$.

Taking into account that $K_\nu(sr)\sim\left(sr/2\right)^{-|\nu|}$ as $r\to 0$, $\nu\neq 0$ and $K_0(sr)\sim\ln(sr\mathbf{T}2)$ as $r\to 0$, we obtain the following estimate for the function $\Phi_1(\nu)$ (9$'$):

$$\Phi_1(\nu) \le C_0\int_0^1 x^{-|\nu|+z_1-1}\,dx = \frac{C_0}{-|\nu|+z_1},$$

where the constant C_0 is independent of ν and γ. Hence, $\Phi_1(\nu)$ is bounded in the strip Π_1: $\left\{\mathrm{Re}|\nu|<3/2\right\}$ for any c and γ. Since $\Pi\subset\Pi_1$, we see that $\Phi_1(\nu)$ is bounded in Π.

As $r\to c+0$, the function $g_2(r)\sim H^{\mathrm{t}}_{\varphi}(r,\gamma)=O((r-c)^{1/2})$ and, as $r\to\infty$, it satisfies the condition of boundary absorption. Hence, $\Phi_2(\nu)$ defined by (9$''$) is an integer function over the entire complex plane ν $\left(|\nu|<\infty\right)$. In view of the fact that

$$K_\nu(sr) \sim \frac{1}{2}\Gamma\left(|\nu|\right)\left(\frac{sr}{2}\right)^{-|\nu|}, \qquad \left|\frac{\nu}{sr}\right| \gg 1,$$

where $\Gamma(\nu)$ is the gamma-function, we find

$$\frac{\Phi_2(\nu)}{\Gamma(-\nu)}\left(\frac{sc}{2}\right)^{-\nu} = O\left(|\nu|^{-3/2}\right) \tag{13}$$

in the domain $\operatorname{Re}\nu < 1/2$ as $|\nu| \to \infty$.

The known term in Eq. (8) is also a regular function in the strip of regularity Π and exponentially vanishes as $e^{-|\nu|\gamma}$ for $|\nu| \to \infty$. Hence, Eq. (8) is true in Π.

Using the formula of representation of the Macdonald function in terms of the modified Bessel functions [3], we rewrite function (9′) as follows [6]:

$$\Phi_1(\nu) = \frac{1}{2}\Phi_1^{+}(\nu)\left(\frac{sc}{2}\right)^{\nu}\Gamma(-\nu) + \frac{1}{2}\Phi_1^{-}(\nu)\left(\frac{sc}{2}\right)^{-\nu}\Gamma(\nu). \tag{14}$$

Here,

$$\Phi_1^{\pm}(\nu) = \Gamma(1\pm\nu)\left(\frac{sc}{2}\right)^{\mp\nu}\sqrt{s}\int_0^c g_1(r) I_{\pm\nu}(sr)\frac{dr}{\sqrt{r}}, \tag{15}$$

and the relation $\Phi_1^{+}(\nu) = \Phi_1^{-}(-\nu)$ is true.

Since the asymptotic estimate

$$I_{\pm\nu}(sr) \sim \frac{1}{\Gamma(1\pm\nu)}\left(\frac{sr}{2}\right)^{\pm\nu}, \qquad \left|\frac{\nu}{sr}\right| \gg 1,$$

is valid for the modified Bessel functions, the relation

$$\Phi_1^{\pm}(\nu) \le C_1\int_0^1 x^{\pm\nu+z_1-1}\,dx = \frac{C_1}{\pm\nu+z_1} \tag{16}$$

is true for functions (15). Here, C_1 is a constant independent of ν and γ.

As follows from relation (16), the functions $\Phi_1^{\pm}(\nu)$ are regular in the half planes $\operatorname{Re}\nu \begin{smallmatrix} > -z_1 \\ < z_1 \end{smallmatrix}$ of the complex plane ν, respectively.

In view of the behavior of $E_r(r,\gamma)$, i.e., the component of the field as $r \to c-0$, we obtain the following estimate for the functions $\Phi_1^{\pm}(\nu)$ as $|\nu| \to \infty$:

$$\Phi_1^{\pm}(\nu) \le C_2\int_0^1 (1-x)^{-1/2} x^{\pm\nu-1/2}\,dx = C_2 B\left(\frac{1}{2}, \frac{1}{2}\pm\nu\right) = O(\nu^{-1/2}),$$

$$\operatorname{Re}\nu \begin{smallmatrix} > -1/2 \\ < 1/2 \end{smallmatrix},$$

where C_2 is a constant and $B(x,y)$ is the beta-function [3].

Further, we factorize function (10):

$$M(\nu) = M_+(\nu)M_-(\nu), \tag{17}$$

where $M_\pm(\nu)$ are regular functions not equal to zero in the half planes $\operatorname{Re}\nu \begin{smallmatrix} > -1/2 \\ < 1/2 \end{smallmatrix}$, respectively. Furthermore, in this case, we have $M_+(\nu) = M_-(-\nu)$ and, as $|\nu| \to \infty$, in the domains of regularity, we get $M_\pm(\nu) = O\left(\nu^{-1/2}\right)$ and

$$M_+(\nu) = B_0 \frac{\prod_{n=1}^{\infty}\left(1+\frac{\nu}{z_n}\right)e^{-\frac{\nu\gamma}{n\pi}}}{\prod_{n=1}^{\infty}\left(1+\frac{\nu}{\nu_n}\right)e^{-\frac{\nu\gamma}{n\pi}}}, \tag{18}$$

where

$$B_0 = 2i\left[\frac{P^1_{-1/2}(\cos\gamma)}{P_{-1/2}(\cos\gamma)}\right]^{1/2}.$$

We now substitute expression (14) in Eq. (8) and multiply the relation obtained as a result by $2(sc/2)^{-\nu}M_-^{-1}(\nu)\Gamma^{-1}(-\nu)$. As a result, we arrive at a functional equation of the form

$$\Phi_1^+(\nu)M_+(\nu) + \Phi_1^-(\nu)M_+(\nu)\left(\frac{sc}{2}\right)^{-2\nu}\frac{\Gamma(\nu)}{\Gamma(-\nu)}$$

$$- \frac{4A_0(sc/2)^{-\nu}K_\nu(sr_0)}{\pi\sin\gamma\,\Gamma(-\nu)\,M_-(\nu)\,P_{\nu-1/2}(\cos\gamma)}$$

$$= \Phi_2(\nu)\frac{2(sc/2)^{-\nu}}{\Gamma(-\nu)M_-(\nu)}. \tag{19}$$

The first term on the left-hand side of Eq. (19) represents a function regular in the right half plane $\operatorname{Re}\nu > -1/2$ and, as $|\nu| \to \infty$, approaches zero as ν^{-1}. The right-hand side of (19) represents a function regular in the domain $\operatorname{Re}\nu < 1/2$ and, since estimate (13) is true, also tends to zero like ν^{-1} as $|\nu| \to \infty$. The remaining terms of this equation are regular in the strip Π, approach zero in this strip as $|\nu| \to \infty$, and, outside this strip, admit singularities of the type of simple poles. In other words, Eq. (19) is a modified Wiener – Hopf equation.

We apply to Eq. (19) the procedure of factorization with respect to a contour placed in the vertical strip Π by the relations [1]

$$[\ldots]^\pm = \mp\frac{1}{2\pi i}\int_{\Gamma^\pm}[\ldots]\frac{d\nu}{\nu-\alpha},$$

where $[\ldots]^{\pm}$ are functions regular in the half planes $\operatorname{Re}\nu \gtrless {}^{-1/2}_{1/2}$, respectively, $\alpha, \nu \in \Pi$, $\operatorname{Re}\alpha > \operatorname{Re}\nu$ for Γ^{+}, and $\operatorname{Re}\alpha < \operatorname{Re}\nu$ for Γ^{-}.

We now group the terms of the obtained integral equation: the functions regular for $\operatorname{Re}\nu > -1/2$ are moved to the left-hand side and the functions regular for $\operatorname{Re}\nu < 1/2$ are moved to the right-hand side. As $|\nu| \to \infty$, they vanish in these half planes like ν^{-1}. Since the equality holds in the common strip of regularity, its right- and left-hand sides form a function regular in the entire complex plane. According to the Liouville theorem, this function is identically equal to zero in the entire complex plane. Thus, the left- and right-hand sides of this equation are equivalent to the integral equations (7′) and (7″), respectively. We write one of these equations, sufficient for the solution of our problem, in the following way:

$$\Phi_1^{+}(\alpha)M_{+}(\alpha) - \frac{1}{2\pi i}\int_{\Gamma^{+}} \Phi_1^{-}(\nu)M_{+}(\nu)\left(\frac{sc}{2}\right)^{-2\nu} \frac{\Gamma(\nu)}{\Gamma(-\nu)}\frac{d\nu}{\nu-\alpha}$$

$$= -\frac{2A_0}{\pi^2 i \sin\gamma}\int_{\Gamma^{+}} \frac{(sc/2)^{-\nu}K_{\nu}(sr_0)}{\Gamma(-\nu)M_{-}(\nu)P_{\nu-1/2}(\cos\gamma)}\frac{d\nu}{\nu-\alpha}. \tag{20}$$

Here, the integrand on the left-hand side has simple poles at the points $\nu = -n$ and $\nu = -\nu_n$, $n = 1, 2, \ldots$, and the integrand on the right-hand side at the points $\nu = -\nu_n$.

We restrict ourselves to the case where $r_0 > c$. Thus, deforming the contour Γ^{+} in (20) into the left half plane and replacing integrals by the series of residues, we obtain the following functional relation:

$$\Phi_1^{+}(\alpha)M_{+}(\alpha) + \sum_{n=1}^{\infty} \frac{(-1)^n \Phi_1^{+}(n)M_{-}(n)\left(\frac{sc}{2}\right)^{2n}}{\Gamma(n)\Gamma(n+1)(n+\alpha)}$$

$$+ \sum_{n=1}^{\infty} \frac{\pi\Phi_1^{+}(\nu_n)\left(\frac{sc}{2}\right)^{2\nu_n}}{\Gamma(\nu_n)\Gamma(\nu_n+1)\sin(\pi\nu_n)\left[M_{-}^{-1}(\nu_n)\right]'(\nu_n+\alpha)}$$

$$= -\frac{4A_0}{\pi\sin\gamma}\sum_{n=1}^{\infty} \frac{K_{\nu_n}(sr_0)\left(\frac{sc}{2}\right)^{\nu_n}}{\Gamma(\nu_n)M_{+}(\nu_n)(\nu_n+\alpha)\frac{\partial}{\partial\nu}\left[P_{\nu-1/2}(\cos\gamma)\right]_{\nu_n}}, \tag{21}$$

where $\Phi_1^{+}(n)$ and $\Phi_1^{+}(\nu_n)$ are unknown values of the function $\Phi_1^{+}(\alpha)$ at discrete points.

We determine the derivative $\frac{\partial}{\partial\nu}\left[P_{\nu-1/2}(\cos\gamma)\right]_{\nu=\nu_n}$ of the Legendre function with respect to its subscript by using the well-known formula [3] and find

$$\left[M_-^{-1}(\nu_n) \right]' \equiv \frac{d}{d\nu}\left[M^{-1}(\nu) \right]_{\nu=\nu_n} = \frac{(\nu^2 - 1/4)M_+(\nu_n)\partial/\partial\nu\left[P_{\nu-1/2}(\cos\gamma) \right]_{\nu_n}}{P^1_{\nu_n-1/2}(\cos\gamma)} .$$

The following theorems are true:

Theorem 1. *For any parameters of the boundary-value* problem (4), (5)*, the functional relation (21) guarantees the possibility of construction of a function* $\Phi_1^+(\alpha)$ *regular in* $\operatorname{Re}\alpha > -1/2$*, approaching zero like* $\alpha^{-1/2}$ *as* $|\alpha| \to \infty$*, and possessing simple poles in the negative part of the real axis at the points* $\alpha = -n$ *and* $\alpha = -z_n$*,* $n = 1, 2, \ldots$ *.*

Proof. In (21), we set $\alpha = p$ and $\alpha = \nu_p$, $p = 1, 2, \ldots$. As a result, we arrive at two infinite systems of linear algebraic equations of the second kind. Further, we introduce vectors of unknowns $X = \{x_n\}_{n=1}^{\infty}$ and $Y = \{y_n\}_{n=1}^{\infty}$, where $x_n = \Phi_1^+(n)M_+(n)$ and $y_n = \Phi_1^+(\nu_n)M_+(\nu_n)$. Then X and Y belong to the Hilbert space ℓ_2 because $x_n, y_n = O(1/n)$ as $n \to \infty$. The matrices of these systems are written as the sum of operators unit and compact in ℓ_2. Hence, the obtained infinite systems of linear algebraic equations admit solutions with given accuracy for arbitrary parameters of the problem (except the points of the spectrum). The assertions of Theorem 1 directly follow from relation (21), Q.E.D.

Theorem 2. *The scalar potential* U *of the field of a radial electric dipole on the axis of the cone* $\boldsymbol{Q} = \boldsymbol{Q}_1 \cup \boldsymbol{Q}_2$ *can be represented as a series in eigenfunctions of the Helmholtz equation for conical domains with perfectly electrically conducting surface for* $r > c$ *and perfectly magnetically conducting surface for* $r < c$ *.*

Proof. Consider the case where $r > c$. Then we determine from Eq. (7′) the transform of the scalar potential of the diffracted field and substitute it in relation (6). In view of relation (14) and expression (3) for the vanishing field, we represent the potential of the total field in the form

$$U^{\mathrm{t}}(r,\theta) = \frac{1}{\pi i\sqrt{sr}} \int\limits_{\Gamma^*} \nu \frac{\Phi_1^+(\nu)\left(\dfrac{sc}{2}\right)^{\nu} \Gamma(-\nu)P_{\nu-1/2}(\cos\theta)}{2(\nu^2 - 1/4)P_{\nu-1/2}(\cos\gamma)} I_\nu(sr)\,d\nu$$

$$+ \frac{1}{\pi i\sqrt{sr}} \int\limits_{\Gamma^*} \nu \frac{\Phi_1^-(\nu)\left(\dfrac{sc}{2}\right)^{-\nu} \Gamma(\nu)P_{\nu-1/2}(\cos\theta)}{2(\nu^2 - 1/4)P_{\nu-1/2}(\cos\gamma)} I_\nu(sr)\,d\nu$$

$$- \frac{A_0}{\pi i\sqrt{sr}} \int\limits_{\Gamma^*} \nu\Big[P_{\nu-1/2}(-\cos\gamma)P_{\nu-1/2}(\cos\theta)$$

$$- P_{\nu-1/2}(\cos\gamma)P_{\nu-1/2}(-\cos\theta) \Big]\Big[\cos(\pi\nu)P_{\nu-1/2}(\cos\gamma) \Big]^{-1} K_\nu(sr_0)I_\nu(sr)\,d\nu . \qquad (22)$$

We replace the first integral in (22) by a series of residues, closing the contour of integration into the right half-plane, where the integrand has simple poles at the points $\nu = n$ and $\nu = \nu_n$, $n = 1, 2, \ldots$, and for

$\nu = 1/2$. We represent the second integral in (22) by a series of residues at the points $\nu = -1/2$, $\nu = -n$, and $\nu = -\nu_n$, $n = 1, 2, \ldots$, closing the contour of integration into the left half plane. Since $I_n(sr) = I_{-n}(sr)$, it is easy to see that the residues at the points $\nu = n$ and $\nu = -n$ are mutually canceled and, by virtue of the relation

$$K_{\nu_n}(sr) = \frac{1}{2}\Gamma(-\nu_n)\Gamma(\nu_n+1)\left[I_{\nu_n}(sr) - I_{-\nu_n}(sr)\right],$$

we can form a series of Macdonald functions from the residues at the points $\nu = \nu_n$ and $\nu = -\nu_n$. We obtain the required representation of the third integral in (22) by rearranging the arguments of the modified Bessel function and Macdonald function in this integral and replacing it by a series of residues at the points $\nu = \nu_n$. The final expression for the potential can be written as

$$U^{\mathrm{t}}(r,\theta) = \frac{2A_0}{\sqrt{sr}}\sum_{n=1}^{\infty}\frac{\nu_n P_{\nu_n-1/2}(-\cos\gamma) I_{\nu_n}(sr_0) P_{\nu_n-1/2}(\cos\theta) K_{\nu_n}(sr)}{\cos(\pi\nu_n)\,\partial/\partial\nu\left[P_{\nu-1/2}(\cos\gamma)\right]_{\nu=\nu_n}}$$

$$-\frac{2}{\sqrt{sr}}\sum_{n=1}^{\infty}\frac{\Phi_1^+(\nu_n)\left(\dfrac{sc}{2}\right)^{\nu_n} P_{\nu_n-1/2}(\cos\theta) K_{\nu_n}(sr)}{(\nu_n^2-1/4)\Gamma(\nu_n)\,\partial/\partial\nu\left[P_{\nu-1/2}(\cos\gamma)\right]_{\nu=\nu_n}}. \tag{23}$$

Here, the first series corresponds to the scalar potential of the complete field of a perfectly conducting semiinfinite cone under the condition of its excitation by the field of radial dipole (2), whereas the second series describes the disturbance introduced to this potential by a finite perfectly magnetically conducting conical surface.

Note that, in (23), we omitted the terms independent of the angle θ and corresponding to the poles of the integrand at the points $\nu = \pm 1/2$. These terms do not make contributions to the expressions for the components of the field.

Further, we obtain an expression for the field potential in the domain $r < c$. By using relations (1) and representation (23) for $\theta = \gamma$, we find

$$\frac{H_\varphi^{\mathrm{t}}(r,\gamma)}{i\omega\varepsilon} = -\frac{2}{\pi\sin\gamma}\frac{A_0}{\pi i\sqrt{sr}}\int_{-i\infty}^{i\infty}\frac{\nu K_\nu(sr_0)}{P_{\nu-1/2}(\cos\gamma)} I_\nu(sr)\,d\nu$$

$$-\frac{2}{\sqrt{sr}}\sum_{n=1}^{\infty}\frac{\Phi_1^+(\nu_n)\left(\dfrac{sc}{2}\right)^{\nu_n} P^1_{\nu_n-1/2}(\cos\gamma)}{(\nu_n^2-1/4)\Gamma(\nu_n)\,\partial/\partial\nu\left[P_{\nu-1/2}(\cos\gamma)\right]_{\nu=\nu_n}} K_{\nu_n}(sr). \tag{24}$$

In deriving (24), we have taken into account that if $P_{\nu_n-1/2}(\cos\gamma) = 0$, then the equality

$$P_{\nu_n-1/2}(-\cos\gamma)P^1_{\nu_n-1/2}(\cos\gamma) = \frac{2\cos\pi\nu_n}{\pi\sin\gamma}$$

is true and the first series in (23) has been represented in the form of an integral.

We now substitute expression (24) in relation (9″) and represent the function $\Phi_2(\alpha)$ in the domain $\operatorname{Re}\alpha > \operatorname{Re}\nu \in \Pi$ as follows:

$$\Phi_2(\alpha) = 2\sum_{n=1}^{\infty} \frac{\Phi_1^+(\nu_n) P^1_{\nu_n-1/2}(\cos\gamma)\left(\frac{sc}{2}\right)^{\nu_n}}{(\nu_n^2 - 1/4)\Gamma(\nu_n)\partial/\partial\nu\left[P_{\nu-1/2}(\cos\gamma)\right]_{\nu=\nu_n}} \frac{scW\left[K_{\nu_n}K_\alpha\right]_{sc}}{\alpha^2-\nu_n^2}$$

$$-\frac{4A_0}{\pi\sin\gamma}\sum_{n=1}^{\infty}\frac{\nu_n K_{\nu_n}(sr_0)}{\partial/\partial\nu\left[P_{\nu-1/2}(\cos\gamma)\right]_{\nu=\nu_n}}\frac{scW[I_{\nu_n}K_\alpha]_{sc}}{\alpha^2-\nu_n^2} - \frac{2A_0K_\alpha(sr_0)}{\pi\sin\gamma P_{\alpha-1/2}(\cos\gamma)}. \tag{25}$$

Here, $W\left[f_\alpha\varphi_\nu\right]_t = f_\alpha(t)\varphi'_\nu(t) - \varphi_\nu(t)f'_\alpha(t)$, where $f_\alpha(t)$ and $\varphi_\nu(t)$ are arbitrary Bessel functions and the primes stand for the derivatives with respect to the argument.

The series in (25) converge absolutely and uniformly with respect to α $(|\alpha|<\infty)$. The function $\Phi_2(\alpha)$ is regular because the singularities of the second sum at the points $\alpha = \pm\nu_n$, where $n = 1, 2, \ldots$, are compensated by the singularities of the third term of (25) at these points.

We can now write the transform of required scalar potential of the diffracted field as follows:

$$F(\nu) = \frac{\Phi_2(\nu)}{P^1_{\nu-1/2}(\cos\gamma)} + \frac{A_0P^1_{\nu-1/2}(-\cos\gamma)K_\nu(sr_0)}{\cos(\pi\nu)P^1_{\nu-1/2}(\cos\gamma)}. \tag{26}$$

Substituting relation (26) in (6) and taking into account expression (25), we obtain an integral along the contour Γ^* whose integrand satisfies the Jordan lemma in the right half plane. Further, we replace this integral by a series of residues and, by virtue of relation (3), obtain the representation of scalar potential of the total field in the form of a series in eigenfunctions of the Helmholtz equation for a conical domain with perfectly magnetically conducting surface:

$$U^{\mathrm{t}}(r,\theta) = \frac{1}{\sqrt{sr}}\sum_{p=1}^{\infty}\frac{2z_p\Psi_pP_{z_p-1/2}(\cos\theta)}{\partial/\partial z\left[P^1_{z_p-1/2}(\cos\gamma)\right]}I_{z_p}(sr). \tag{27}$$

Here,

$$\Psi_p = -2\sum_{n=1}^{\infty}\frac{\Phi_1^+(\nu_n)(sc/2)^{\nu_n}P^1_{\nu_n-1/2}(\cos\gamma)scW\left[K_{\nu_n}K_{z_p}\right]_{sc}}{(\nu_n^2-1/4)\Gamma(\nu_n)\partial/\partial\nu\left[P_{\nu_n-1/2}(\cos\gamma)\right](z_p^2-\nu_n^2)}$$

$$+\frac{4A_0}{\pi\sin\gamma\sqrt{sr}}\sum_{n=1}^{\infty}\frac{\nu_nK_{\nu_n}(sr_0)scW\left[I_{\nu_n}K_{z_p}\right]_{sc}}{\partial/\partial\nu\left[P_{\nu_n-1/2}(\cos\gamma)\right](z_p^2-\nu_n^2)}. \tag{28}$$

Representations (23) and (27) complete the proof of Theorem 2.

Corollary. *The values of the function* $\Phi_1^+(\nu)$ *at the points* $\nu = \nu_n$, $n = 1, 2, \ldots$, *guarantee the possibility of construction of the solution of the Wiener–Hopf equation* (8) *and, hence, of the original diffraction problem.*

Field in the Radiation Zone. As $r \to \infty$, relations (1) and (23) yield

$$E_\theta^t(r,\theta) = ZH_\varphi^t(r,\theta) = D(\theta)\frac{e^{ikr}}{r}. \tag{29}$$

Here, $D(\theta)$ is the directional pattern of the cone $\boldsymbol{Q}$:

$$D(\theta) = D_1(\theta) + D_2(\theta),$$

where

$$D_1(\theta) = -\frac{2A_0}{\sin\gamma}\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\frac{\nu_n I_{\nu_n}(sr_0)P^1_{\nu_n-1/2}(\cos\theta)}{P^1_{\nu_n-1/2}(\cos\gamma)\partial/\partial\nu\left[P_{\nu_n-1/2}(\cos\gamma)\right]}, \tag{30}$$

$$D_2(\theta) = 2\sqrt{\frac{\pi}{2}}\sum_{n=1}^{\infty}\frac{\Phi_1^+(\nu_n)\left(\dfrac{sc}{2}\right)^{\nu_n}P^1_{\nu_n-1/2}(\cos\theta)}{(\nu_n^2-1/4)\Gamma(\nu_n)\partial/\partial\nu\left[P_{\nu-1/2}(\cos\gamma)\right]_{\nu=\nu_n}}. \tag{31}$$

Relations (30) and (31) enable us to get, within the framework of the Macdonald model [9], an expression for the directional pattern of a semiinfinite cone with "black" vertex:

$$\hat{D}(\theta) = \frac{D_1(\theta) + D(\theta)}{2}. \tag{32}$$

Special Cases. In the static limit $(s \to 0)$, the functional equation (21) admits the following exact solution:

$$\Phi_1^+(\nu_n) = -\frac{2A_0}{\pi\sin\gamma M_+(\nu_n)}\sum_{p=1}^{\infty}\frac{(c/r_0)^{\nu_p}}{M_+(\nu_p)(\nu_p+\nu_n)\partial/\partial\nu\left[P_{\nu_p-1/2}(\cos\gamma)\right]}, \tag{33}$$

where series (33) converges absolutely and uniformly in γ.

Now let $\gamma = \pi/2$. Under this condition, we get $\nu_n = 2n - 1/2$. Then

$$M_+(\nu_n) = \frac{i}{\sqrt{2n}}\frac{\Gamma(n+1/2)}{\Gamma(n)}, \tag{34}$$

$$P^1_{\nu_n-1/2}(0) = \frac{(-1)^n 2\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n)}, \tag{35}$$

$$\frac{\partial P_{\nu_n-1/2}(0)}{\partial\nu} = \frac{(-1)^n\sqrt{\pi}\,\Gamma(n)}{2\Gamma(n+1/2)}. \tag{36}$$

In view of relations (34)–(36), we rewrite expression (33) as follows:

$$\Phi_1^+(\nu_n) = \frac{4A_0\Gamma(n+1)}{\pi^{3/2}(c/r_0)^{1/2}\Gamma(n+1/2)}\sum_{p=1}^{\infty}\frac{(-1)^p p(c/r_0)^{2p}}{p+n-1/2}. \tag{37}$$

If $|sc/2| < 1$, then, by using relations (34)–(37), we obtain the following approximations for the radiation field (30), (31) in the low-frequency range:

$$D_1(\theta) = -2A_0\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\frac{\left(\frac{sr_0}{2}\right)^{2n-\frac{1}{2}}P_{2n-1}^1(\cos\theta)}{\Gamma(2n-1/2)}, \tag{38}$$

$$D_2(\theta) = \frac{4A_0}{\pi}\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\frac{(-1)^n/^{2n-\frac{1}{2}}P_{2n-1}^1(\cos\theta)}{(2n-1)\Gamma(2n-1/2)}\left[\sum_{p=1}^{\infty}\frac{(-1)^p p(c/r_0)^{2p-1/2}}{(p+n-1/2)}\right]. \tag{39}$$

If $c/r_0 \ll 1$, then, in the analysis of (38) and (39), we restrict ourselves to the single-mode approximation:

$$\begin{aligned} D_1(\theta) &= -2iI_r^e hkZ\sin\theta + O\left(\left(\frac{kr_0}{2}\right)^3\right), \\ D_2(\theta) &= \frac{8iI_r^e hkZ}{3\pi}\left[\left(\frac{c}{r_0}\right)^3 + O\left(\frac{c}{r_0}\right)^5\right]\sin\theta + O\left(\left(\frac{kr_0}{2}\right)^3\left(\frac{c}{r_0}\right)^5\right). \end{aligned} \tag{40}$$

In view of relations (40), we represent the field scattered by the conducting plane with "black" circle of small radius in the form

$$\hat{D}(\theta) = -2iI_r^e hkZ\left[1 - \frac{2}{3\pi}\left(\frac{c}{r_0}\right)^3\right]\sin\theta. \tag{41}$$

Relation (41) illustrates the effect of decay of the field because $|\hat{D}(\theta)| < |D_1(\theta)|$. As follows from this formula, the level of efficiency of absorbing disk in the plane is determined by the cube of the ratio c/r_0 and increases as the source approaches the vertex.

CONCLUSIONS

We solve the problem of diffraction of the field of a radial electric dipole on a cone whose surface is formed by finite perfectly magnetically conducting and truncated perfectly electrically conducting surfaces. Using the Wiener – Hopf technique together with the Kontorovich–Lebedev integral transformation, we obtain the solution guaranteeing that all necessary conditions of the boundary-value problem are satisfied for arbitrary values of the parameters. We deduce the representations of the field via the series in eigenfunctions. The exact solution of the problem is obtained in the static limit and its approximate solution is obtained in the low-frequency range. We propose an expression for the directional pattern of the cone with perfectly absorbing vertex. In the low-frequency case, it is shown that if the conducting plane with circular absorbing coating (the case $\gamma = \pi/2$) is excited by a radial electric dipole, then the efficiency of absorption is determined by the cube of the ratio of radius of the circle c to the radial coordinate of the dipole r_0 .

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