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ON CONSTRUCTION OF THE REFINED EQUATIONS OF VIBRATION OF ELASTIC PLATES

I. T. Selezov UDC 539.3

We describe the key stages of the development of generalized dynamic theories of bending vibrations of the bars, plates, and shells, based on the shear model proposed by S. P. Timoshenko, outstanding Ukrainian scientist, in 1921 ("On the correction for shear of the differential equation for transverse vibrations of prismatic bar," *Phil. Mag., Ser. 6,* **41**, No. 245, 744–746). We present the mathematical construction of equations of the theory of plates derived as hyperbolic approximations of the elastodynamic problem for a layer. The analytic expression for the coefficient of thickness shear is obtained. Some contemporary investigations are also discussed. As an example, we consider the process of wave propagation along an elastically restrained strip.

Introduction

The refined equations for plates were first written by Uflyand in 1948 [14]. A fundamental contribution to the development of the Timoshenko theory was made in 1951 by Mindlin [28], who generalized the Timoshenko theory of beams to plates. A fundamental survey of investigations carried out in this field up to 1972 was proposed by Grigolyuk and Selezov [4]. In 1960, both the known refined equations of the theory of plates and new equations of higher order were constructed as hyperbolic approximations of the elastodynamic problem for a layer in [13]. In the present work, we describe the main stages of development of the method of power series {starting from Cauchy (1828) [16] and Poisson (1829) [30]} and its applications. We deduce and present a generalized equation of the theory of plates including the known equations and new (more exact) equations without introducing physical hypotheses and correcting shear coefficients. We also reveal the correspondence between the equations obtained in [28] and more exact equations deduced in [13] and show the difference between these equations.

To construct refined theories, the researchers applied the method of power series, asymptotic and some other methods. The asymptotic methods in the dynamics of bars, plates, and shells began to develop much later than in the other natural sciences. All known methods are, in fact, reduced to decreasing, in one or another way, the dimension of the three-dimensional problem of the theory of elasticity, i.e., to the replacement of a three-dimensional problem with a two-dimensional problem.

Parallel with analytic methods, the computational approaches become more and more extensively used at present. Reliable computer programs have been developed. However, the application of numerical analysis, especially in the dimensional form, often decreases the possibilities of establishing the regularities of the investigated processes. As one of the ways of decreasing the dimension of the boundary-value problems, one can mention its reduction to boundary integral equations. Another way is connected with its transformation to the problem of finding the minimum of a functional defined on the boundary of the domain [3]. The application of

Institute of Hydromechanics, Ukrainian National Academy of Sciences, Kiev, Ukraine.

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spline functions in the problem of reduction enables one to consider higher modes, which enables us to construct a very efficient algorithm occupying, in fact, an intermediate position between the theory of plates and the theory of elasticity. However, this is not a method for the development of models but just an application of computational mathematics.

1. On the Development of the Cauchy–Poisson Method for the Reduction of Three-Dimensional Problems to Two-Dimensional Problems

The method of power series was applied for the first time in the works of Cauchy and Poisson, outstanding mathematicians of the 19th century [16, 30].[1] They derived the approximate equations of plates proceeding from the fact that a plate represents a domain in the three-dimensional space one dimension of which (thickness) is small as compared with the other dimensions and, therefore, the solutions were sought as power series in the thickness coordinate. This method was further developed and applied to the construction of refined models of plates and shells. In Fig. 1, we show the applications of the method of power series to the construction of approximate models of plates and shells.

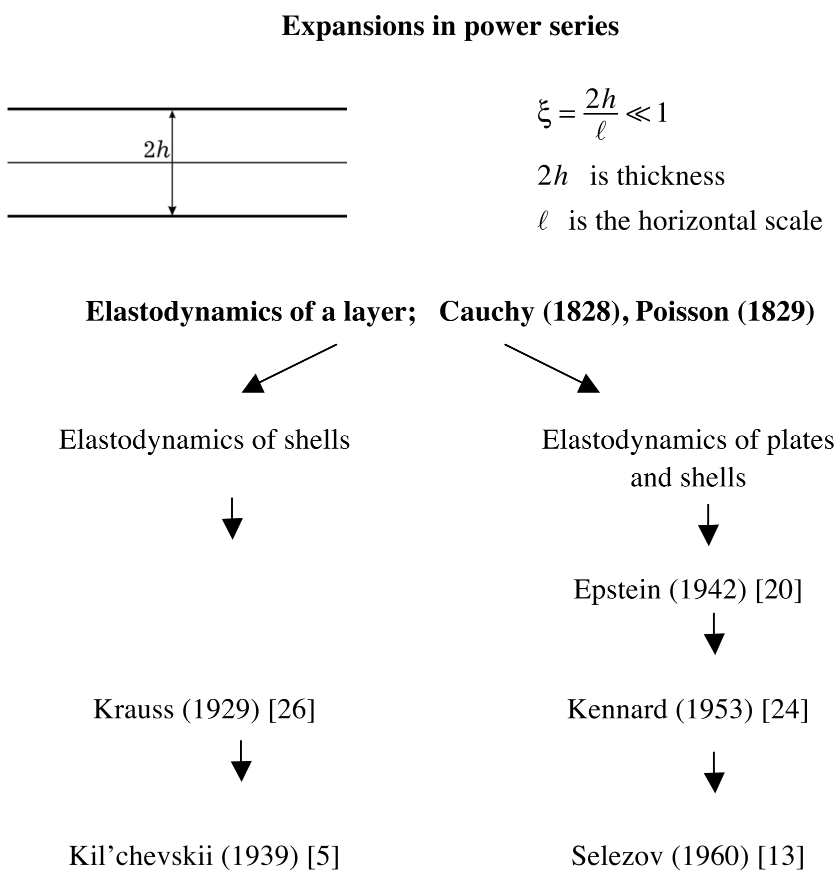

Fig. 1. Evolution of the Cauchy–Poisson algorithm.

[1] The author managed to read these works in 1958 at the Lenin Library in Moscow.

As follows from Fig. 1, this method was first used in the statics of shells (left column) and then in the dynamics of plates and shells (right column).

2. Refined Known and New Equations of Higher Orders as Hyperbolic Approximations of the Elastodynamic Problem for a Layer

Consider the reduction of a three-dimensional elastodynamic problem for a layer to a two-dimensional problem based on the method of power series in the thickness coordinate x_3 for hyperbolic models. The reduction of the set of mathematical approximations obtained in this case enables one to construct refined models improving the classical models. The algorithm used in this case leads to the degeneracy of the original initial-boundary problem with respect to the small thickness coordinate ξ and decreases the dimension of the problem:

$$(x_1, x_2, x_3, t) \to (x_1, x_2, t)\,. \tag{1}$$

However, this is attained by the degeneracy of the spectrum of the original problem.

Consider an elastodynamic problem for an infinite layer of thickness $2h$ bounded by the surfaces $x_3 = \pm h$ in a domain

$$\Omega = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1, x_2 (-\infty, \infty), x_3 \in \left[-\frac{\xi}{2}, \frac{\xi}{2} \right] \right\}, \qquad \xi = \frac{2h}{\ell}\,, \tag{2}$$

of the Cartesian coordinate system (x_1, x_2, x_3), where ℓ is the horizontal scale of the layer. The initial-boundary-value problem for the displacement vector $\mathbf{u} = (u_1, u_2, u_3 = w)$ is formulated as follows: It is necessary to find a vector function $\mathbf{u} = \mathbf{u}(x_1, x_2, x_3, t)$ as a solution in the domain $\Omega \times [0, T]$, $T > 0$ of the hyperbolic equations

$$\nabla^2 u_k + \left(1 + \frac{\lambda}{G} \right) \partial_k (\nabla \cdot \mathbf{u}) = \partial_{tt} u_k\,, \qquad k = 1,2,3\,, \tag{3}$$

satisfying the following boundary conditions for the components of the stress tensor on the surfaces of the layer $x_3 = \pm \xi/2$:

$$\sigma_{33}\big|_{x_3 = \xi/2} = q^{\pm}(x_1, x_2, t)\,,$$

$$\sigma_{3i}\big|_{x_3 = -\xi/2} = p_i^{\pm}(x_1, x_2, t), \qquad i = 1,2\,, \tag{4}$$

and the initial conditions at $t = 0$:

$$u_k\big| = 0, \qquad \partial_t u_k\big| = 0, \qquad k = 1,2,3\,. \tag{5}$$

The dimensionless quantities are introduced everywhere and the following characteristic scales are used for this purpose: length ℓ, shear modulus G, and the velocity of propagation of shear waves $c_s = \sqrt{G/\rho}$. We assume that λ and G are constants.

In what follows, it is assumed that the thickness of the layer is small, $\xi \ll 1$, and hence, it is natural to use expansions in the dimensionless coordinate x_3 relative to the middle surface $x_3 = 0$, decreasing in this way the dimension of our problem [35, 36]. This leads to the degeneracy of the original hyperbolic model. In this case, there are three possible cases of degeneracy resulting in equations of the parabolic, hyperbolic, and mixed types. Note that only the degeneracy of a hyperbolic model into a hyperbolic model is correct and has a physical meaning provided that the velocity of propagation of disturbances is finite [10].

We represent the unknown functions in the form of power series:

$$u_k(x_1, x_2, x_3, t) = \sum_{m=0}^{\infty} u_{km}(x_1, x_2, t) x_3^m, \qquad k = 1,2,3 . \tag{6}$$

As a result, the original problem (2)–(5) is reduced to finding infinitely many functions u_{km} satisfying an infinite system of differential equations and recursive relations. In turn, this infinite system is decomposed into two independent subsystems, corresponding to the symmetric (plane) and asymmetric (bending) deformation of the middle surface $x_3 = 0$. The case of symmetric vibrations is considered in [37].

Consider the case of asymmetric deformations (summation is carried out from $s = 0$ to $s = \infty$):

$$e(x_1, x_2, x_3, t) = \sum_{s=0}^{\infty} \tilde{e}_{(2s+1)}(x_1, x_2, t) x_3^{2s+1}, \qquad e = u_{i,i}, \qquad i = 1,2 ,$$

$$w(x_1, x_2, x_3, t) = \sum_{s=0}^{\infty} \tilde{w}_{(2s)}(x_1, x_2, t) x_3^{2s} ,$$

$$\begin{gathered}
\sum_{s=0}^{\infty} [(2s+1)\tilde{e}_{2s+1} + \nabla^2 \tilde{w}_{2s}] 2^{-2s} \xi^{2s} = \frac{\partial}{\partial x_1} \frac{1}{2}(p_1^+ + p_1^-) + \frac{\partial}{\partial x_2} \frac{1}{2}(p_2^+ + p_2^-) , \\
\sum_{s=0}^{\infty} \left[-\tilde{e}_{2s+1} - \frac{1}{2s+1} L_s \tilde{w}_{2s} \right] 2^{-(2s+1)} \xi^{2s+1} = \frac{1}{2}(q^+ - q^-) , \\
\tilde{w}_{2s+2} = -\frac{1+\lambda/G}{(2s+2)(2+\lambda/G)} \tilde{e}_{2s+1} - \frac{1}{(2s+1)(2s+2)(2+\lambda/G)} L_s \tilde{w}_{2s} , \\
\tilde{e}_{2s+3} = \frac{1}{(2s+2)(2s+3)} \left[-L_e + \frac{1+\lambda/G}{2+\lambda/G} \nabla^2 \right] \tilde{e}_{2s+1} + \frac{1}{(2s+1)(2s+2)(2s+3)} \frac{1+\lambda/G}{2+\lambda/G} \nabla^2 + L_s \tilde{w}_{2s} ,
\end{gathered} \tag{7}$$

where

$$L_s = c_s^2\nabla^2 - \frac{\partial^2}{\partial t^2}, \qquad L_e = c_e^2\nabla^2 - \frac{\partial^2}{\partial t^2},$$

e is the divergence of planar displacements, w is a deviation, $p_i^{\pm}$ and $q^{\pm}$ are the shear and normal loads along the faces of the layers, and c_s and c_e are the velocities of propagation of shear and dilatation waves.

Equations (7) give the exact solution of our problem. The reduction of this system enables one to obtain a series of approximations of different kinds. The hyperbolic degeneracy of the initial-boundary-value problem for a finite hyperbolic system of equations of any order in $\mathbb{R}^n$ based on the method of power series was considered in [36]. Thus, the necessary and sufficient conditions for the degeneracy were established. They include the completeness of the reduced system and the preservation of all space-time differential operators up to a certain order.

The truncation of Eqs. (7) up to the seventh order, inclusively, leads to a three-mode approximation (thickness wave modes), which can be reduced to the following equation:

$$\left\{\left[\left\langle\left(\xi\frac{\partial^2}{\partial t^2} + \xi^3 a_1\nabla^2\nabla^2\right)_K - \xi^3 a_2\frac{\partial^2}{\partial t^2}\nabla^2\right\rangle_R + \xi^3 a_3\frac{\partial^4}{\partial t^4}\right]_{TM} - \xi^5 b_1\nabla^2\nabla^2\nabla^2\right.$$

$$\left. + \xi^5 b_2\frac{\partial^2}{\partial t^2}\nabla^2\nabla^2 - \xi^5 b_3\frac{\partial^4}{\partial t^4}\nabla^2 + \xi^5 b_4\frac{\partial^6}{\partial t^6}\right\}_{TMS} w$$

$$= \left\{\left[1 - \xi^2 d_1\nabla^2 + \xi^2 d_2\frac{\partial^2}{\partial t^2}\right]_{TM}\right.$$

$$\left. + \xi^4 d_3\nabla^2\nabla^2 - \xi^4 d_4\frac{\partial^2}{\partial t^2}\nabla^2 + \xi^4 d_5\frac{\partial^4}{\partial t^4}\right\}_{TMS} (q^+ - q^-)\,. \tag{8}$$

Timoshenko [40] generalized the Bernoulli–Euler parabolic model of bending vibrations of a beam to a hyperbolic model on the phenomenological basis by introducing corrections responsible for thickness shear strains and rotary inertia. On this basis, Mindlin [28] generalized the Kirchhoff parabolic model of the bending vibrations of the plates [25] [operator K in Eq. (8)] to a hyperbolic model (double-mode model—TM operators). A more general hyperbolic model [13] was constructed as a mathematical approximation without introducing phenomenological assumptions (three-mode approximation—TMS operators), including a two-dimensional system as a special case.

3. Shear Coefficient

It is worth noting that the coefficients a_p, b_q, and d_r in Eq. (8) depend only on Poisson's ratio ν. This fact enables us to find the exact value of the shear coefficient

$$k^2 = \frac{2}{2 - \nu + \sqrt{1/2 + \nu^2}} \tag{9}$$

by comparing Eq. (10) given below with the correcting shear coefficient and equation $[\cdot]_{TM}$ in (8) obtained as a mathematical approximation. A more exact value of the shear coefficient can be found from the complete operator $\{\cdot\}_{TMS}$ in (8).

The Mindlin refined equations of vibration of the plates are based on the same distribution of displacements as in the Timoshenko model. The resolving equation for the deflection w_0 has the form

$$\left\{\frac{\partial^2}{\partial t^2} + \frac{D}{\rho h}\nabla^2\nabla^2 - \frac{D}{k^2 Gh} + \frac{I}{h}\frac{\partial^2}{\partial t^2}\nabla^2 + \frac{\rho I}{k^2 Gh}\frac{\partial^4}{\partial t^4}\right\} w_0 = \left\{\frac{1}{\rho h} + \frac{I}{k^2 Gh^2}\frac{\partial^2}{\partial t^2} - \frac{D}{k^2 G\rho h^2}\nabla^2\right\} q\,, \tag{10}$$

where $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity, ρ is density, h is thickness, k^2 is the shear coefficient, G is the shear modulus, I is the moment of inertia, E is the Young modulus, ν is Poisson's ratio, t is time, q is the transverse load, and ∇^2 is the Laplacian.

Comparing Eq. (10) with a more exact equation (8), we see their difference: Eq. (8) takes into account three wave modes unlike two modes in (10).

In the Timoshenko model applied to plates, the analysis of thickness shear stresses is performed by rejecting the hypothesis of normality of a linear element to the middle surface of the plate. At the same time, it is supposed that the element originally linear and normal to the middle surface remains linear after deformation. This does not agree with the parabolic law of changes in the stresses of thickness shear. In the case of static bending of the plate, Vlasov [2] removed this disagreement by bending of an originally rectilinear element of the plate. The corresponding analog in the theory of bars was considered earlier [9]. Another attempt to take into account transverse shear more exactly was made in [34].

The refined theory of layered composite plates taking into account the actual (parabolic) distribution of transverse stresses over the thickness of the plate was developed by Reddy in [32]. Later, the shear strains of higher orders were taken into account for layered composite plates in [33]. It should be emphasized that the derived system of three equations in displacements is quite cumbersome and, hence, their application to the solution of problems is possible only with the use of computer calculations.

4. Correctness of the Formulation of Problems and Their Solvability

In the solution of boundary-value problems, the refined equations are supplemented by the boundary conditions corresponding to the Timoshenko model. In this case, the problem of correctness of the boundary conditions is very important. The correct boundary conditions were proposed in [41], where one can also find the proof of existence of the second spectrum (see also [4, p. 80]). Formerly, in some earlier and contemporary works, the equations of Timoshenko beam were solved with the boundary conditions of the classical theory corresponding to the single-mode approximation [43]. Therefore, all subsequent considerations and conclusions concerning the frequency spectra and the meaning of the second spectrum remain debatable [1, 8, 15, 18, 29].

We also emphasize that there exists a principal difference between the Rayleigh equation (single-mode approximation)

$$\xi \frac{\partial^2 w}{\partial t^2} + \xi^3 a_1 \frac{\partial^4 w}{\partial x^4} - \xi^3 a_2 \frac{\partial^4 w}{\partial t^2 \partial x^2} = q^+ - q^- \tag{11}$$

and the Timoshenko equation (double-mode approximation)

$$\xi \frac{\partial^2 w}{\partial t^2} + \xi^3 a_1 \frac{\partial^4 w}{\partial x^4} - \xi^3 a_2 \frac{\partial^4 w}{\partial t^2 \partial x^2} + \xi^3 a_3 \frac{\partial^4 w}{\partial t^4} = \left(1 - \xi^2 d_1 \frac{\partial^2}{\partial x^2} + \xi^2 d_2 \frac{\partial^2}{\partial t^2}\right)(q^+ - q^-)\,. \tag{12}$$

The classical method of separation of the variables [11]

$$w(x,t) = W(x)T(t)$$

leads to the complete separation of variables in the first case [Eq. (11)]; however, the variables are not separated in the second case [Eq. (12)].

It is also worth noting that the dynamic reaction of anisotropic layered composite plates to stationary random excitations was studied in [17] on the basis of the Elishakoff monograph [19].

The models of the dynamics of bars, plates, and shells, taking into account shear strains and rotary inertia, are used in the contemporary analysis of elements of different structures [6, 7, 21–23, 27, 31, 38, 39, 42, 44].

5. Wave Propagation Along a Strip with Elastically Restrained Edges

Consider the problem of propagation of harmonic waves in an infinite elastic plate of width b with elastically restrained edges. The motion of elastic plate is described by the Mindlin refined theory taking into account rotary inertia and the strains of transverse shear [12]:

$$\left[\frac{\partial^2}{\partial t^2} + \frac{D}{\rho h}\nabla^4 - \left(\frac{D}{k_s^2 G h} + \frac{I}{h}\right)\frac{\partial^2}{\partial t^2}\nabla^2 + \frac{\rho I}{k_s^2 G h}\frac{\partial^4}{\partial t^4}\right] w = 0\,, \tag{13}$$

$$D(1-\nu)\frac{\partial^2 \psi_1}{\partial x_2^2} + \left[D(1-\nu)\frac{\partial^2}{\partial x_1^2} - 2k_s^2 G h - 2\rho I \frac{\partial^2}{\partial t^2}\right]\psi_1$$

$$= \left\{\frac{\partial}{\partial x_1}\left[D(1+\nu)\nabla^2 + 2k_s^2 G h - \frac{\rho D(1+\nu)}{k_s^2 G}\frac{\partial^2}{\partial t^2}\right]\right\} w\,, \tag{14}$$

$$-D(1-\nu)\frac{\partial^2 \psi_1}{\partial x_1 \partial x_2} + \left[D(1-\nu)\frac{\partial^2}{\partial x_1^2} - 2k_s^2 G h - 2\rho I \frac{\partial^2}{\partial t^2}\right]\psi_2$$

$$= 2\left[\frac{\partial}{\partial x_2}\left(D\nabla^2 + k_s^2 G h - \frac{D\rho}{k_s^2 G}\frac{\partial^2}{\partial t^2}\right)\right] w\,. \tag{15}$$

The boundary conditions corresponding to elastic restraint are taken in the form

$$w|_{x_2=0} = 0, \qquad \left.\frac{\partial\psi_2}{\partial x_2}\right|_{x_2=0} = R_1\psi_2\,,$$
$$w|_{x_2=b} = 0, \qquad \left.\frac{\partial\psi_2}{\partial x_2}\right|_{x_2=b} = -R_2\psi_2\,, \tag{16}$$

where w is a deflection, ψ is a rotation angle, $I = h^3/12$ is the cross-sectional moment of inertia, h is thickness, R_1 and R_2 are the stiffnesses of elastic restraints for the rotation of the edges, ν is Poisson's ratio, E is Young's modulus, $G = E/[2(1+\nu)]$ is the Lamé constant, and $k_s^2 = 2/(2-\nu+\sqrt{0.5+\nu^2})$ is the shear coefficient [13]. In Eqs. (13)–(16) and in what follows, we use dimensionless quantities introduced by the following relations (the asterisks are omitted):

$$(x_1^*, x_2^*, w^*, h^*) = \frac{1}{\ell}(x_1, x_2, w, h), \qquad t^* = t\frac{v}{\ell}, \qquad \rho^* = \frac{\rho}{\rho_0}\,,$$

$$D^* = \frac{D}{\ell^3\rho v^3}, \qquad G^* = \frac{G}{\rho v^2}, \qquad I^* = \frac{I}{\ell^3}, \qquad \{R_1^*, R_2^*\} = \ell\{R_1, R_2\}\,,$$

where ℓ and v are the characteristic length and velocity, respectively.

We seek the solution in the class of running waves:

$$f(x_1, x_2, t) = \bar{f}(x_2)e^{i(kx_1-\omega t)}\,. \tag{17}$$

Then Eqs. (13)–(15) with regard for (17) take the form

$$D\frac{d^4\bar{w}}{dx_2^4} + \left(\frac{D}{k_s^2Gh} + \frac{I}{h}\right)\omega^2\frac{d^2\bar{w}}{dx_2^2} + \left\{\frac{Dk^4}{\rho h} + \frac{\rho I\omega^4}{k_s^2Gh} - \left[\omega^2 + \left(\frac{D}{k_s^2Gh} + \frac{I}{h}\right)\omega^2k^2\right]\right\}\bar{w} = 0\,, \tag{18}$$

$$D(1-\nu)\frac{d^2\bar{\psi}_1}{dx_2^2} + [-k^2D(1-\nu) - 2k_s^2Gh + 2\rho I\omega^2]\bar{\psi}_1 = ik\left[D(1-\nu)\frac{d^2}{dx_2^2}\right.$$
$$\left. + k^2D(1+\nu) + 2k_s^2Gh + \frac{\rho\omega^2D(1+\nu)}{k_s^2G}\right]\bar{w}\,, \tag{19}$$

$$[-k^2D(1-\nu) - 2k_s^2Gh + 2\rho\omega^2I]\bar{\psi}_2 = 2\left[\frac{d}{dx_2}\left(D\frac{d^2}{dx_2^2} - k^2D\right.\right.$$
$$\left.\left. + 2k_s^2Gh + \frac{\omega^2\rho D}{k_s^2G}\right)\right]\bar{w} + ikD(1-\nu)\frac{d\bar{\psi}_1}{dx_2}\,. \tag{20}$$

In the general case, the solution of the problem under consideration has the form

$$\bar{w} = C_1 e^{-\alpha_1 x_2} + C_2 e^{\alpha_1 x_2} + C_3 e^{-\alpha_3 x_2} + C_4 e^{\alpha_3 x_2}, \tag{21}$$

$$\bar{\psi}_1 = \frac{\eta_1 e^{-\alpha_1 x_2}}{\alpha_1^2 - \eta^2} C_1 + \frac{\eta_1 e^{\alpha_1 x_2}}{\alpha_1^2 - \eta^2} C_2 + \frac{\eta_3 e^{-\alpha_3 x_2}}{\alpha_3^2 - \eta^2} C_3 + \frac{\eta_3 e^{\alpha_3 x_2}}{\alpha_3^2 - \eta^2} C_4, \tag{22}$$

$$\bar{\psi}_2 = \xi_1 \alpha_1 e^{-\alpha_1 x_2} C_1 - \xi_1 \alpha_1 e^{\alpha_1 x_2} C_2 + \xi_3 \alpha_3 e^{-\alpha_3 x_2} C_3 - \xi_3 \alpha_3 e^{\alpha_3 x_2} C_4, \tag{23}$$

where

$$\alpha_{1,3} = -\beta \pm \sqrt{-k^4 + \beta_2 k^2 + \beta_3},$$

$$\eta_{1,3} = ik \frac{1+\nu}{1-\nu} \left(\alpha_{1,3}^2 - k^2 + \frac{2k_s^2 Gh}{D(1+\nu)} + \frac{\rho\omega^2}{k_s^2 G} \right),$$

$$\xi_{1,3} = \frac{2}{\eta^2 (1-\nu)} \left[\alpha_{1,3}^2 - k^2 + \frac{ik\eta_{1,3}(1-\nu)}{2(\alpha_{1,3}^2 - \eta^2)} + \frac{k_s^2 Gh}{D} + \frac{\omega^2 \rho}{k_s^2 G} \right],$$

$$\beta_1 = \frac{\rho h}{2D} \left(\frac{D}{k_s^2} + \frac{I}{h} \right) \omega^2, \qquad \beta_2 = 2\beta_1, \qquad \eta = k^2 - \frac{2(\rho\omega^2 I - k_s^2 Gh)}{D(1-\nu)},$$

$$\beta_3 = \frac{\rho^2 h^2}{4D^2} \left[\left(\frac{D}{k_s^2 Gh} + \frac{I}{h} \right)^2 - \frac{4DI}{k_s^2 Gh} \right] \omega^4 + \frac{\rho h \omega^2}{D}.$$

Substituting solutions (21)–(23) in the boundary conditions (16), we arrive at a dispersion equation

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ (\alpha_1^2 + R_1\alpha_1)\xi_1 & (\alpha_1^2 - R_1\alpha_1)\xi_1 & (\alpha_3^2 + R_1\alpha_3)\xi_3 & (\alpha_3^2 - R_1\alpha_3)\xi_3 \\ e^{-\alpha_1 b} & e^{\alpha_1 b} & e^{-\alpha_3 b} & e^{\alpha_3 b} \\ (\alpha_1^2 - R_2\alpha_1)\xi_1 e^{-\alpha_1 b} & (\alpha_1^2 + R_2\alpha_1)\xi_1 e^{\alpha_1 b} & (\alpha_3^2 - R_2\alpha_3)\xi_3 e^{-\alpha_3 b} & (\alpha_3^2 + R_2\alpha_3)\xi_3 e^{\alpha_3 b} \end{vmatrix} = 0. \tag{24}$$

The numerical determination of the roots of the dispersion equation (24) was carried out for the wave numbers $kb \in [0.01, 10]$ and the following dimensionless parameters: characteristic length $b = 1$, $\rho = 1$, $D = 2.974 \cdot 10^{-10}$, $c_r = 1.231$, $I = 8.333 \cdot 10^{-11}$, $R_1 = -10$, and $R_2 = 10$. As the characteristic values, we took the rate of wave propagation according to the Timoshenko theory $v = c_r = \sqrt{Gk_s^2/\rho}$ and the width of the plate $\ell = b$. The results of calculation of the dependence of phase velocity $c_p = \omega/k$ on the frequency ω for different thicknesses of the plate are presented in Fig. 2.

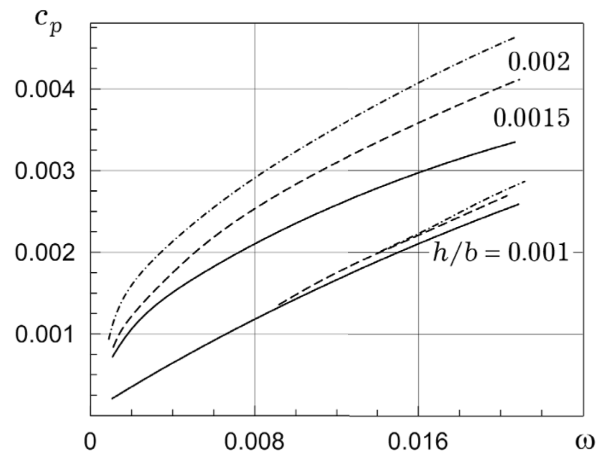

Fig. 2. Dependences of the phase velocity c_p on the circular frequency ω for different thicknesses.

The performed analysis shows that, within the analyzed interval of wave numbers, the dispersion equation (24) has two real roots one of which weakly depends on the thickness h, whereas the second root strongly depends on h. The presented curves illustrate the dispersion properties of the analyzed system and we see, in particular, that, the dispersion increases with the thickness of the plate.

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