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ABSENCE OF C^1- Ω-EXPLOSION IN THE SPACE OF SMOOTH SIMPLEST SKEW PRODUCTS

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ABSTRACT. We give a detailed proof of absence of a C^1- Ω-explosion in the space of C^1-regular simplest skew products of mappings of an interval (i.e., skew products of mappings of an interval with a closed set of periodic points). We study the influence of C^1-perturbations (of the class of skew products) to the set of periods of the periodic points of C^1-regular simplest skew products, and describe the peculiarities of period doubling bifurcations of the periodic points.

1. Introduction

Different aspects of Ω-explosion phenomenon in dynamical systems are considered, in particular, in [2, 4, 6, 7, 12, 13, 16, 17, 25, 26, 28, 30].

In this paper we consider the influence of C^1-perturbations (of the class of skew products) to a nonwandering set of C^1-regular *simplest* skew products of mappings of an interval (i.e., skew products of mappings of an interval with a closed set of periodic points). Our results should be probably treated in the context of studies of the general problem concerning perturbations of dynamical systems of the class of skew products formulated by Anosov in [1].

This paper is a sequel to [4, 12]. It contains a detailed proof of impossibility of a C^1- Ω-explosion in C^1-regular simplest skew products of mappings of an interval (this property was announced in [4]); we study the influence of C^1-perturbations (of the class of skew products) to the set of periods of the periodic points of C^1-regular simplest skew products and describe the peculiarities of period doubling bifurcations of the periodic points (see [13]).

We consider a skew product of mappings of an interval, i.e., a dynamical system (d.s.) $F : I \to I$, where $I = I_1 \times I_2$ is a closed rectangle in the plane (I_1 and I_2 are segments) such that

$$F(x, y) = (f(x), g_x(y)), \quad \text{where} \quad g_x(y) = g(x, y),\ (x; y) \in I. \tag{1.1}$$

A mapping $f : I_1 \to I_1$ is called *the factor mapping* (*factor*) of d.s. (1.1), and the mapping $g_x : I_2 \to I_2$ for each $x \in I_1$ is called the mapping *acting in the layer over the point* x.

Due to (1.1), there holds

$$F^n(x, y) = (f^n(x), g_{x,n}(y)), \text{ where } \quad g_{x,n} = g_{f^{n-1}(x)} \circ \ldots \circ g_x. \tag{1.2}$$

We use the notation $\widetilde{g}_x$ for the mapping $g_{x,n}$ if x is a periodic point of f ($x \in Per(f)$) and n is its (smallest) period.

By $T^0(I)$ ($T^1(I)$), we denote the space of all continuous (C^1-regular) skew products of mappings of an interval with a standard C^0 norm $\|\cdot\|_0$ (C^1 norm $\|\cdot\|_1$).

For an arbitrary mapping $F \in T^0(I)$ ($F \in T^1(I)$), the C^0 (C^1) norm is defined by the formula

$$\|F\|_0 = \max\{\sup_{x \in I_1} |f(x)|, \sup_{(x,y) \in I} |g_x(y)|\}$$

$$(\|F\|_1 = \max\{\|F\|_0, \|DF\|_0\}), \tag{1.3}$$

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$$\text{where}\quad ||DF||_0 = \max\left\{\sup_{x\in I_1}|f'(x)|,\ \sup_{(x,y)\in I}\left(\left|\frac{\partial g_x(y)}{\partial x}\right| + \left|\frac{\partial g_x(y)}{\partial y}\right|\right)\right\},$$

$DF : I \to I$ is the differential of F).

The base of topology in the space $T^r(I)$ ($r = 0$ or 1) is given by the set of ε-balls $B^r_\varepsilon(F)$ with a center F for all $\varepsilon > 0$ and all $F \in T^r(I)$.

Different functional spaces are related to an arbitrary skew product $F \in T^r(I)$. Thus we use the space $C^r(I_k)$ ($k = 1, 2$) of all continuous mappings of the segment I_k into itself with a C^0 norm $||\cdot||_{0,k}$, where

$$||f||_{0,k} = \sup_{t\in I_k}|f(t)|,$$

for $r = 0$ and of all C^1-regular mappings of the segment I_k into itself with a C^1 norm

$$||f||_{1,k} = \max\{\sup_{t\in I_k}|f(t)|, \sup_{t\in I_k}|f'(t)|\} \tag{1.4}$$

for $r = 1$. In these cases the base of topology in $C^r(I_k)$ is given by the set of ε-balls $B^r_{k,\varepsilon}(f)$ for each $\varepsilon > 0$ and each $f \in C^r(I_k)$.

We also need spaces $C^r(I, I_2)$ of continuous mappings of a rectangle I into the segment I_2 with a C^0 norm $||\cdot||_{0,(1,2)}$ such that

$$||g||_{0,(1,2)} = \sup_{(x;y)\in I}|g_x(y)|$$

for $r = 0$ and of C^1-regular mappings from I into I_2 with a C^1 norm $||\cdot||_{1,(1,2)}$ for $r = 1$.

Let $Dg : I \to I_2$ be the differential of the mapping $g \in C^1(I, I_2)$. Put

$$||Dg||_{0,(1,2)} = \sup_{(x,y)\in I}\left(\left|\frac{\partial g_x(y)}{\partial x}\right| + \left|\frac{\partial g_x(y)}{\partial y}\right|\right).$$

We define the C^1 norm $||\cdot||_{1,(1,2)}$ setting

$$||g||_{1,(1,2)} = \max\{||g||_{0,(1,2)},\ ||Dg||_{0,(1,2)}\}. \tag{1.5}$$

The base of the standard topology in $C^r(I, I_2)$ is given by the set of ε-balls

$$B^r_{(1,2),\varepsilon}(g) = \{\psi \in C^r(I, I_2) : ||g - \psi||_{r,(1,2)} < \varepsilon\}$$

for each $g \in C^r(I, I_2)$ and each $\varepsilon > 0$.

We need the following inequality for the skew product $F \in T^1(I)$, which immediately follows from (1.3)–(1.5):

$$\max\{||f||_{1,1},\ ||g_x||_{1,2}\} \le \max\{||f||_{1,1},\ ||g||_{1,(1,2)}\} = ||F||_1 \tag{1.6}$$

for each $x \in I_1$.

We assign the functional mapping $\rho_1 : I_1 \to C^r(I, I_2)$ called the C^r-representation (se, e.g., [23]) (here $r = 0$ or 1) such that

$$\rho_1(x) = g_x \text{ for all } x \in I_1$$

to a skew product of mappings of the interval $F \in T^r(I)$.

Recall that the C^r-representation $\rho_1 : I_1 \to C^r(I, I_2)$ is *continuous* at the point $x' \in I_1$ if for each $\varepsilon > 0$ there exists a positive number $\delta = \delta(x', \varepsilon)$ such that for each $x \in I_1$ satisfying inequality $|x - x'| < \delta$ there holds

$$||g_x - g_{x'}||_{r,(1,2)} < \varepsilon.$$

Note that $F \in T^r(I)$ iff $f \in C^r(I_1)$, and $\rho_1 : I_1 \to C^r(I, I_2)$ is continuous on the segment I_1 (i.e., is continuous at each point of this segment in the aforementioned sense) [20, Ch. 2, Sec. 20, VII; Ch. 4, Sec. 44, IV].

We need both a definition of a nonwandering set of a dynamical system (see [18, Part I, Ch. 3, Sec. 3.3]) of form (1.1) and that of a C^r- Ω-explosion (see [2, Ch. 1, Sec. 4]) for $r = 0$ and 1.

Definition 1.1. A point $z^0(x^0;\, y^0) \in I$ is called a *nonwandering point* of a mapping $F \in T^0(I)$ if for each its neighborhood $U(z^0)$ in I there exists a natural number $n = n(z^0)$ such that

$$U(z^0) \cap F^n(U(z^0)) \neq \varnothing.$$

The set of all nonwandering points of the d.s. (1.1) is called the *nonwandering set* and denoted as $\Omega(F)$. Points of the phase space that are not nonwandering are called *wandering* ones.

Definition 1.2. We say that a mapping $F \in T^r(I)$ $(r = 0 \,\text{or}\, 1)$ *admits a C^r- Ω-explosion* if there exists a $\delta > 0$ such that each ε-neighborhood $B^r_\varepsilon(F)$ of the mapping F in the space $T^r(I)$ contains a mapping Φ such that $\Omega(\Phi) \not\subset U_\delta(\Omega(F))$, where $U_\delta(\Omega(F))$ is the δ-neighborhood of the nonwandering set $\Omega(F)$ of the mapping F in the rectangle I.

Note that if mappings of the space $T^1(I)$ are considered as elements of the space $T^0(I)$ with a C^0 norm, one can speak about the phenomenon of a C^0- Ω-explosion in C^1-regular skew products of mappings of the interval (see [4, 12]).

The main results of the paper are: Theorem 3.1 on absence of a C^1- Ω-explosion in the space of C^1-regular simplest skew products of mappings of an interval (Sec. 3); Theorem 3.2 on an estimate of periods of periodic points of such mappings from a certain neighborhood of the mapping under consideration (in $T^1(I)$) (Sec. 3); Theorems 4.1 and 4.2 on peculiarities of period doubling bifurcations of the periodic points in C^1-regular skew products (Sec. 4). For the sake of comparison with the assertion of Theorem 3.1, note that at the same time C^1-regular simplest skew products of mappings of an interval do admit a C^0- Ω-explosion (see [4, 12]).

2. Preliminaries

In order to formulate and prove the main results of the paper, we need some statements from [5, 15, 19, 21, 27]. Thus, the following proposition on coexistence of periods for periodic points of continuous skew products of interval mappings is proved in [19].

Proposition 2.1. *If a mapping $F \in T^0(I)$ contains a periodic orbit of period m, then it also contains periodic orbits of each period n such that n precedes m $(n \prec m)$ in the Sharkovsky order*:

$$1 \prec 2 \prec 2^2 \prec 2^3 \prec \ldots \prec 2^2 \cdot 5 \prec 2^2 \cdot 3 \ldots\ldots \prec 2 \cdot 5 \prec 2 \cdot 3 \prec \ldots 9 \prec 7 \prec 5 \prec 3$$

We also use the auxiliary statement[1] proved in [4].

Lemma 2.1. *If the set $Per(F)$ of a C^1-regular mapping F is closed, then the set $\tau(F)$ of the (smallest) periods of periodic points of F is bounded.*

Proposition 2.1 and Lemma 2.1 imply that for an arbitrary simplest mapping $F \in T^1(I)$ one can find an integer $0 \leq \nu < +\infty$ such that $\tau(F) = \{1, 2, \ldots, 2^\nu\}$.

It is important to note that each C^1-regular mapping of a segment into itself (even in the case of a bounded set of (the smallest) periods of its periodic points) can be approximated with any precision in the C^0 norm by a C^1-regular mapping of the segment that has periodic points with arbitrary large periods (see, e.g., [29, Ch. 2, Sec. 2]) (this property is a cause, though not the only one [4, 12], of possibility of a C^0- Ω-explosion in C^1-regular simplest skew products of interval mappings). At the

[1] An analogous result is contained in paper [8]. Note that [8] affirms the existence of a C^∞-regular skew product of interval mappings of the type $\prec 2^\infty$ that has a one-dimensional attracting set. But the skew product itself is realized as a shift mapping along the trajectories of the respective nonautonomous system of differential equations with C^∞-regular right-hand sides. The latter means that the system is considered in $\mathbb{R}^3$, and oscillations of the trajectory near the limit set are "distributed" along the unbounded axis t. When we consider the skew product in a rectangle of the plane xOy it is impossible to "distribute" oscillations of a trajectory with a one-dimensional attracting set. This leads to oscillations of the partial derivative $\dfrac{\partial}{\partial x} g_x(y)$ and to its unboundedness near the attracting set, though in this case the mapping $g_x(y)$ can be of class C^∞ with respect to y (but not with respect to the entirety of variables) [11].

same time, as the following statement from [21] shows, such approximation is impossible in the C^1 norm.

Proposition 2.2. *If a mapping $\varphi \in C^1(I_k)$ ($k = 1$ or 2) has no periodic point with period 2^i, then there exists a ε-neighborhood $B^1_{k,\varepsilon}(\varphi)$ of φ in $C^1(I_k)$ such that each mapping from $B^1_{k,\varepsilon}(\varphi)$ has no periodic points with period 2^{i+1}, whatever $i \geq 1$ is.*

An effective instrument for studying the phenomenon of a C^0- Ω-explosion in different classes of dynamical systems are chain recurrent points (see, e.g., [7, Ch. I, Sec. 2-3], [6]). Thus, in [12] chain recurrent points are used to prove the criterion of realizability of a C^0- Ω-explosion in C^1-regular simplest skew products of interval mappings. At the same time, in continuous (or C^1-regular) mappings of the segment with a closed set of periodic points the phenomenon of C^0- Ω-explosion is impossible.

In fact, there holds

Proposition 2.3. *For a mapping $\varphi \in C^0(I_k)$ ($k = 1$ or 2), the following statements are equivalent*:

(2.3.1) *the set $Per(\varphi)$ of periodic points of φ is closed*;
(2.3.2) $\Omega(\varphi) = Per(\varphi)$ [27];
(2.3.3) $CR(\varphi) = Per(\varphi)$, *where $CR(\cdot)$ is the set of chain recurrent points*[2] *of the mapping* (*see* [5, Ch. 6, Sec. 1, 4]).

Moreover, in [6] the following statement is proved.

Proposition 2.4. *A continuous mapping Φ of a compact X into itself admits an Ω-explosion in the space of continuous embeddings of X into itself iff for sets of chain recurrent points $CR(\Phi)$ and of nonwandering ones $\Omega(\Phi)$ of the mapping Φ one has*

$$CR(\Phi) \neq \Omega(\Phi).$$

Propositions 2.3 and 2.4 imply impossibility of a C^0- Ω-explosion in continuous (or C^1-regular) mappings of a segment with a closed set of periodic points. Since $C^1(I_k) \subset C^0(I_k)$ and $B^1_{k,\varepsilon}(\varphi) \subset B^0_{k,\varepsilon}(\varphi)$ for all $\varepsilon > 0$ and all $\varphi \in C^1(I_k)$, using the definition of a C^1- Ω-explosion in C^1-regular mappings of a segment (see Definition 1.2), we obtain for the sequel

Proposition 2.5. *Any mapping $\varphi \in C^1(I_k)$ ($k = 1$ or 2) with a closed set $Per(\varphi)$ does not admit a C^1- Ω-explosion.*

To conclude this section, we give a proof of the following statement that uses the ideas of [15].

Theorem 2.1. *For a C^1-regular skew product of interval mappings, the following statements are equivalent*:

(2.1.1) $\Omega(F) = Per(F)$, *where $Per(\cdot)$ is the set of periodic points of the mapping*;
(2.1.2) *the set of points $Per(F)$ is closed.*

For the proof of Theorem 2.1, we use special multivalued functions related to an arbitrary continuous skew product of interval mappings.

Definition 2.1 (see [10]). The Ω-*function* of a mapping $F \in T^0(I)$ is the function $\zeta^F : \Omega(f) \to 2^{I_2}$ such that for any $x \in \Omega(f)$ one has

$$\zeta^F(x) = (\Omega(F))(x),$$

where $(\Omega(F))(x) = \{y \in I_2 : (x; y) \in \Omega(F)\}$ is the cut of the nonwandering set $\Omega(F)$ by a vertical layer over the point x, 2^{I_2} is the topological space of closed subsets of I_2 with exponential topology [20, Ch. 1, Sec. 17, I].

[2]A point $z \in I_k$ is called chain recurrent for a mapping $\varphi \in C^0(I_k)$ if for each $\varepsilon > 0$ there exists a ε-chain with respect to the mapping φ, which connects z with itself. Here a ε-chain with respect to the mapping φ connecting points z_1 and z_2 is a finite set of points $\{u_k\}_{k=0}^n$ such that $u_0 = z_1$, $u_n = z_2$, and $|\varphi(u_{k-1}) - u_k| < \varepsilon$ for $k = 1, 2, \ldots, n$ (see, e.g., [7, Ch. I, Sec. 2])

Note that the Ω-function of the mapping $F \in T^0(I)$ has a real dynamic sense: its graph in I coincides with the nonwandering set $\Omega(F)$ of F[10].

For any mapping $F \in T^0(I)$ and any natural number n, introduce a skew product $F_n(x, y) = (id(x), g_{x,n}(y))$ and a direct product $F_{n,1}(x, y) = (f^n(x), id(y))$, where $id(x)$ is the identical mapping of I_1 and $id(y)$ is that of I_2, $F_n, F_{n,1} : I \to I$. Then the mapping F_n "stops motion" in the base of I_1 (any point $x \in I_1$ is fixed for its factor mapping $id(x)$), and this leads to F_n-invariance of each vertical layer $\{x\} \times I_2$; and the mapping $F_{n,1}$ "stops motion" in the vertical layers (any point $y \in I_2$ is fixed for the mapping $id(y)$ in an arbitrary vertical layer), and this leads to $F_{n,1}$-invariance of horizontal layers $I_1 \times \{y\}$.

It is important to note that

$$F^n = F_{n,1} \circ F_n. \tag{2.1}$$

Formula (2.1) allows to relate to each iteration of F new multivalued functions such that with the help of their graphs one can form a nonwandering set of the skew product $F \in T^0(I)$ (or, equivalently, the graph of the Ω-function of F).

Let the factor mapping f of the skew product $F \in T^0(I)$ have a closed set of periodic points $Per(f)$. Then one has $\Omega(f^n) = \Omega(f) = Per(f)$ (see [27]).

Following [10], define *auxiliary multivalued functions* $\eta_n : \Omega(f) \to 2^{I_2}$, setting for any $x \in \Omega(f)$

$$\eta_n(x) = \Omega(g_{x,n}).$$

The closures $\overline{\eta}_n$ of the graphs of functions η_n have a real dynamic sense: for any $n \geq 1$, the closure $\overline{\eta}_n$ coincides with the nonwandering set of the restriction $F_{n|\Omega(f) \times I_2}$ (see [14, 15]).

After defining functions η_n for each $n \geq 1$, each point $(x;\, y)$ of the graph η_n should be moved to $(f^n(x);\, y)$ by the direct product $F_{n,1}$ (see Eq. (2.1)). Thus we naturally obtain multivalued functions $\eta_{n,1} : \Omega(f) \to 2^{I_2}$ $(n \geq 1)$ defined by equalities

$$\eta_{n,1}(x) = (F_{n,1}(\eta_n))(x)$$

for each $x \in \Omega(f)$; here η_n is the graph of the respective multivalued function in I, and $(F_{n,1}(\eta_n))(x)$ is the cut of the set $F_{n,1}(\eta_n)$ by a layer over the point $x \in \Omega(f)$.

Since we arrive to each point $(x;\, y)$ on the graph of $\eta_{n,1}$ using $F_{n,1}$ from each point $(\overline{x};\, y)$, where $\overline{x}$ is an arbitrary point of the nth complete preimage of x with respect to the mapping $f_{|\Omega(f)}$, the following equality holds:

$$\eta_{n,1}(x) = \bigcup_{\overline{x} \in \{f^{-n}(x)\}} \eta_n(\overline{x}), \tag{2.2}$$

where $\{f^{-n}(x)\}$ is the nth complete preimage of the point x with respect to $f_{|\Omega(f)}$.

Multivalued functions η_n and $\eta_{n,1}$ admit natural extensions η_n^{ex} to the segment I_1 and $\eta_{n,1}^{ex}$ to $f^n(I_1)$ $(n \geq 1)$ respectively if $\Omega(f) \neq I_1$. In this case, whatever $n \geq 1$ is, one has

$$\eta_n^{ex}(x) = \Omega(g_{x,n})$$

for all $x \in I_1$ and

$$\eta_{n,1}^{ex}(x) = (F_{n,1}(\eta_n^{ex}))(x)$$

for all $x \in f^n(I_1)$, where in the latter equality η_n^{ex} is the graph of the respective multivalued function in I.

Denote by $Per(f, l_n)$ the set of all f-periodic points such that the (smallest) period of each such point is a divisor of $l_n = 2^n$, $n \geq 0$ (i.e., the (smallest) periods of such points form the set $\{1, 2, 2^2, \ldots, 2^n\}$).

For each $n \geq 0$, set[3]

$$\eta'_{l_n|Per(f,l_n)} = \bigcup_{i=0}^{n} \eta_{l_i|Per(f,l_i)}; \qquad \eta'_{l_n,1|Per(f,l_n)} = \bigcup_{i=0}^{n} \eta_{l_i,1|Per(f,l_i)};$$
$$\eta^{ex'}{}_{l_n,1} = \bigcup_{i=0}^{n} \eta^{ex}_{l_i,1}, \tag{2.3}$$

where the last relation in (2.3) holds for all $x \in \bigcap_{i=0}^{n} f^{l_i}(I_1)$.

The proof of Theorem 2.1 is based on the following statement from [15].

Proposition 2.6. *Let the factor mapping f of the skew product of interval mappings $F \in T^0(I)$ have a closed set of periodic points $Per(f)$. Then there exist mutually equal topological limits* (*see* [20, Ch. 2, Sec. 29, VI]) $\operatorname*{Lim}_{n\to+\infty} \eta'_{l_n,1|Per(f,l_n)}$ *and* $\operatorname*{Lim}_{n\to+\infty} \eta'_{l_n|Per(f,l_n)}$, *and there holds the following formula*:

$$\zeta^{F_{|Per(f)\times I_2}} = \operatorname*{Lim}_{n\to+\infty} \eta'_{l_n,1|Per(f,l_n)} = \operatorname*{Lim}_{n\to+\infty} \eta'_{l_n|Per(f,l_n)} = \overline{\bigcup_{x\in Per(f)} \{x\} \times \Omega(\widetilde{g}_x)},$$

where $\eta'_{l_n,1|Per(f,l_n)}$, $\eta'_{l_n|Per(f,l_n)}$, and $\zeta^{F_{|Per(f)\times I_2}}$ are the graphs of respective multivalued functions in I, and $\overline{(\cdot)}$ is the closure of a set.

Moreover, if x is a periodic point of f with the (*smallest*) *period $l(x)$, which is a limit point for nonperiodic points of f, then*

$$\zeta^F(x) = \operatorname*{Ls}_{n\to+\infty} \eta^{ex'}_{l(x)n,1|U_{1,\varepsilon_{l(x)n}}(x)}, \tag{2.4}$$

where $\zeta^F(x)$ is the value of the Ω function at x (*see Definition* 2.1), $\operatorname*{Ls}_{n\to+\infty}(\cdot)$ *is the upper topological limit of a sequence of sets* [20, Ch. 16, Sec. 29, III-IV], $\eta^{ex'}_{l(x)n,1|U_{1,\varepsilon_{l(x)n}}(x)}$ *are graphs of respective multivalued functions in I, and $U_{1,\varepsilon_{l(x)n}}(x)$ is the $\varepsilon_{l(x)n}$-neighborhood of point $x \in Per(f)$ such that* $\lim_{n\to+\infty} \varepsilon_{l(x)n} = 0$.

We pass to the proof of Theorem 2.1.

Proof of Theorem 2.1. Since $\Omega(F)$ is a closed set, property (2.1.1) implies (2.1.2). Therefore we check that property (2.1.2) implies (2.1.1). In fact, by Lemma 2.1 for some $\nu < +\infty$ one has $\tau(F) = \{1, 2, \dots, 2^\nu\}$.

1. Consider the case $\nu = 0$, i.e., $\tau(F) = \{1\}$. Using Proposition 2.2, we find, firstly, a ε_1-neighborhood $B^1_{1,\varepsilon_1}(f)$ of the mapping f in $C^1(I_1)$ such that for any $\varphi \in B^1_{1,\varepsilon_1}(f)$ one has $\tau(\varphi) \subseteq \{1;\, 2\}$, and, secondly, for each $n \geq 2$ and $x \in Per(f)$ a $\varepsilon_n(x)$-neighborhood $B^1_{2,\varepsilon_n(x)}(g^n_x)$ of the mapping $g^n_x = g_{x,n}$ in $C^1(I_2)$ such that for any mapping $\theta \in B^1_{2,\varepsilon_n(x)}(g^n_x)$ one has the inclusion $\tau(\theta) \subseteq \{1;\, 2\}$.

For all $n \geq 2$ and $x \in Per(f)$, we use neighborhoods $B^1_{(1,2),\varepsilon_n(x)}(g^n_x)$ of C^1-regular (with respect to the entirety of variables) mappings in layers $\{g_{x,n}\}_{x\in I_1}$ for F^n. By inequality (1.6), in particular, for each $x \in Per(f)$ ($Per(f) = Per(f^n)$) one has

$$B^1_{(1,2),\varepsilon_n(x)}(g^n_x) \subset B^1_{2,\varepsilon_n(x)}(g^n_x).$$

Therefore, for each $x' \in I_1$ such that $g_{x',n} \in B^1_{(1,2),\varepsilon(x)}(g^n_x)$, one has the inclusion $\tau(g_{x',n}) \subseteq \{1;\, 2\}$.

For each $n \geq 2$, we use the C^1-representation $\rho_n : I_1 \to C^1(I, I_2)$, where $\rho_n(x) = g_{x,n}$ for each $x \in I_1$. The continuity of ρ_n and the compactness of $Per(f)$ imply the compactness of the image $\rho_n(Per(f)) = \{g_{x,n}\}_{x\in Per(f)}$ in the family of C^1-regular mappings in layers $\{g_{x,n}\}_{x\in I_1}$ for the mapping F^n. The

[3]Introducing functions η'_{l_n} and $\eta'_{l_n,1}$ allows to avoid difficulties related to the possible failure of the equality $\Omega(\widetilde{g}^{2^i}_x) = \Omega(\widetilde{g}^{2^{i-1}}_x)$ (see [9]).

family of neighborhoods $\{B^1_{(1,2),\varepsilon_n(x)}(g^n_x)\}_{x\in Per(f)}$ is an open covering of the compact $\rho_n(Per(f))$, infinite iff $Per(f)$ is an infinite set. In case of an infinite set $Per(f)$ we extract a finite subcovering $\{B^1_{(1,2),\varepsilon_n(x_i)}(g^n_{x_i})\}_{i=1}^{r(n)}$ from the open covering $\{B^1_{(1,2),\varepsilon_n(x)}(g^n_x)\}_{x\in Per(f)}$ of the compact set $\rho_n(Per(f))$. Put $\varepsilon_n = \min_{1\le i\le r(n)}\{\varepsilon_n(x_i)\}$. For $\varepsilon_n > 0$, using the equicontinuity of the C^1-representation $\rho_n : I_1 \to C^1(I, I_2)$, we choose a positive number $\delta_n < \varepsilon_n/2$ so that for each $x, x' \in I_1$ such that $|x - x'| < \delta_n$ there holds $||g_{x,n} - g_{x',n}||_{1,(1,2)} < \varepsilon_n/2$.

For each $n \ge 2$, choose and fix an arbitrary $0 < \delta'_n < \delta_n$. Denote by $\overline{U}_{1,\delta'_n}(Fix(f))$ the closure of the δ'_n-neighborhood in I_1 of the set of f-fixed points $Fix(f)$ (in item 1 we have $Fix(f) = Per(f)$) and consider multivalued functions $\eta^{ex}_{n\,|\overline{U}_{1,\delta'_n}(Fix(f))}$ $(n \ge 2)$. By the above, the graph of each such function coincides with the closed set of periodic points of the restriction $F_{n|(\overline{U}_{1,\delta'_n}(Fix(f)))\times I_2}$. Using the closedness of the set $Per(F_{n|(\overline{U}_{1,\delta'_n}(Fix(f)))\times I_2})$, the compactness of I_2 and the Hausdorff property of I_1, hence we get that each function $\eta^{ex}_{n\,|\overline{U}_{1,\delta'_n}(Fix(f))}$ $(n \ge 2)$ is upper semicontinuous. Choose and fix an arbitrary point $x \in \Omega(f) = Fix(f)$ (see Proposition 2.3). Using the upper semicontinuity $\eta^{ex}_{n\,|\overline{U}_{1,\delta'_n}(Fix(f))}$ (at x), for each $n \ge 2$ and for $\varepsilon_n > 0$ we find $0 < \delta''_n < \delta'_n$ so that for all $x' \in I_1$ such that $|x' - x| < \delta''_n$ there holds the inclusion $\eta^{ex}_n(x') \subset U_{2,\varepsilon_n}(Per(\widetilde{g}_x))$, where $U_{2,\varepsilon_n}(Per(\widetilde{g}_x))$ is the ε_n-neighborhood of the set $Per(\widetilde{g}_x)$ in I_2.

Choose a sequence $\{\varepsilon_n\}_{n\ge2}$ so that $\lim_{n\to\infty}\varepsilon_n = 0$. Then, using the continuity of f, formula (2.2) and Proposition 2.6 (see Eq. (2.4)), we get that $\zeta^F(x) \subset Per(\widetilde{g}_x)$. The latter implies the validity of property (2.1.1) for $\nu = 0$.

2. Let $\nu \ge 1$. Then $\Omega(F^{2^\nu}) = Per(F^{2^\nu}) = Per(F)$ (see item 1). From Def. 1.1 it follows that $\Omega(F^{2^\nu}) \subset \Omega(F)$. Note that in the case under consideration the opposite inclusion

$$\Omega(F) \subset \Omega(F^{2^\nu}) \tag{2.5}$$

is also valid. In fact, take an arbitrary point $(x^0; y^0) \in \Omega(F)$. By Proposition 2.6, we have $x^0 \in Per(f)$. Denote by $n(x^0)$ the (smallest) period of the point x^0. Since $Per(f)$ is a closed set, we can find a neighborhood $U_1(x^0) \subset I_1$ of x^0 with the following property: $U_1(x^0)\bigcap f^n(U_1(x^0)) \ne \varnothing$ iff n is a multiple of the period $n(x^0) = 2^{i_0}$ of x^0 (see [24]). Hence, using formula (1.1), we get that $(x^0; y^0) \in \Omega(F)$ iff $(x^0; y^0) \in \Omega(F^{n(x^0)})$. Therefore, inclusion (2.5) holds for $n(x^0) = 2^\nu$.

Let $n(x^0) = 2^{\nu-1}$. If $(x^0; y^0)$ is a limit point for the set $\Omega(F)$ such that x^0 is a limit point for points of the set $Per(f)$ such that the smallest period of each such point equals 2^ν, then, using the closedness of $\Omega(F)$, we get from the above that $(x^0; y^0) \in \Omega(F^{2^\nu})$. Suppose that $(x^0; y^0)$ does not have this property. In this case x^0 is either an isolated point of $Per(f)$ or a limit point for such points of $Per(f)$ that the smallest period of each such point equals $2^{\nu-1}$ (but not 2^ν). Therefore, there exists a neighborhood $U_1(x^0)$ of x^0 in I_1 that does not contain points from $Per(f)$, such that the smallest period of each such point equals 2^ν. Repeating the arguments from item 1 for $\left(f_{|\overline{U_1}(x^0)}\right)^{2^\nu}$, we find that $(x^0; y^0) \in Per(F) = \Omega(F^{2^\nu})$. Repeating these arguments consecutively for the cases $n(x^0) = 2^{\nu-p}$ if $2 \le p \le \nu - 1$, we find that $(x^0; y^0) \in Per(F) = \Omega(F^{2^\nu})$. Therefore, one has inclusion (2.5). This and the opposite inclusion prove the validity of property (2.1.1) for $\nu \ge 1$. Theorem 2.1 is proved. □

3. Theorem on Absence of a C^1- Ω-explosion. Estimate of the Set of Periods of Periodic Points

One of the main results of this section is

Theorem 3.1. *Let $F \in T^1(I)$, and let $Per(F)$ be a closed set. Then F does not admit a Ω-explosion in $T^1(I)$, and there exists a neighborhood $B^1_\varepsilon(F)$ of the mapping F in the space $T^1(I)$ such that any mapping $\Phi \in B^1_\varepsilon(F)$ has a closed set of periodic points.*

The proof of Theorem 3.1 consists of two stages separated into Propositions 3.1 and 3.2.

Proposition 3.1. *Let $F \in T^1(I)$, and let $Per(F)$ be a closed set. Then there exists a neighborhood $B^1_\varepsilon(F)$ of the mapping F in the space $T^1(I)$ such that any mapping $\Phi \in B^1_\varepsilon(F)$ has a closed set of periodic points.*

Proof. 1. Let at first $\tau(F) = \{1\}$ (in this case $\tau(F^2) = \{1\}$ as well). Using Proposition 2.2, we find firstly a ε_1-neighborhood $B^1_{1,\varepsilon_1}(f)$ of the mapping f in the space $C^1(I_1)$ such that for any $\varphi \in B^1_{1,\varepsilon_1}(f)$ one has $\tau(\varphi) \subseteq \{1;\, 2\}$, and, moreover, for each $x \in Per(f)$ one has $\varepsilon_2(x)$-neighborhood $B^1_{2,\varepsilon_2(x)}(g^2_x)$ of the mapping $g^2_x = g_{x,2}$ the space $C^1(I_2)$ such that for any mapping $\theta \in B^1_{2,\varepsilon_2(x)}(g^2_x)$ one has the inclusion $\tau(\theta) \subseteq \{1;\, 2\}$.

For all $x \in Per(f)$, we use neighborhoods $B^1_{(1,2),\varepsilon_2(x)}(g^2_x)$ of C^1-regular (with respect to the entirety of variables) mappings in the layers $\{g_{x,2}\}_{x\in I_1}$ for F^2. By inequality (1.6), in particular, for each $x \in Per(f)$ ($Per(f) = Per(f^2)$) one has $B^1_{(1,2),\varepsilon_2(x)}(g^2_x) \subset B^1_{2,\varepsilon_2(x)}(g^2_x)$. Therefore, for each $x' \in I_1$ such that $g_{x',2} \in B^1_{(1,2),\varepsilon(x)}(g^2_x)$ one has the inclusion $\tau(g_{x',2}) \subseteq \{1; 2\}$.

We make use of the C^1-representation $\rho_2 : I_1 \to C^1(I, I_2)$, where $\rho_2(x) = g_{x,2}$ for each $x \in I_1$ (see Sec. 1). Continuity of ρ_2 and compactness of $Per(f)$ imply compactness of the image $\rho_2(Per(f)) = \{g_{x,2}\}_{x\in Per(f)}$ in the family of C^1-regular mappings in the layers $\{g_{x,2}\}_{x\in I_1}$ for the mapping F^2. The family of neighborhoods $\{B^1_{(1,2),\varepsilon_2(x)}(g^2_x)\}_{x\in Per(f)}$ is an open covering of the compact set $\rho_2(Per(f))$, which is infinite iff $Per(f)$ is an infinite set. In the case of an infinite set $Per(f)$ we extract a finite subcovering $\{B^1_{(1,2),\varepsilon_2(x_i)}(g^2_{x_i})\}^r_{i=1}$ from the open covering $\{B^1_{(1,2),\varepsilon_2(x)}(g^2_x)\}_{x\in Per(f)}$ of the compact set $\rho_2(Per(f))$. We put $\varepsilon_2 = \min\limits_{1\le i\le r}\{\varepsilon_2(x_i)\}$. For $\varepsilon_2 > 0$, using the equicontinuity of the C^1-representation $\rho_2 : I_1 \to C^1(I, I_2)$, we find a positive number $\delta_2 < \varepsilon_2/2$ such that for any $x, x' \in I_1$ such that $|x - x'| < \delta_2$ there holds the inequality $||g^2_x - g^2_{x'}||_{1,(1,2)} < \varepsilon_2/2$.

We choose and fix an arbitrary positive number $\delta_3 < \delta_2$. Using the absence of C^1- Ω-explosion of regular mappings of a segment with a closed set of periodic points (see Proposition 2.5), for a number $\delta_3 > 0$ we find an ε_3-neighborhood $B^1_{1,\varepsilon_3}(f)$ of the mapping f in the space $C^1(I_1)$ so that for each mapping $\varphi \in B^1_{1,\varepsilon_3}(f)$ one has $Per(\varphi) \subset U_{1,\delta_3}(Per(f))$.

Let a positive number ε be chosen from $\varepsilon_* = 1/2\min\{\varepsilon_1, \varepsilon_3, \delta_3\}$ ($\varepsilon < \varepsilon_*$) so that for each $\Phi \in B^1_\varepsilon(F)$ there holds $\Phi^2 \in B^1_{\varepsilon_*}(F^2)$, where $\Phi(x, y) = (\varphi(x), \psi_x(y))$. Then due to the choice of the number $\varepsilon_* > 0$ and to inequality (1.6) one has: $\varphi \in B^1_{1,\varepsilon_1}(f)$, and for any $x' \in Per(\varphi)$ there is a point $x \in Per(f)$ such that $|x - x'| < \delta_2$, and therefore, $\psi_{x',2} \in B^1_{(1,2),\varepsilon_2}(g^2_x)$. Hence we immediately get the inclusion $\tau(\Phi) \subseteq \{1, 2, 2^2\}$. Boundedness of the set $\tau(\Phi)$ implies closedness of the set $Per(\Phi)$.

2. Now let $\tau(F) = \{1, 2, \ldots, 2^\nu\}$ for some $0 < \nu < +\infty$. Then by item 1 $\tau(F^{2^\nu}) = \{1\}$, and there exists a neighborhood $B^1_\varepsilon(F^{2^\nu})$ of the mapping F^{2^ν} in the space $T^1(I)$ such that for any $\Phi \in B^1_\varepsilon(F)$ one has $\tau(\Phi) \subseteq \{1, 2, 2^2\}$. Choose a positive number ε' from ε so that for each mapping $\Phi \in B^1_{\varepsilon'}(F)$ there holds $\Phi^{2^\nu} \in B^1_\varepsilon(F^{2^\nu})$. Then each $\Phi \in B^1_{\varepsilon'}(F)$ has a bounded set of (the smallest) periods of periodic points (one has the inclusion $\tau(\Phi) \subseteq \{1, 2, \ldots, 2^{\nu+2}\}$) and, hence, a closed set $Per(\Phi)$.

Proposition 3.1 is proved. □

We give an upper estimate of the set of (the smallest) periods of periodic points of the simplest mappings in $T^1(I)$.

Corollary 3.1. *Let $F \in T^1(I)$ be one of the simplest mappings such that $\tau(F) = \{1, 2, \ldots, 2^\nu\}$ for some $0 \le \nu < +\infty$. Then there exists a neighborhood $B^1_\varepsilon(F)$ of the mapping F in the space $T^1(I)$ with the following property: for each $\Phi \in B^1_\varepsilon(F)$ one has*[4] *$\tau(\Phi) \subseteq \{1, 2, \ldots, 2^{\nu+2}\}$.*

[4]Note for the sequel that an upper estimate of the set of (the smallest) periods of periodic points of the simplest mappings from $T^1(I)$ given in Corollary 3.1 can be improved. A sharp (unimprovable) estimate of this set is obtained in Theorem 3.2.

Proposition 3.2. *Let $F \in T^1(I)$, and let the set $Per(F)$ be closed. Then F does not admit an Ω-explosion in the space $T^1(I)$.*

Proof. Suppose the converse. Let there exist the simplest mapping $F_* \in T^1(I)$ that admits an Ω-explosion in the space $T^1(I)$.

1. At first assume that $Per(F_*) = Fix(F_*)$. Here $\tau(F_*) = \{1\}$. Using Corollary 3.1 for $\nu = 0$, we find a ε_0-neighborhood $B^1_{\varepsilon_0}(F_*)$ of the mapping F_* in the space $T^1(I)$ so that for an arbitrary skew product $\Phi \in B^1_{\varepsilon_0}(F_*)$ one has $\tau(\Phi) \subseteq \{1, 2, 2^2\}$. By Definition 1.2 and Theorem 2.1, there exists a $\delta > 0$ such that for any positive $\varepsilon < \varepsilon_0$ one can find a $\Phi \in B^1_\varepsilon(F_*)$ with the following property: for each point $w_0 \in Per(\Phi)$, one has $w_0 \notin U_\delta(Fix(F_*))$.

2. Let a sequence of positive numbers $\{\varepsilon_n\}_{n=1}^{+\infty}$ be such that $\lim\limits_{n\to+\infty} \varepsilon_n = 0$. Then one can find a $n_0 \geq 1$ such that for any $n \geq n_0$ one has $\varepsilon_n < \varepsilon_0$. Therefore, for each $n \geq n_0$ one can find a mapping $\Phi_n \in B^1_{\varepsilon_n}(F_*)$ such that for some point $w_n \in Per(\Phi_n)$ one has $w_n \notin U_\delta(Fix(F_*))$.

Using the compactness of I, from sequence $\{w_n\}_{n\geq n_0}$ we extract a subsequence $\{w_{n_k}\}_{k\geq1}$ convergent to some point $w_*(x_*, y_*)$. Then $w_* \notin U_\delta(Fix(F_*))$. Show that $w_* \in P(F_*) \setminus Fix(F_*)$, where $P(F_*)$ is the set of Poisson stable points[5].

In fact, choose and fix an arbitrary positive number $\varepsilon < \varepsilon_0$. Using equicontinuity of F_* on the compact set I, for ε and each $i \geq 1$ we find a positive number $\eta_i < \varepsilon$ so that, firstly, for arbitrary points $z', z'' \in I$ such that $d(z', z'') < \eta_i$ (here d is a metric in I generating the product topology), one has the inequality

$$d(F_*^{4^i}(z'), F_*^{4^i}(z'')) < \varepsilon, \tag{3.1}$$

and, secondly, for each $\Phi \in B^1_{\eta_i}(F_*)$ one has

$$\Phi^{4^i} \in B^1_\varepsilon(F_*^{4^i}). \tag{3.2}$$

Let a natural number $k(i)$ be chosen so that one has the inequality $d(w_{n_{k(i)}}, w_*) < \eta_i$. Then, using relations (3.1) and (3.2) and the equality $\Phi^{4^i}(w_{n_{k(i)}}) = w_{n_{k(i)}}$, where $\Phi \in B^1_{\eta_i}(F_*)$, we have

$$\begin{aligned} d(w_*, F_*^{4^i}(w_*)) &\leq d(w_*, w_{n_{k(i)}}) + d(\Phi^{4^i}(w_{n_{k(i)}}), F_*^{4^i}(w_{n_{k(i)}})) \\ &\quad + d(F_*^{4^i}(w_{n_{k(i)}}), F_*^{4^i}(w_*)) < 3\varepsilon. \end{aligned} \tag{3.3}$$

Inequality (3.3) is equivalent to simultaneous validity of the following two inequalities for the factor mapping and the mapping in the layer over the point x_* of the skew product F_*:

$$|x_* - f^{4^i}(x_*)| < 3\varepsilon, \quad |y_* - g_{x_*,4^i}(y_*)| < 3\varepsilon,$$

and for $w_* \notin U_\delta(Fix(F_*))$ one has either $f^{4^i}(x_*) \neq x_*$ or $y_* \neq g_{x_*,4^i}(y_*)$. Hence, $w_* \in P(F_*) \setminus Per(F_*)$. The latter is impossible, since Theorem 2.1 implies that $P(F_*) = Per(F_*)$. Thus, the assumption is wrong, and each simplest mapping $F \in T^1(I)$ such that $\tau(F) = \{1\}$ does not admit an Ω-explosion in the space $T^1(I)$.

3. Now let $\tau(F) = \{1, 2, \ldots, 2^\nu\}$ for some $0 < \nu < +\infty$. Then $\tau(F^{2^\nu}) = \{1\}$, and by the above, the mapping F^{2^ν} does not admit an Ω-explosion in the space $T^1(I)$. Since $\Omega(F^{2^\nu}) = Per(F^{2^\nu}) = Per(F) = \Omega(F)$ (see Theorem 2.1), using the definition of the absence of an Ω-explosion in the space $T^1(I)$, hence we get that the mapping F does not admit an Ω-explosion in the space $T^1(I)$ either. Proposition 3.2 is proved. □

[5] A point $z^0(x^0, y^0) \in I$ is called Poisson stable (see [22, Ch. V, Sec. 5]) for the mapping F_*, if there exists a sequence of natural numbers $n_1 < n_2 < \ldots < n_i < \ldots$ such that there holds the equality

$$\lim_{i\to+\infty} F_*^{n_i}(z^0) = z^0.$$

Validity of Theorem 3.1 follows from Propositions 3.1 and 3.2. Theorem 3.1 is proved.

To conclude Sec. 3, we extend Proposition 2.2 to the case of mappings from $T^1(I)$ improving the upper estimate for the set of (the smallest) periods of periodic points of regular simplest skew products given in Corollary 3.1.

Theorem 3.2. *If a mapping $F \in T^1(I)$ does not contain a periodic point with period 2^i, there exists an ε-neighborhood $B^1_\varepsilon(F)$ of F in $T^1(I)$ such that any mapping from $B^1_\varepsilon(F)$ does not contain periodic points with period 2^{i+1} for any $i \geq 1$.*

Proof[6]. 1. Let F be an arbitrary mapping from $T^1(I)$ that does not contain a periodic point with period 2^i for some $i \geq 1$. By the results of [19], under the hypotheses of Theorem 3.2 we have: $\tau(F) = \{1, 2, \dots, 2^{i-1}\}$ for some $1 \leq i < +\infty$. Then for the mapping $G = F^{2^{i-1}}$ we have $\tau(G) = \{1\}$. Using Corollary 3.1, we find a neighborhood $B^1_{\varepsilon'}(G)$ of G in the space $T^1(I)$ such that for any $\Phi \in B^1_{\varepsilon'}(G)$ one has $\tau(\Phi) \subseteq \{1, 2, 2^2\}$.

2. Show that there exists a subneighborhood $B^1_{\varepsilon''}(G)$ of the neighborhood $B^1_{\varepsilon'}(G)$ of the mapping G such that an arbitrary mapping from $B^1_{\varepsilon''}(G)$ does not contain periodic points with period 4. Suppose the converse. Then there exists a sequence of mappings $\{\Phi_n\}_{n\geq 1}$ such that

$$\lim_{n\to+\infty} ||\Phi_n - G||_1 = 0, \tag{3.4}$$

and the mapping Φ_n for any $n \geq 1$ contains a periodic point with period 4. Due to [19], a projection of a periodic orbit of the mapping Φ_n with period 4 to the axis Ox is a periodic orbit of the factor mapping of the skew product Φ_n with period 2. Let this periodic orbit consist of points

$$(x_1^n, y_1^n), (x_2^n, y_2^n), (x_1^n, y_3^n), (x_2^n, y_4^n), \quad \text{and} \quad y_3^n \neq y_1^n, \quad y_4^n \neq y_2^n.$$

Let $\Phi_n(x, y) = (f_n(x), (g_n)_x(y))$, $n \geq 1$. We construct graphs of functions $\overline{y} = (g_n)_{x_1^n}(y)$ and $\overline{y} = (g_n)_{x_2^n}(y)$ in the plane $YO\overline{Y}$. We connect points $(y_1^n;\, y_2^n)$ with $(y_3^n;\, y_4^n)$ on the graph of the function $\overline{y} = (g_n)_{x_1^n}(y)$ and $(y_2^n;\, y_3^n)$ with $(y_4^n;\, y_1^n)$ on the graph of the function $\overline{y} = (g_n)_{x_2^n}(y)$ by straight line segments.

We write out the equations of each line containing these segments. Thus, points $(y_1^n;\, y_2^n)$ and $(y_3^n;\, y_4^n)$ are connected by the line l_1^n given by the equation

$$\overline{y} = \frac{y_4^n - y_2^n}{y_3^n - y_1^n}(y - y_1^n) + y_2^n,$$

and points $(y_2^n;\, y_3^n)$ and $(y_4^n;\, y_1^n)$ — by the line l_2^n given by the equation

$$\overline{y} = \frac{y_3^n - y_1^n}{y_2^n - y_4^n}(y - y_4^n) + y_1^n.$$

Put $k_1^n = \dfrac{y_4^n - y_2^n}{y_3^n - y_1^n}$, $k_2^n = \dfrac{y_3^n - y_1^n}{y_2^n - y_4^n}$. Then we have

$$k_1^n k_2^n = -1, \tag{3.5}$$

i.e., the lines l_1^n and l_2^n are mutually perpendicular (for each $n \geq 1$).

At the same time, one has the equalities

$$k_1^n = \frac{(g_n)_{x_1^n}(y_3^n) - (g_n)_{x_1^n}(y_1^n)}{y_3^n - y_1^n}, \quad k_2^n = \frac{(g_n)_{x_2^n}(y_2^n) - (g_n)_{x_2^n}(y_4^n)}{y_2^n - y_4^n}.$$

Define intervals

$$J_{1,3}^n = \{x_1^n\} \times (\min\{y_1^n, y_3^n\},\ \max\{y_1^n, y_3^n\}),$$
$$J_{2,4}^n = \{x_2^n\} \times (\min\{y_2^n, y_4^n\},\ \max\{y_2^n, y_4^n\}).$$

[6] Analogous arguments for one-parametric families of C^1-regular skew products of interval mappings were given by Blinova and the author in [4].

Using the classical Lagrange theorem for a function of one variable, we find points $\xi_1^n \in J_{1,3}^n$ and $\xi_2^n \in J_{2,4}^n$ such that one has

$$\frac{\partial}{\partial y}(g_n)_{x_1^n}(\xi_1^n) = k_1^n, \quad \frac{\partial}{\partial y}(g_n)_{x_2^n}(\xi_2^n) = k_2^n.$$

Using (3.5), hence we get

$$\left|\frac{\partial}{\partial y}(g_n)_{x_1^n}(\xi_1^n) - \frac{\partial}{\partial y}(g_n)_{x_2^n}(\xi_2^n)\right| = |k_1^n - k_2^n| = \left|k_1^n + \frac{1}{k_1^n}\right| \geq 2. \tag{3.6}$$

On the other hand, by Theorem 3.1 the mapping G does not admit an Ω-explosion in the space $T^1(I)$. Use Definition 1.2 and take an appropriate sequence of positive numbers $\{\delta_m\}_{m\geq 1}$, which converges to 0. Then for each $m \geq 1$, there exists a positive number $\varepsilon(\delta_m)$ such that for any mapping $\Phi \in B^1_{\varepsilon(\delta_m)}(G)$ one has $\Omega(\Phi) \subset U_{\delta_m}(\Omega(G))$; and, in particular, by (3.4) for each $m \geq 1$ there exists a mapping $\Phi_{n_m} \in B^1_{\varepsilon(\delta_m)}(G)$ that contains a periodic orbit of period 4 and such that $\Omega(\Phi_{n_m}) \subset U_{\delta_m}(\Omega(G))$. Thus, any convergent (as $m \to \infty$) sequence of points from $\Omega(\Phi_{n_m})$ has a unique limit point, which is fixed with respect to the mapping G. Hence, using Eq. (3.4) and continuity of the mappings G and Φ_{n_m}, we immediately obtain that if a subsequence $\{(x_1^{n_{m_k}}, y_1^{n_{m_k}})\}_{k\geq 1}$ converges and

$$\lim_{k\to+\infty}(x_1^{n_{m_k}}, y_1^{n_{m_k}}) = (x^0, y^0), \quad \text{where} \quad G(x^0, y^0) = (x^0, y^0),$$

then so does each sequence $\{(x_1^{n_{m_k}}, y_3^{n_{m_k}})\}_{k\geq 1}$ and $\{(x_2^{n_{m_k}}, y_s^{n_{m_k}})\}_{k\geq 1}$ for $s = 2, 4$, and there hold equalities

$$(x^0, y^0) = \lim_{k\to+\infty}(x_1^{n_{m_k}}, y_3^{n_{m_k}}) = \lim_{k\to+\infty}(x_2^{n_{m_k}}, y_s^{n_{m_k}}).$$

Then, using the obtained relations, Eq. (3.4) and C^1 regularity of the mappings Φ_{n_m}, we get

$$\frac{\partial}{\partial y}g_{x^0,2^{i-1}}(y^0) = \lim_{k\to+\infty}\frac{\partial}{\partial y}(g_{n_{m_k}})_{x_1^{n_{m_k}}}(\xi_1^{n_{m_k}}) = \lim_{k\to+\infty}\frac{\partial}{\partial y}(g_{n_{m_k}})_{x_2^{n_{m_k}}}(\xi_2^{n_{m_k}}).$$

Therefore,

$$\lim_{k\to+\infty}\left|\frac{\partial}{\partial y}(g_{n_{m_k}})_{x_1^{n_{m_k}}}(\xi_1^{n_{m_k}}) - \frac{\partial}{\partial y}(g_{n_{m_k}})_{x_2^{n_{m_k}}}(\xi_2^{n_{m_k}})\right| = 0,$$

and at the same time inequality (3.6) implies

$$\lim_{k\to+\infty}\left|\frac{\partial}{\partial y}(g_{n_{m_k}})_{x_1^{n_{m_k}}}(\xi_1^{n_{m_k}}) - \frac{\partial}{\partial y}(g_{n_{m_k}})_{x_2^{n_{m_k}}}(\xi_2^{n_{m_k}})\right| \geq 2.$$

Thus, inequality (3.6) contradicts C^1 regularity of the mappings Φ_{n_m}. Therefore, our assumption is wrong, and there exists a subneighborhood $B^1_{\varepsilon''}(G)$ of the neighborhood $B^1_{\varepsilon'}(G)$ of the mapping G such that any mapping from $B^1_{\varepsilon''}(G)$ contains no periodic points with period 4 (though it may contain ones with period 2).

3. Now let $\tau(F) = \{1, 2, \ldots, 2^{i-1}\}$ for some $2 \leq i < +\infty$. Using the number $\varepsilon'' > 0$ (see item 2), we choose a positive number ε so that for any mapping $\Psi \in B^1_\varepsilon(F)$ it holds $\Psi^{2^{i-1}} \in B^1_{\varepsilon''}(G)$. Then $\Psi^{2^{i-1}}$ contains no periodic points with period 4, and hence, Ψ contains no periodic points with period 2^{i+1} (though it may contain ones with period 2^i). Theorem 3.2 is proved. □

As the example below shows, the estimate of the set of (the smallest) periods of periodic points for C^1-regular skew products of interval mappings from some neighborhood of one of the simplest mappings in the space $T^1(I)$ obtained in Theorem 3.2 is unimprovable.

Example 3.1. Consider a mapping $F_1 \in T^1([0,1]^2)$ such that its factor mapping $f(x)$ is defined by the following equality:

$$f(x) = \begin{cases} x & \text{for} \quad x \in \left(\dfrac{1}{2^{2j+1}}, \dfrac{1}{2^{2j}}\right]; \\ x + \dfrac{1}{2^{2j+5}} \sin^2 \pi(2^{2j+2}x - 1) & \text{for} \quad x \in \left(\dfrac{1}{2^{2j+2}}, \dfrac{1}{2^{2j+1}}\right], \; j \geq 0; \\ 0 & \text{for} \quad x = 0. \end{cases} \tag{3.7}$$

Hence the set of f-fixed points $Fix(f)$ is perfect (and has the cardinality of continuum). We pass to definition of mappings in layers.

Denote by λ_i $(i \geq 1)$ the bifurcation value of the parameter in the family of logistic mappings $\overline{x} = \lambda x(1-x)$ such that for $\lambda = \lambda_i$ the logistic mapping has a nonrough attracting periodic orbit with period 2^{i-1}, and for $\lambda_i < \lambda < \lambda_{i+1}$ logistic mappings have a periodic orbit with period 2^i (see, e.g., [29, Ch. 1, Sec. 2] and [3]). Define mappings in the layers $g_x(y)$ of the skew product F_1, putting $g_x(y) = ((\lambda_i - \lambda_{i-1})x + \lambda_{i-1})y(1-y)$, $i \geq 2$.

Then $\tau(F_1) = \{1, 2, \dots, 2^{i-1}\}$. Since f is a growing diffeomorphism (see Eq. (3.7)), there exists a neighborhood $B^1_{1,\varepsilon_1}(f)$ of the mapping f in the space $C^1(I_1)$ such that each mapping $\varphi \in B^1_{1,\varepsilon_1}(f)$ is also a growing diffeomorphism (i.e., $\tau(\varphi) = \{1\}$). By Proposition 2.2, for each $x \in Fix(f)$ there exists a neighborhood $B^1_{2,\varepsilon_2(x)}(g_x)$ such that an arbitrary mapping $g \in B^1_{2,\varepsilon_2(x)}(g_x)$ does not contains periodic orbits with period 2^i for $x = 0$ (in this case $\{1, 2, \dots, 2^{i-2}\} \subseteq \tau(g) \subseteq \{1, 2, \dots, 2^{i-1}\}$) or those with period 2^{i+1} for $x \in (0, 1]$ (here $\{1, 2, \dots, 2^{i-1}\} \subseteq \tau(g) \subseteq \{1, 2, \dots, 2^i\}$).

We use neighborhoods $B^1_{(1,2),\varepsilon_2(x)}(g_x) \subset B^1_{2,\varepsilon_2(x)}(g_x)$ of C^1 regular (with respect to the entirety of variables) mappings in the layers of the skew product F_1 for all $x \in Fix(f)$. Then for each $x' \in [0,1]$ such that $g_{x'} \in B^1_{(1,2),\varepsilon_2(x)}(g_x)$ for some $x \in Fix(f)$ the set $\tau(g_{x'})$ does not contain the natural number 2^{i+1}.

Continuity of the C^1-representation $\rho_1 : [0,1] \to C^1([0,1]^2, [0,1])$ (see Sec. 1) and compactness of $Fix(f)$ imply compactness of $\rho_1(Fix(f)) = \{g_x\}_{x \in Fix(f)}$ in the family of C^1 regular mappings in layers $\{g_x\}_{x \in [0,1]}$ of the mapping F_1. The family of neighborhoods $\{B^1_{(1,2),\varepsilon_2(x)}(g_x)\}_{x \in Fix(f)}$ is an infinite open coverage of the compact set $\rho_1(Fix(f))$. Let the neighborhoods $\{B^1_{(1,2),\varepsilon_2(x_i)}(g_{x_i})\}_{i=1}^r$ form a finite subcovering of the compact set $\rho_1(Fix(f))$. Put $\varepsilon_2 = \min\limits_{1 \leq i \leq r}\{\varepsilon_2(x_i)\}$.

Using equicontinuity of the C^1 representation ρ_1, for $\varepsilon_2 > 0$ we find a positive number $\delta_2 < \varepsilon_2/2$ so that for any $x, x' \in [0,1]$ such that $|x - x'| < \delta_2$ one has the inequality $||g_x - g_{x'}||_{1,(1,2)} < \varepsilon_2/2$.

We choose and fix an arbitrary number $\delta_3 < \delta_2$. Using the absence of C^1- Ω-explosions of segment mappings with a closed sets of periodic points in $C^1([0,1])$, for a number $\delta_3 > 0$ we find an ε_3-neighborhood $B^1_{1,\varepsilon_3}(f)$ of the mapping f in the space $C^1([0,1])$ such that $\varepsilon_3 < \varepsilon_1$, and for each mapping $\varphi \in B^1_{1,\varepsilon_3}(f)$ there holds $Fix(\varphi) \subset U_{1,\delta_3}(Fix(f))$.

Put $\varepsilon_* = 1/2 \min\{\varepsilon_3, \delta_3\}$. Then any mapping $\Phi \in B^1_{\varepsilon_*}(F_1)$ does not contain periodic points with (the smallest) period equal to 2^{i+1}. At he same time, let ε be an arbitrary positive number such that $\varepsilon < \min\{\varepsilon_*, \lambda_{i+1} - \lambda_i\}$. Put

$$\Phi_\varepsilon(x, y) = (f(x), (\lambda_i - \lambda_{i-1})x + \lambda_{i-1} + \varepsilon)y(1-y)).$$

Then $\Phi_\varepsilon \in B^1_{\varepsilon_*}(F_1)$ and $\tau(\Phi_\varepsilon) = \{1, 2, \dots, 2^i\}$.

4. Note on Peculiarities of Period Doubling Bifurcations in the Simplest C^1 Regular Skew Products

To conclude the paper, we note some distinctive peculiarities of period doubling bifurcations for periodic points of regular skew products of interval mappings, where the multiplier $\lambda_1((x^0; y^0))$ of the fixed point $(x^0; y^0)$ passes -1, and the multiplier $\lambda_2((x^0; y^0))$ of the same point passes either 1 or -1. Recall that in regular mappings of a segment into itself whenever a multiplier of a fixed point passes -1, there appears a unique periodic orbit with period 2 formed by sinks (see, e.g., [29, Ch. 2, Sec. 3] and [3]), where the fixed point becomes a source (though it was an attracting point at the moment of bifurcation).

Theorem 4.1. *Let $F_\alpha : I \to I$, $F_\alpha(x,y) = (f_\alpha(x), g_{\alpha,x}(y))$, be a one-parameter family of C^3 regular skew products of interval mappings, with a C^1 regular dependence on the parameter $\alpha \in (\alpha_1, \alpha_2)$, $(x^0; y^0)$ — a fixed point of the skew product F_{α_0}, where $\lambda_1((x^0; y^0)) = -1$ and $\lambda_2((x^0; y^0)) = 1$. Let for $\alpha = \alpha_0$ at the fixed point $(x^0; y^0)$ the following inequalities hold:*

(4.1.1) $\dfrac{\partial^3}{\partial x^3}(f_\alpha^2(x)) < 0, \quad \dfrac{\partial^2}{\partial y^2}(g_{\alpha,x,2}(y)) > 0,$

(4.1.2) $\dfrac{\partial}{\partial \alpha}(f_\alpha^2(x)) < 0, \quad \dfrac{\partial}{\partial \alpha}(g_{\alpha,x,2}(y)) < 0.$

Then there exist $\varepsilon > 0$ and $\delta > 0$ such that

(4.1.a) *for $\alpha \in (\alpha_0 - \delta, \alpha_0$, the skew product F_α has no fixed points in the open square $(x^0 - \varepsilon, x^0 + \varepsilon) \times (y^0 - \varepsilon, y^0 + \varepsilon)$;*

(4.1.b) *for $\alpha \in (\alpha_0, \alpha_0 + \delta)$, the skew product F_α in the open square $(x^0 - \varepsilon, x^0 + \varepsilon) \times (y^0 - \varepsilon, y^0 + \varepsilon)$ has two fixed points with the same fixed point of the factor mapping f_α as the projection, so that one of the fixed points F_α is a source, and the other is a saddle; and a pair of periodic orbits with period 2 with the same periodic orbit with period 2 of the factor mapping f_α as the projection, where one of the periodic orbits with period 2 of the skew product F_α consists of sinks, and the other one of saddle periodic points.*

Proof. Consider a family of two equations such that its solutions are fixed points and periodic points with period 2 of the skew product F_α:

$$\begin{cases} f_\alpha^2(x) - x = 0, \\ g_{\alpha,x,2}(y) - y = 0. \end{cases} \tag{4.1}$$

Apply the first inequalities (4.1.1) and (4.1.2) to the first equation (4.1). Then by the results of [29, Ch. 2, Sec. 3] one can find $\varepsilon' > 0$ and $\delta' > 0$ such that the first equation (4.1)

(1) has a unique solution for $\alpha \in (\alpha_0 - \delta', \alpha_0)$ from the interval $(x_0 - \varepsilon', x_0 + \varepsilon')$ that corresponds to the sink of the factor equation f_α;
(2) has three solutions for $\alpha \in (\alpha_0, \alpha_0 + \delta')$ belonging to the interval $(x_0 - \varepsilon', x_0 + \varepsilon')$, such that one of them, namely $x_+^0 = x_+^0(\alpha)$, corresponds to the source of the factor mapping f_α, and the other two form a sink of period 2.

Before passing to the second equation in (4.1) note that for $\alpha = \alpha_0$ at the fixed point $(x^0; y^0)$ one has equalities

$$\frac{\partial}{\partial y}(g_{\alpha,x}(y)) = \frac{\partial}{\partial y}(g_{\alpha,x,2}(y)) = 1.$$

Apply the second inequalities in (4.1.1) and (4.1.2) to the second equation in (4.1). Using the results of [29, Ch. 2, Sec. 3], we find $\varepsilon'' > 0$ and $\delta'' > 0$ such that the second equation in (4.1)

(3) has no solutions with respect to y in the interval $(y_0 - \varepsilon'', y_0 + \varepsilon'')$ for $\alpha \in (\alpha_0 - \delta'', \alpha_0)$ and each $x \in (x_0 - \varepsilon'', x_0 + \varepsilon'')$;
(4) has 2 solutions $\alpha = \alpha_{x,i}(y)$ $(i = 1, 2)$ for $y \in (y_0 - \varepsilon'', y_0 + \varepsilon'')$, $\alpha \in (\alpha_0, \alpha_0 + \delta'')$ and each $x \in (x_0 - \varepsilon'', x_0 + \varepsilon'')$, where one value of y is a sink, and another one is a source for the mapping $g_{\alpha,x,2} : I_2 \to I_2$; for $x = x_+^0(\alpha)$, each of the two values y such that $\alpha = \alpha_{x,i}(y)$ is a fixed point of the mapping $g_{\alpha,x}$ (if the points $\alpha_{x,i}(y)$ formed a periodic orbit with period 2 with respect to y in the interval I_2, then the interval bounded by these points would have contained also a fixed point $g_{\alpha,x}$ for $x = x_+^0(\alpha)$, and the second equation in (4.1) would have 3 solutions).

Then for $\varepsilon = \min\{\varepsilon', \varepsilon''\}$, $\delta = \min\{\delta', \delta''\}$ each of the properties (4.1.a) and (4.1.b) holds.

Theorem 4.1 is proved. ☐

Arguing similarly to the proof of the previous Theorem 4.1, we can see that the following statement holds.

Theorem 4.2. *Let $F_\alpha : I \to I$, $F_\alpha(x, y) = (f_\alpha(x), g_{\alpha,x}(y))$, be a one-parameter family of C^3-regular skew products of interval mappings, C^1-regularly dependent on a parameter $\alpha \in (\alpha_1, \alpha_2)$, $(x^0; y^0)$ — a fixed point of the skew product F_{α_0}, where $\lambda_1((x^0; y^0)) = -1$ and $\lambda_2((x^0; y^0)) = -1$. Let for $\alpha = \alpha_0$ at the fixed point $(x^0; y^0)$ the following inequalities hold:*

(4.2.1) $\dfrac{\partial^3}{\partial x^3}(f^2_\alpha(x)) < 0, \quad \dfrac{\partial^3}{\partial y^3}(g_{\alpha,x,2}(y)) < 0,$

(4.2.2) $\dfrac{\partial}{\partial \alpha}(f^2_\alpha(x)) < 0, \quad \dfrac{\partial}{\partial \alpha}(g_{\alpha,x,2}(y)) < 0.$

Then there exist $\varepsilon > 0$ and $\delta > 0$ such that

(4.2.a) *for $\alpha \in (\alpha_0 - \delta, \alpha_0)$ the skew product F_α has only one fixed point in the open square $(x^0 - \varepsilon, x^0 + \varepsilon) \times (y^0 - \varepsilon, y^0 + \varepsilon)$, and this point is a sink;*

(4.2.b) *for $\alpha \in (\alpha_0, \alpha_0 + \delta)$, F_α in the open square $(x^0 - \varepsilon, x^0 + \varepsilon) \times (y^0 - \varepsilon, y^0 + \varepsilon)$ has only one fixed point — a source and 4 periodic orbits with period 2, where one of the orbits with period 2 with the same projection as the fixed point is formed by saddle periodic points; one of the three periodic orbits with period 2 of the mapping F_α with the same periodic orbit with period 2 of the factor mapping f_α as a projection is formed by saddle periodic points, and the two other ones by sinks.*

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