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ON A PROBLEM OF CONSTRUCTIVE THEORY OF HARMONIC MAPPINGS

S. I. Bezrodnykh and V. I. Vlasov UDC 517.57

Abstract. The problem of irremovable error appears in finite difference realization of the Winslow approach in the constructive theory of harmonic mappings. As an example, we consider the well-known Roache–Steinberg problem and demonstrate a new approach, which allows us to construct harmonic mappings of complicated domains effectively and with high precision. This possibility is given by the analytic-numerical method of multipoles with exponential convergence rate. It guarantees effective construction of a harmonic mapping with precision controlled by an a posteriori estimate in a uniform norm with respect to the domain.

1. Introduction

1.1. Harmonic mappings of plain domains. A homeomorphism of a domain $\mathcal{Z} \subset \mathbb{C}$ onto $\mathcal{W}$ is called harmonic [27] and denoted $\mathcal{F} : \mathcal{Z} \overset{harm}{\longrightarrow} \mathcal{W}$ if it is realized by a complex-valued function $w = \mathcal{F}(z)$, which is harmonic in $\mathcal{Z}$. Here $z = x + iy$ and $w = u + iv$ are complex variables, while $\mathbb{C}$ is the complex plane. The components $u(x, y)$ and $v(x, y)$ of the function $\mathcal{F}(z)$ do not need to satisfy the Cauchy–Riemann conditions.

Some authors, including Radó [67], Kneser [48], and Choquet [24], understand harmonic mapping as a harmonic continuation $w = \mathcal{F}(z)$ of a given homeomorphism $\mathcal{B} : \partial\mathcal{Z} \overset{Hom}{\longrightarrow} \partial\mathcal{W}$ of boundaries of domains $\mathcal{Z}$ and $\mathcal{W}$ onto the domain $\mathcal{Z}$. In this case, the question whether the mapping $\mathcal{F}$ is a homeomorphism between domains $\mathcal{Z}$ and $\mathcal{W}$ remains open.

The concept of harmonic mapping of domain $\mathcal{Z} \subset \mathbb{C}$ also appears in the literature (see, e.g., [51]). It is a mapping of this domain by a complex-valued harmonic function $w = \mathcal{F}(z)$ defined in $\mathcal{Z}$, where the image $\mathcal{F}(\mathcal{Z})$ is not given a priori.

Harmonic mappings (in all these senses) are widely studied (see, e.g., monographs and reviews [22, 27, 29, 36, 43, 46, 59, 70, 86, 95]). The development of the theory of planar harmonic mappings started in the 1920s. Firstly, it was related to their important role in differential geometry, especially in the theory of minimal surfaces [12, 37, 63, 64, 68, 69]. Secondly, the interest in them appeared in connection with the theory of quasiconformal mappings, which was started in [1, 34, 52, 53, 62, 81, 82] (see also [2, 4, 10, 41, 54–56, 61, 83, 98]): harmonic mappings are an important special case of those. Also, in the 1920s as well, the attention of researchers was attracted to the problem of homeomorphism of closed domains $\overline{\mathcal{Z}}$ and $\overline{\mathcal{W}}$ under a harmonic continuation of a given homeomorphism of the boundaries $\mathcal{B} : \partial\mathcal{Z} \overset{Hom}{\longrightarrow} \partial\mathcal{W}$ onto $\mathcal{Z}$. This problem was studied in the papers by Radó [67], Kneser [48], Choquet [24], and Kudryavtsev [51]. We also note a paper by Lewy on a local homeomorphism [57].

The interest in harmonic mappings got a new push in the 1980s, when analogues of classical results from the geometrical theory of analytic functions [33] were obtained in papers [26, 35, 75]. These papers contained analogues of growth, distortion, and covering theorems, estimates of coefficients for one-sheeted mappings, an extremal "harmonic Koebe function" (an analogue of the well-known Koebe function for one-sheeted analytic functions [33]), and solutions or formulations of several extremal problems for harmonic mappings. We also note the important problem of extension of the Riemann

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theorem to harmonic mappings, which caught the attention of many researchers (see [13, 16, 17, 21, 28, 38, 39, 61, 70, 95]).

Even earlier, in the 1960s, a burst of interest in the subject occurred after the progress in the constructive theory of harmonic mappings due to Winslow [96]. This paper gave a new approach to numerical construction of harmonic mappings of complicated domains with application to generation of calculation grids on this basis. It started a rapidly growing trend of generation of high quality calculation grids in complicated domains, including adaptive ones [11, 18, 20, 25, 32, 43, 45, 50, 58, 59, 65, 77, 79, 80, 84–86, 92–94, 97]. The results of this trend helped to improve numerical methods, including finite difference and finite element ones, and gave new possibilities for their application in industry [5, 19, 30–32, 42, 73, 78, 99–102].

On the other hand, in papers [6, 43–45, 49, 66, 71, 74, 76], etc., it was established that construction of harmonic mappings and generation of related grids by difference methods based on Winslow's approach can result in large errors in the numerical mapping, up to the loss of homeomorphism property despite theoretical predictions [24, 48, 67]. In this situation, there often appears the effect of an "irremovable" error, that is, neither the growth of approximation order of the difference scheme nor refining of the grid result in substantial reduction of the error. A typical problem of such kind is the problem of a harmonic mapping of a "horseshoe-like" domain onto a square, well-known from papers [6, 44, 49, 71]. We will call it the Roache–Steinberg problem, from the names of the authors of the pioneering paper [71].

In the present paper, this problem is used as an example which shows that an effective construction of a harmonic mapping of complicated domains with arbitrary high precision is possible. This opportunity is given by the analytic-numerical multipole method [88, 90, 91], which enables one to effectively solve boundary-value problems for the Laplace equation with precision guaranteed by an a posteriori estimate in a uniform norm with respect to the domain and with exponential convergence rate. Thus, this method opens new possibilities in the constructive theory of harmonic mappings.

The solution of the Roache–Steinberg problem with relative precision 10^{-9} obtained in this paper by the multipole method is used as a test for a detailed study of the effect of "irremovable" error, which appeared in papers [6, 49, 71] in the course of solving this problem by difference methods based on Winslow's approach.

The multipole method is exposed in Sec. 2. A high-precision solution of the Roache–Steinberg problem obtained by the multipole method combined with results from [14, 15] is given in Sec. 3. The effect of irremovable error is studied in the case of this problem in Sec. 4.

1.2. Some facts from the theory of harmonic mappings. It is known (see, e.g., [27, 46]) that harmonic mappings $w = \mathcal{F}(z)$, which realize orientation-preserving homeomorphisms $\mathcal{Z} \overset{Hom}{\longrightarrow} \mathcal{W}$ of the mapped domains, as quasiconformal mappings in general, obey (the first) Beltrami equation. In terms of the Riemann derivatives

$$f_z := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad f_{\overline{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right),$$

it can be written as

$$\mathcal{F}_{\overline{z}}(z) = \mu(z)\,\mathcal{F}_z(z).$$

Here $\mu(z)$ is the (first) complex dilatation of the mapping $\mathcal{F}(z)$ that corresponds to inequality $|\mu(z)| < 1$. Dilatation $K(z)$ is the ratio of the greater and smaller axes of an infinitesimal ellipse, which is the image of an infinitesimal disk under the mapping $\mathcal{F}$. Dilatation $K(z)$ is connected with the complex dilatation $\mu(z)$ by the formula

$$K(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}.$$

If $\mu(z) = 0$ in the whole domain, then $\mathcal{F}$ is a conformal mapping. The Jacobian of a quasiconformal (in particular, harmonic) mapping $\mathcal{F}$ is denoted by $J(\mathcal{F})$ and can be expressed as a function of the

Riemannian derivatives by the formula $J(\mathcal{F}) = |\mathcal{F}_z|^2 - |\mathcal{F}_{\overline{z}}|^2$, which implies $J(\mathcal{F}) = |\mathcal{F}_z|^2 \left[1 - |\mu(z)|^2\right]$. Then, as follows from $|\mu(z)| < 1$, for quasiconformal mappings one has $J(\mathcal{F}) > 0$, and therefore they, as was mentioned above, are always orientation-preserving. For the harmonic mappings under consideration, it is possible that either $J(\mathcal{F}(z)) > 0$, $z \in \mathcal{Z}$, which makes the mapping $\mathcal{F}$ orientation-preserving, or $J(\mathcal{F}(z)) < 0$, $z \in \mathcal{Z}$, which makes the mapping $\mathcal{F}$ orientation-reversing.

In the theory of harmonic mappings, it is often more convenient to consider not the first Beltrami equation mentioned above, but the second one: $\overline{\mathcal{F}_{\overline{z}}}(z) = \omega(z)\, \mathcal{F}_z(z)$, where $\omega(z)$ is the second complex dilatation, which is an analytic function in $\mathcal{Z}$ for harmonic mappings. For a harmonic mapping $\mathcal{F}$ in a simply connected domain $\mathcal{Z} \subset \mathbb{C}$, the canonical representation $\mathcal{F} = \mathcal{H} + \overline{\mathcal{G}}$ is valid, where $\mathcal{H}(z)$ and $\mathcal{G}(z)$ are analytic functions in $\mathcal{Z}$. By using this representation, the Jacobian of the harmonic mapping $\mathcal{F}$ can be written in the form $J(\mathcal{F}) = |\mathcal{H}'|^2 - |\mathcal{G}'|^2$, where the prime denotes complex differentiation.

Note that though harmonic mappings have much in common with conformal ones, there are many differences as well. For example, a composition of conformal mappings is a conformal mapping itself, while a composition of harmonic mappings or the composition $f \circ g$ of a conformal mapping f and a harmonic one g is, generally speaking, not a harmonic mapping; similarly, an inverse mapping of a harmonic one is generally not harmonic.

For harmonic mappings, as for conformal ones, local one-sheet property at a point z is equivalent [57] to the fact that $J(\mathcal{F}(z)) \neq 0$. But the behavior of harmonic mappings on the boundary is more complicated than that of conformal mappings. For example, for a conformal mapping $\mathcal{Z} \overset{conf}{\longrightarrow} \mathcal{W}$ between simply connected domains $\mathcal{Z}$ and $\mathcal{W}$ the Jordan property of domains $\mathcal{Z}$ and $\mathcal{W}$ is a necessary and sufficient condition [23, 60] of homeomorphism $\overline{\mathcal{Z}} \overset{Hom}{\longrightarrow} \overline{\mathcal{W}}$, but for harmonic mappings this is not true. In fact, a function

$$\mathcal{F}(z) = \frac{1}{\pi} \sum_{k=0}^{N-1} \exp\big(2\pi i\, k/N\big) \arg\left\{\frac{z - \exp\left[\pi i(2k+1)/N\right]}{z - \exp\left[\pi i(2k-1)/N\right]}\right\}$$

(mentioned in [27]), which realizes an orientation-preserving harmonic homeomorphism between the disk $\mathbb{U} := \{|z| < 1\}$ and the regular N-sided polygon $\mathcal{W}_N$ with vertices at the Nth roots of 1, does not realize a homeomorphism between boundaries, since it takes arcs

$$\Gamma_k := \big\{z = e^{i\varphi} : \varphi \in \big((2k-1)/N, (2k+1)/N\big)\big\}$$

of the circle $\partial\mathbb{U}$ to vertices $\exp\big(2\pi i\, k/N\big)$ of the polygon $\mathcal{W}_N$, also denoted by Γ_k, and points $\gamma_k := \exp\big(\pi i\,(2k-1)/N\big)$ of the circle $\partial\mathbb{U}$ onto respective sides of the polygon $\mathcal{W}_N$ that connect its vertices Γ_k and Γ_{k+1}. Figure 1 illustrates this mapping for $N = 5$; it contains a depiction of domains, arcs, and points mentioned above, as well as that of the polar grid in the disk and its image in the polygon $\mathcal{W}_5$.

1.3. Harmonic extension. For the given homomorphism $\mathcal{B} : \partial\mathcal{Z} \overset{Hom}{\longrightarrow} \partial\mathcal{W}$ of the boundaries of Jordan domains $\mathcal{Z}$ and $\mathcal{W}$, the harmonic extension $w = \mathcal{F}(z)$ onto domain $\mathcal{Z}$ is defined (see [24, 27, 48, 67]) as the classical solution of the following Dirichlet problem:

$$\Delta\, \mathcal{F}(z) \,=\, 0, \qquad z \in\, \mathcal{Z}, \tag{1.1}$$

$$\mathcal{F}(z) \,=\, \mathcal{B}(z), \qquad z \in\, \partial\mathcal{Z}. \tag{1.2}$$

A complex-valued harmonic function $\mathcal{F}(z)$ defined in such a way does not necessarily realize a homeomorphism $\mathcal{Z} \overset{Hom}{\longrightarrow} \mathcal{W}$ between domains.

We give an example (see [6]) of a harmonic extension, which does not provide a homeomorphism $\mathcal{Z} \overset{Hom}{\longrightarrow} \mathcal{W}$ (further examples are given in [3, 22, 43, 65]). As domains $\mathcal{Z}$ and $\mathcal{W}$ we choose, respectively, the unit square and the arrow-like domain depicted in Fig. 2. A homeomorphism $\mathcal{B} : \partial\mathcal{Z} \overset{Hom}{\longrightarrow} \partial\mathcal{W}$ between their boundaries can be defined by the formula

$$\mathcal{B}(z) = \frac{3}{8}\left(z^2 - \overline{z}^2\right) + \frac{1}{4}\Big[(1+i)\, z + 3\,(1-i)\, \overline{z}\Big] + \frac{i}{2}, \qquad z \in \partial\mathcal{Z}. \tag{1.3}$$

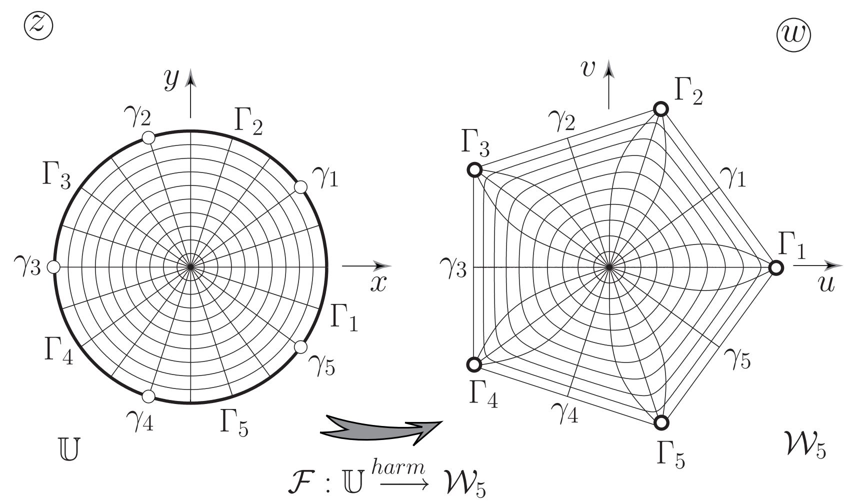

Fig. 1

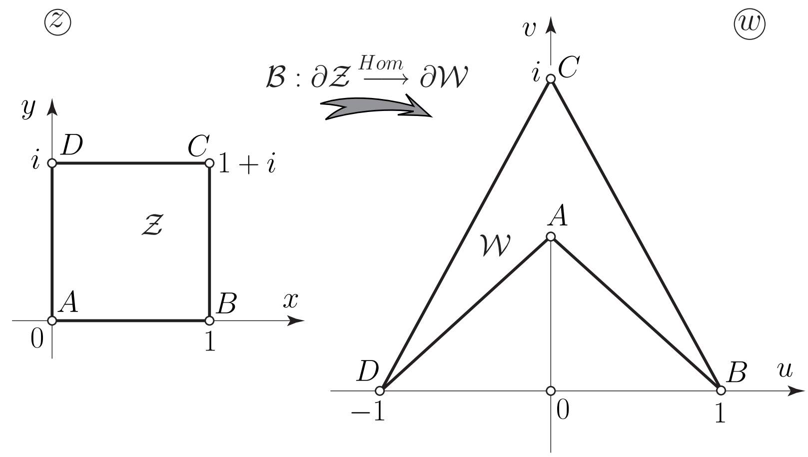

Fig. 2

In Fig. 2, points on different planes that are taken into each other by homeomorphism (1.3) are denoted by the same letters. The solution $\mathcal{F}(z)$ of the Dirichlet problem (1.1), (1.2) with such boundary function $\mathcal{B}(z)$ is given by the right-hand side of expression (1.3), where z takes values in $\mathcal{Z}$.

Calculating the Jacobian of the function $\mathcal{F}(z)$:

$$J(\mathcal{F}) = \frac{3}{2}\,(x+y) - 1,$$

we see that it vanishes on the segment connecting the point $D' = \dfrac{2}{3}$ with $B' = \dfrac{2}{3}\,i$, see Fig. 3(a). Hence, by the Lewy theorem [27, 57], this function does not realize a homeomorphism $\mathcal{Z} \overset{Hom}{\longrightarrow} \mathcal{W}$ of domains. A more detailed analysis shows that the triangle $[AD'B']$ cut by the segment $(D'B')$, where the Jacobian is negative, is taken to a curvilinear triangle with the same notation, where two sides are straight lines, and the third one, $(D'B')$, is an arc of a parabola (see Fig. 3). The remaining part of the square, the pentagon $[B'BCDD']$, where the Jacobian is positive, is taken to a pentagon with the same notation, where four sides are straight lines, and the fifth one, $(D'B')$, is the same parabolic arc as above. Thus the constructed harmonic extension of the boundary homeomorphism (1.3) onto the square $\mathcal{Z}$ maps it not onto the domain $\mathcal{W}$ inside the contour $\partial\mathcal{W} = \mathcal{F}(\partial\mathcal{Z})$, but onto the two-sheeted manifold $\widetilde{\mathcal{W}}$ depicted in Fig. 3(b). One of its sheets is the curvilinear pentagon $[B'BCDD']$, and the other one is the curvilinear triangle $[AD'B']$. In Fig. 3(b), one can see the image in $\widetilde{\mathcal{W}}$ of the Cartesian grid in $\mathcal{Z}$.

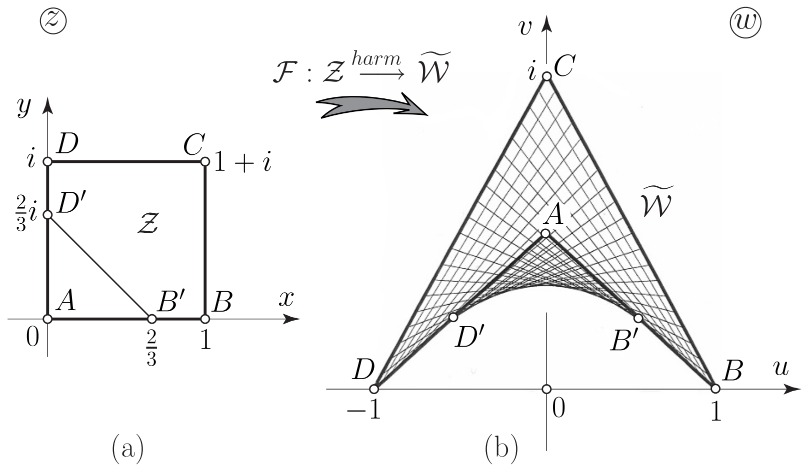

Fig. 3

We give another similar example. We choose a trapezium with vertices $A = -1 - i$, $C = 1 - i$, $E = 1/2 + 3\,i/4$, and $G = -1/2 + 3\,i/4$ as the domain $\mathcal{Z}$ and the interior of the contour $\mathcal{F}(\partial\mathcal{Z})$ as $\mathcal{W}$, which is the image of the boundary $\partial\mathcal{Z}$ of the trapezium under the mapping

$$\mathcal{F}(z) = \frac{1}{2}\left[\, i\,(z^2 + \overline{z}^{\,2}) + z + \overline{z}\,\right]. \tag{1.4}$$

These domains are depicted in Fig. 4. A homeomorphism $\mathcal{B} : \partial\mathcal{Z} \xrightarrow{Hom} \partial\mathcal{W}$ between the boundaries of these domains can be given by the right-hand side of formula (1.4), where z takes values in $\partial\mathcal{Z}$. The function (1.4) for $z \in \mathcal{Z}$ is a harmonic extension of the mentioned above boundary homeomorphism onto the domain $\mathcal{Z}$.

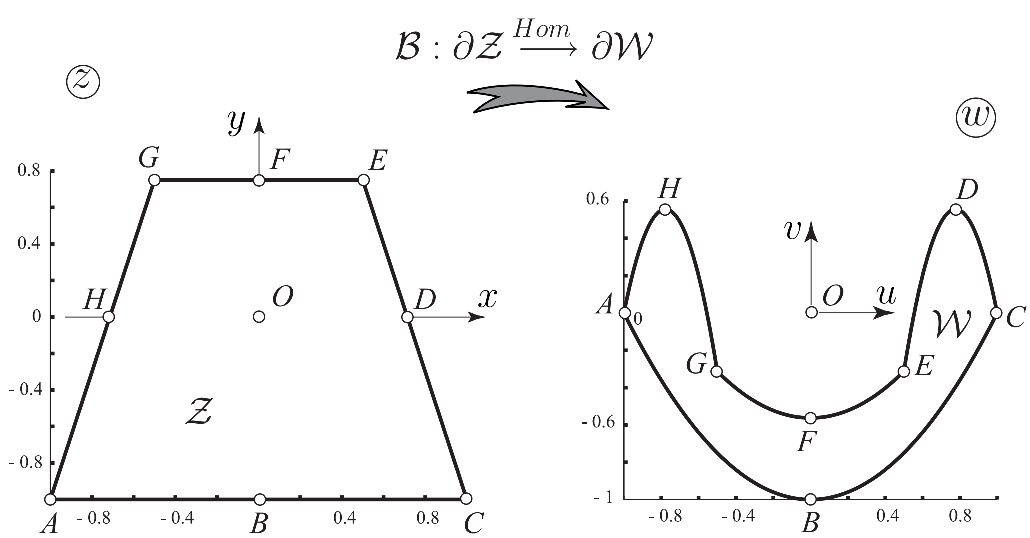

Fig. 4

Calculating the Jacobian $J(\mathcal{F}) = -2y$, we see that it vanishes on the segment (HOD) of the real axis, which lies (except for its ends) in the domain $\mathcal{Z}$. Therefore, function (1.4) does not realize a homeomorphism of domains $\mathcal{Z} \overset{Hom}{\longrightarrow} \mathcal{W}$. Analysis shows that mapping (1.4) takes the domain $\mathcal{Z}$ not onto the domain $\mathcal{W}$ but onto a two-sheeted manifold $\widetilde{\mathcal{W}} \supset \mathcal{W}$, which consists of two plain domains $\mathcal{W}^- = [ABCDOHA]$ and $\mathcal{W}^+ = [HGFEDOH]$ connected by an arc interval $\mathrm{int}(DOH)$, which are one-sheeted images of the lower $\mathcal{Z}^- := \mathcal{Z} \cap \mathbb{H}^-$ and upper $\mathcal{Z}^+ := \mathcal{Z} \cap \mathbb{H}^+$ halves of the trapezium, respectively, where $\mathbb{H}^-$ and $\mathbb{H}^+$ are the lower and upper half-planes. The Jacobian of the mapping is positive in $\mathcal{Z}^-$ and negative in $\mathcal{Z}^+$. In Fig. 5, one can see the trapezium $\mathcal{Z}$, the two-sheeted manifold $\widetilde{\mathcal{W}}$, a grid in $\mathcal{Z}$, and its image in $\widetilde{\mathcal{W}}$.

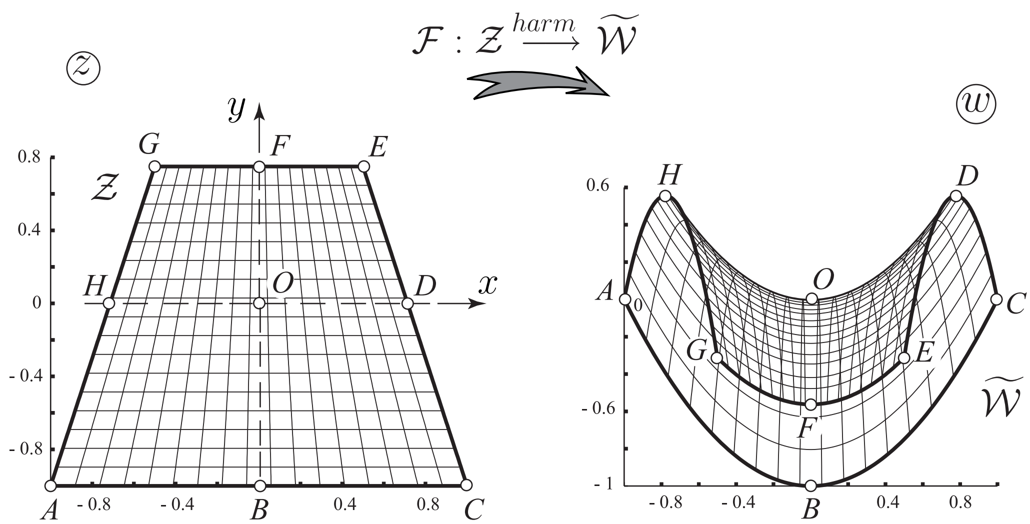

Fig. 5

1.4. Homeomorphism property of a harmonic extension. The problem of homeomorphism property of a harmonic extension, as the problem of homeomorphism property in general, is one of the central problems in the theory of harmonic mappings. It was studied in many papers, including [3, 24, 27, 40, 46, 48, 51, 57, 67, 86].

A sufficient condition for homeomorphism property of a harmonic extension is established in the theorem formulated by Radó [67] and proved by Kneser [48] in 1926; later, in 1945, it was rediscovered and proved in a different way by Choquet [24].

Theorem 1.1 (Radó, Kneser, Choquet). *Let $\mathcal{Z}$ and $\mathcal{W}$ be Jordan domains, and let $\mathcal{W}$ be convex. Then the harmonic extension $\mathcal{F}(z)$ of any boundary homeomorphism $\mathcal{B} : \partial\mathcal{Z} \overset{Hom}{\longrightarrow} \partial\mathcal{W}$ onto domain $\mathcal{Z}$, i.e., the solution of problem* (1.1), (1.2), *realizes a homeomorphism $\mathcal{F} : \overline{\mathcal{Z}} \overset{Hom}{\longrightarrow} \overline{\mathcal{W}}$ of the closures of the domains and is a harmonic mapping of $\mathcal{Z}$ onto $\mathcal{W}$.*

In [24], a "converse" theorem was also given.

Theorem 1.2 (Choquet). *For any pair of Jordan domains $\mathcal{Z}$ and $\mathcal{W}$, where $\mathcal{W}$ is not convex, there exists a boundary homeomorphism $\mathcal{B} : \partial\mathcal{Z} \overset{Hom}{\longrightarrow} \partial\mathcal{W}$ such that its harmonic extension $\mathcal{F}$ onto $\mathcal{Z}$ is not a homeomorphism of the domain $\mathcal{Z}$ onto $\mathcal{W}$.*

Note that convexity of the image (domain $\mathcal{W}$) is not a necessary condition for a harmonic extension $\mathcal{F}$ of a boundary homeomorphism $\mathcal{B}$ to be a homeomorphism of domain $\mathcal{Z}$ onto $\mathcal{W}$.

Necessary and sufficient conditions under which a boundary homeomorphism implies a homeomorphism of the closures of the domains, i.e.,

$$\mathcal{B} : \partial\mathcal{Z} \overset{Hom}{\longrightarrow} \partial\mathcal{W} \quad \Longrightarrow \quad \mathcal{F} : \overline{\mathcal{Z}} \overset{Hom}{\longrightarrow} \overline{\mathcal{W}}, \tag{1.5}$$

were established by Kudryavtsev in [51] for the case where $\mathcal{Z}$ is a disk $\mathbb{U}$ and $\mathcal{W}$ is a Jordan domain with a Lyapunov boundary. To formulate this result, we represent the boundary homeomorphism in the form $\mathcal{B}(e^{i\theta}) = a(\theta) + ib(\theta)$, where $e^{i\theta}$ is a point on $\partial\mathbb{U}$ and $\theta = \arg z$ is the polar angle, and introduce a function $A(\theta) := e^{-i\theta}\left[\widetilde{a}'(\theta) - ia'(\theta)\right]$, where the prime denotes differentiation with respect to θ while the tilde denotes the Hilbert transform.

Theorem 1.3 (Kudryavtsev, 1955). *Let the index of the function $A(\theta)$ be equal to zero, i.e.,*

$$\int_0^{2\pi} d\,\mathrm{Arg}\,A(\theta) = 0.$$

Then statement (1.5) *is valid if and only if the "boundary Jacobian" of the mapping $\mathcal{F}$ does not vanish, i.e.,*

$$a'(\theta)\,\widetilde{b}'(\theta) - \widetilde{a}'(\theta)\,b'(\theta) \neq 0, \qquad \theta \in [0, 2\pi].$$

1.5. The structure of the image of the domain under a harmonic mapping. In [51], the structure of the image of the domain $\mathcal{Z} \subset \mathbb{C}$ under its harmonic mapping $w = \mathcal{F}(z)$ was also studied. It was shown that the set $\mathcal{Z}_0 := \{z \in \mathcal{Z} : J(\mathcal{F}(z)) = 0\}$ where the Jacobian of the mapping vanishes, consists of no more than a countable number of analytic arcs, their endpoints as singular points (where $u_z = v_z = 0$), and possibly a countable set $\mathcal{A}$ of isolated points that has no limit points in $\mathcal{Z}$.

In the same paper, "canonical components" of the domain $\mathcal{Z}$ with respect to the mapping $\mathcal{F}$ were introduced. They were defined as connected components of the open set $(\mathcal{Z} \setminus \mathcal{Z}_0) \cup \mathcal{A}$, and it was shown that the harmonic mapping $\mathcal{F}(z)$ has many common properties with conformal ones on these components. For example, on each canonical component the Jacobian of the mapping has a constant sign and may vanish only at isolated points, on two neighboring components $J(\mathcal{F})$ it has opposite signs, and the following analogue of the boundary correspondence and domain preservation principles holds. If $\mathcal{D}$ is a Jordan subdomain of the canonical component such that the mapping $\mathcal{F}$ is a homeomorphism on $\partial\mathcal{D}$, then $\mathcal{F}$ is a homeomorphism of the closed domain $\overline{\mathcal{D}}$.

Figure 6 illustrates these statements on the example of the harmonic mapping of the disk $\{|z| < 4\}$ under the function $\mathcal{F}(z) = z + \overline{z}^2/2$. Note that this function minimizes the area of the disk's image $\mathbb{U}$ in the class of one-sheeted and orientation-preserving harmonic functions $\mathcal{F} = \mathcal{H} + \overline{\mathcal{G}}$ in $\mathbb{U}$ that obey conditions $\mathcal{H}(0) = \mathcal{G}(0) = \mathcal{G}'(0) = 0$, $\mathcal{H}'(0) = 1$; the minimum equals $\pi/2$. The Jacobian of this mapping $J(\mathcal{F}) = 1 - |z|$ vanishes on the circle $\{|z| = 1\}$, which splits the original domain $\mathcal{Z} := \{|z| < 4\}$ into two canonical components $\mathcal{Z}_1 = \mathbb{U}$ and $\mathcal{Z}_2 = \{1 < |z| < 4\}$ such that the Jacobian is positive (and therefore the mapping is orientation-preserving) in the first one and negative (and therefore the mapping is orientation-reversing) in the second one.

In Fig. 6(a), one can see: the domain $\mathcal{Z}$ (the disk of radius 4), the unit circle where $J(\mathcal{F}) = 0$, and canonical components $\mathcal{Z}_1$ (the unit disk) and $\mathcal{Z}_2$ (the annulus $\mathcal{Z}_2 = \{1 < |z| < 4\}$); a polar grid is also given (in each component in its own way).

In Fig. 6(b), one can see the image $\mathcal{W}_1 = \mathcal{F}(\mathcal{Z}_1)$ of the first canonical component, which is a curvilinear triangle with zero angles. Here we also give the image of the polar grid $\mathcal{W}_1$, namely, the images of the circles $\{|z| = r_k\}$, where $r_k = \dfrac{1}{10}k$, $k = \overline{1, 10}$, and those of the radial lines $\Big\{z : |z| \le 1$, $\arg z = \dfrac{1}{12}\,\pi k\Big\}$, $k = \overline{0, 23}$. Note that the images of the radial lines reach the sides of the triangle $\mathcal{W}_1$ with angles distinct from zero (with the exception of the angular points) and from $\pi/2$ (with the exception of three lines). We also point out that the mapping $\mathcal{F}$ is a homeomorphism between $\overline{\mathcal{Z}}_1$ and $\overline{\mathcal{W}}_1$.

In Fig. 6(c), we see the image $\mathcal{W}_2 = \mathcal{F}(\mathcal{Z}_2)$ of the second canonical component, which is a three-sheeted manifold and hence not a plain domain. Here one can also see the images of the circles

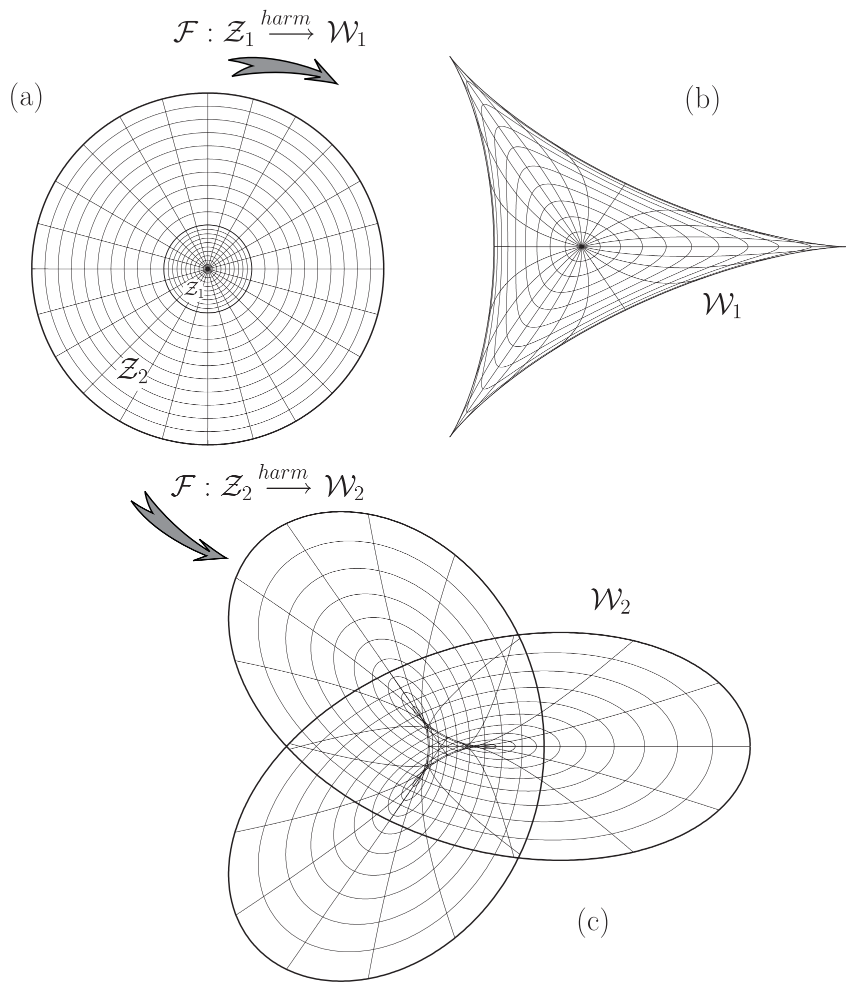

Fig. 6

$\{|z| = r_k\}$, where $r_k = 1 + 0.3\,k$, $k = \overline{0,\,10}$, and of the radial lines $\Big\{z :\ 1 \le |z| \le 4,\ \arg z\ =\ \dfrac{1}{12}\,\pi k\Big\}$, $k = \overline{0,\,23}$.

1.6. Winslow's method and harmonic grids. We turn to an important application of harmonic mappings already mentioned in Sec. 1.1, namely to generation of calculation grids adapted to domains $\mathcal{Z}$ with complicated configuration (see, e.g., [11, 43, 45, 59, 73, 84–87, 99, 102]).

One of the methods of grid generation using a harmonic mapping consists of the following. Choosing the unit square $\mathcal{Q} := [0,1] \times [0,1]$ as the domain $\mathcal{W}$ and constructing a harmonic mapping $\mathcal{F}$: $\mathcal{Z} \overset{harm}{\longrightarrow} \mathcal{Q}$, we obtain the required grid $\mathfrak{Z}_h$ by taking the Cartesian grid (uniform with step h) $\mathfrak{Q}_h$, which is natural for the square, into domain $\mathcal{Z}$ by this mapping, i.e., $\mathfrak{Z}_h = \mathcal{F}^{-1}(\mathfrak{Q}_h)$. Since the domain $\mathcal{Q}$ is convex, the Radó–Kneser–Choquet theorem guarantees that this mapping is one-sheeted. The grid obtained in this way is called harmonic.

Recall the obvious definition of the grid $\mathfrak{Q}_h$ in the square $\mathcal{Q}$ with a step $h = 1/N$ and some $N \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers:

$$\mathfrak{Q}_h := \Big\{z_{m,\,n} = (m-1)h + i(n-1)h\Big\}_{m=1,\,n=1}^{N+1,\,N+1}. \tag{1.6}$$

In this method of generation of the harmonic grid, the boundary homeomorphism $\mathfrak{B} : \partial\mathcal{Z} \overset{Hom}{\longrightarrow} \partial\mathcal{Q}$ is chosen in such a way that on the preimages $l_n := \mathfrak{B}^{-1}(L_n)$ of the sides L_n of the square $\mathcal{Q}$ the "boundary derivative" dS/ds of the mapping $w = \mathfrak{B}(z)$ is constant, i.e.,

$$dS(w)/ds = |l_n|^{-1}, \qquad z \in l_n. \tag{1.7}$$

Here $s(z)$ and $S(w)$ are arc lengths on $\partial\mathcal{Z}$ and on $\partial\mathcal{Q}$ measured in the positive direction (so that the domain remains on the left) from the points $z = \mathfrak{B}^{-1}(0)$ and $w = 0$, respectively, and L_1, L_2, L_3, and L_4 are, respectively, the lower, the right, the upper, and the left sides of the square $\mathcal{Q}$. Thus, for the mapping $w = \mathcal{F}(z)$, the Dirichlet problem (1.1), (1.2) takes the following form:

$$\Delta\mathcal{F}(z) = 0, \qquad z \in \mathcal{Z}, \tag{1.8}$$

$$\mathfrak{B}(z) = w_n - (i)^{n+1}\,|l_n|^{-1}\big[s(z) - \sigma_n\big], \qquad z \in l_n, \quad n = \overline{1,\,4}. \tag{1.9}$$

Here $\sigma_n := \sum\limits_{k=1}^{n-1} |l_k|$ and w_n are the vertices of the square $\mathcal{Q}$ defined by equalities

$$w_1 = 0, \quad w_2 = 1, \quad w_3 = 1 + i, \quad w_4 = i.$$

The regular rectangular grid $\mathfrak{Z}_h$ (i.e., the grid with rectangular cells) will be adapted to the domain $\mathcal{Z}$, i.e., all its nodes will lie in the closure $\overline{\mathcal{Z}}$, and some of them necessarily on the boundary $\partial\mathcal{Z}$, and the cells of the grid itself (if their size h is small enough) will not overlap. These properties follow from the fact that the mapping $\mathcal{F} : \mathcal{Z} \overset{harm}{\longrightarrow} \mathcal{Q}$, as was already mentioned above, is by the Radó–Kneser–Choquet theorem a homeomorphism between the closures $\overline{\mathcal{Z}}$ and $\overline{\mathcal{Q}}$.

In order to construct the required mapping $\mathcal{F}$, one should solve the Dirichlet problem (1.8), (1.9). But since the domain $\mathcal{Z}$ has a complicated form, one should use numerical methods to solve this problem, and their implementation, in its turn, requires a grid adapted to $\mathcal{Z}$. It turns out that in order to construct a grid adapted to the domain, one must already have such a grid!

The way out of this vicious circle was shown by Winslow in [96], who proposed to construct, instead of the direct harmonic mapping $\mathcal{F} : \mathcal{Z} \overset{harm}{\longrightarrow} \mathcal{Q}$, the inverse mapping

$$\mathcal{F}^{-1}(w) \;=\; x(u,\,v) + iy(u,\,v) : \; \mathcal{Q} \overset{harm^{-1}}{\longrightarrow} \mathcal{Z}.$$

The components $x(u,\,v)$ and $y(u,\,v)$ of such a mapping, as is shown in [96], solve the system of equations

$$\begin{cases} (x_v^2 + y_v^2)\,x_{uu} - 2\,(x_u x_v + y_u y_v)\,x_{uv} + (x_u^2 + y_u^2)\,x_{vv} = 0, \\ (x_v^2 + y_v^2)\,y_{uu} - 2\,(x_u x_v + y_u y_v)\,y_{uv} + (x_u^2 + y_u^2)\,y_{vv} = 0, \end{cases} \tag{1.10}$$

in the square $\mathcal{Q}$ with Dirichlet boundary conditions defined by the inverse boundary homeomorphism to (1.9) $\mathfrak{B}^{-1} : \partial\mathcal{Q} \overset{Hom}{\longrightarrow} \partial\mathcal{Z}$. These conditions have the form

$$x(u,v) = \operatorname{Re}\mathfrak{B}^{-1}(w), \quad y(u,v) = \operatorname{Im}\mathfrak{B}^{-1}(w), \qquad w \in \partial\mathcal{Q}. \tag{1.11}$$

The lower subscripts in (1.10) denote the respective partial derivatives of the functions $x(u,v)$ and $y(u,v)$, while $\operatorname{Re} a$ and $\operatorname{Im} a$ denote the real and the imaginary part of the complex number a.

To construct the mapping $\mathcal{F}^{-1} : \mathcal{Q} \overset{harm^{-1}}{\longrightarrow} \mathcal{Z}$, a variational approach related to Winslow's method is also used. It was suggested in [20, 97] and developed in [30, 43, 45, 71, 93], etc. It consists of minimization of the Dirichlet integral for system (1.10):

$$I(\mathcal{F}^{-1}) = \int\limits_{Q} \frac{g_{11} + g_{22}}{g_{11}g_{22} - g_{12}^2}\, du\, dv, \tag{1.12}$$

where the quantities $g_{kl}(u,v)$ defined by equalities

$$g_{11}(u,v) := x_u^2 + y_u^2, \quad g_{22}(u,v) := x_v^2 + y_v^2, \quad g_{12}(u,v) := x_u y_u + x_v y_v,$$

have the geometrical sense as the elements of the metric tensor g for the mapping $\mathcal{F}^{-1}(w)$. On the basis of this interpretation, the integrand in (1.12) can be written in the coordinate-free form $\operatorname{tr}(g)/\sqrt{\det g}$, which is convenient for different modifications and generalizations [6, 18, 42, 43, 79, 80, 92, 99]; here, as usual, we use symbols tr and det to denote the trace and the determinant of a matrix.

For numerical implementation of the minimization of the functional (1.15), one usually applies the two-dimensional rectangle formula for approximation of an integral and the simplest approximation of derivatives in this functional by finite differences [43].

Thus, the approach proposed by Winslow to consider not the harmonic mapping $\mathcal{F} : \mathcal{Z} \stackrel{harm}{\longrightarrow} \mathcal{Q}$ but the inverse mapping $\mathcal{F}^{-1} : \mathcal{Q} \stackrel{harm^{-1}}{\longrightarrow} \mathcal{Z}$ turned out to be very flexible, amenable to broad generalizations and modifications, and effective enough for a large range of problems. Its effectiveness comes, in particular, from the fact that the square $\mathcal{Q}$, where the system of Eqs. (1.10) must be solved, has a natural Cartesian grid $\mathfrak{Q}_h$, where a finite difference approximation of the differential operator is easy to construct [72] and can be used in combination with iteration methods (see, e.g., [7, 32]) to find an approximate solution $\mathcal{F}_h^{-1}$ of nonlinear system (1.10). Recall that here h is the step of the grid $\mathfrak{Q}_h$. Besides, Winslow's approach is effective due to the fact that the obtained harmonic grid is regular and adapted to the domain $\mathcal{Z}$, as was mentioned before. Winslow's approach and methods based on it are widely applied for grid generation [6, 32, 43, 50, 59, 65, 77, 84–86].

1.7. Effect of irremovable error. Nevertheless, as was mentioned in Sec. 1.1, for many problems, application of difference methods based on Winslow's approach results in such an error of the approximate mapping $\mathcal{F}_h^{-1} : \mathcal{Q} \stackrel{harm^{-1}}{\longrightarrow} \mathcal{Z}$ that it is not a homeomorphism (despite the Radó–Kneser–Choquet theorem), and the respective grid is not adapted to the domain; e.g., it can contain overlapping cells, or its part can lie outside of the domain $\mathcal{Z}$. Such phenomena were mentioned in [6, 19, 20, 43–45, 49, 66, 71, 74, 76, 85] and other papers.

As was mentioned in Sec. 1.1, the Roache–Steinberg problem [6, 44, 49, 71] on the construction of a harmonic mapping

$$F(z) = u(x, y) + iv(x, y) : \mathcal{S} \stackrel{harm}{\longrightarrow} \mathcal{Q} \tag{1.13}$$

of a horseshoe-like domain $\mathcal{S}$ onto a square $\mathcal{Q}$ (see Fig. 7) is very typical in this sense (or, in the framework of Winslow's approach, on the construction of a mapping

$$F^{-1}(w) = x(u, v) + iy(u, v) : \mathcal{Q} \stackrel{harm^{-1}}{\longrightarrow} \mathcal{S}, \tag{1.14}$$

which is inverse to the harmonic one and takes the square $\mathcal{Q}$ to the horseshoe-like domain $\mathcal{S}$).

The domain $\mathcal{S}$ is defined as the interior of the contour $\partial\mathcal{S}$ consisting of four lines l_k, $k = \overline{1, 4}$, where two lines are segments of the real axis $l_1 := [1, 2]$ and $l_3 := [-2, -1]$, and the other two are the half-ellipse l_2 and the semicircle l_4 defined by the formula

$$l_2 := \left\{z : \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2A}\right)^2 = 1, \quad \operatorname{Im} z \geq 0\right\}, \qquad l_4 := \{z : |z| = 1, \operatorname{Im} z \geq 0\}, \tag{1.15}$$

where A is the ratio of the axes of the ellipse.

The boundary homeomorphism $B^{-1} : \partial\mathcal{Q} \stackrel{Hom}{\longrightarrow} \partial\mathcal{S}$ necessary for the construction of the mapping F^{-1} can be written componentwise in the form

$$x(u, v) = \operatorname{Re} B^{-1}(w), \quad y(u, v) = \operatorname{Im} B^{-1}(w), \qquad w \in \partial\mathcal{Q}, \tag{1.16}$$

and takes the sides of the square to the segments of the boundary $\partial\mathcal{S}$, i.e., $l_n = B^{-1}(L_n)$. It is defined by inversion of formula (1.9) with B replaced by $\mathfrak{B}$. In our case, in formula (1.9) $s(z)$ is the arc length on $\partial\mathcal{S}$ measured in the positive direction from the point $B^{-1}(0) = 1$, $|l_n|$ are lengths of the segments l_n, and w_n are the vertices of the square.

In Winslow's approach, the mapping $F^{-1}(w) = x(u, v) + iy(u, v)$ was found from the boundary-value problem for system (1.10) with boundary conditions (1.16). Since difference methods were used for its solution, in fact an approximate mapping F_h^{-1} was found, and the required grid $\mathfrak{S}_h$ in

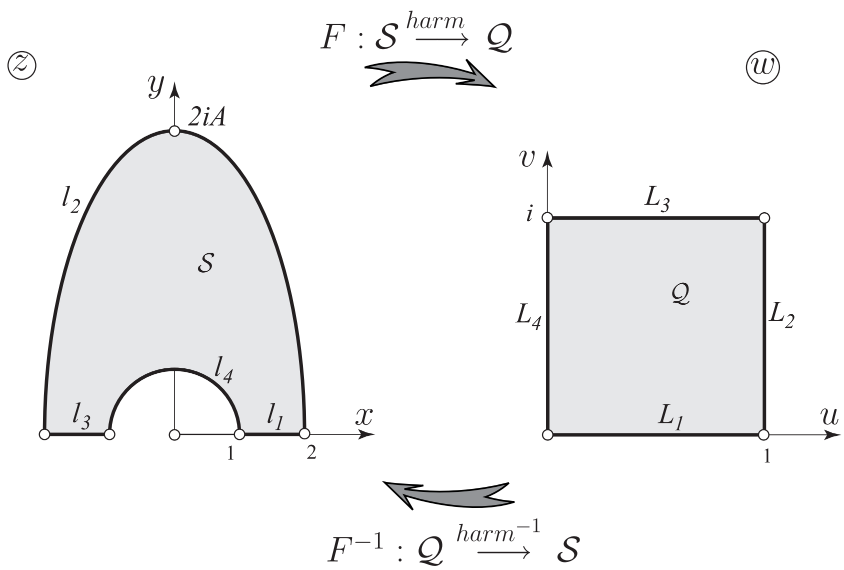

Fig. 7

the horseshoe-like domain $\mathcal{S}$ was obtained as the image of the Cartesian grid $\mathfrak{Q}_h$ under the found mapping, i.e.,

$$\mathfrak{S}_h = F_h^{-1}(\mathfrak{Q}_h). \tag{1.17}$$

The problem under consideration was solved in this way by Roache and Steinberg in [71] in 1985. Then its results were reproduced by Knupp and Luczak [49] in 1995. It turned out that for a large ratio of the axes of the ellipse ($A > 4$), the mapping was not a homeomorphism, namely, the grid contained overlaps and partly lay outside of the domain $\mathcal{S}$.

In paper by Azarenok [6], the numerical solution of the Roache–Steinberg problem was implemented in detail by several difference methods based on Winslow's approach. The paper [6] confirmed the results of [49, 71]. It was shown that neither refining the grid step h, nor growth of approximation order, nor replacing the differential problem (1.10) by the variational one (1.12) result in essential improvement of the grid, which remains unusable for calculation. Thus, it was established that for the problem under consideration (and hence, taking our previous remarks into account, for a certain class of problems of grid generation in the framework of Winslow's approach), an effect of irremovable error arises.

In Sec. 4 of the present paper, this effect is studied in detail on the example of the Roache–Steinberg problem by comparing numerical results from [6] with a high-precision mapping $F(z)$ found in Sec. 3.

2. Multipole Method

2.1. The Hardy-type space $e_2(g, \Gamma)$ and the formulation of the Dirichlet problem. Let the boundary ∂g of a simply connected domain g on the complex plane z consist of two arcs, Γ and γ, such that the first one is piecewise smooth and its smooth parts join each other at points z_q, $q = \overline{1, Q-1}$, with internal angles with respect to g equal to $\pi\alpha_q$. The second part of the contour, the arc γ, in some neighborhoods of its endpoints z_0 and z_Q is a Lyapunov curve and joins the arc Γ with angles $\pi\alpha_0$ and $\pi\alpha_Q$. These angles obey the inclusion $\alpha_q \in (0, 2)$, $q = \overline{0, Q}$. Then we will say that the domain g satisfies the (γ, Γ) condition.

By z' we denote the points of the arc Γ, and by z those of the set $g \cup \operatorname{int}\gamma$, where $\operatorname{int}\gamma$ means the arc γ without endpoints.

Let this domain g have an extension G across the arc Γ, that is, a simply connected domain G such that the following holds: (1) $G \supset g$, (2) $\partial G \supset \gamma$, (3) $G \supset \operatorname{int}\Gamma$, and the arc $\partial G \setminus \operatorname{int}\gamma$ is smooth in some half-neighborhoods of its endpoints z_0 and z_Q and forms angles $\pi\beta_0$ and $\pi\beta_Q$ with the arc γ in these points, where $\beta_0 > \alpha_0$ and $\beta_Q > \alpha_Q$. For such an extension G of domain g, we will write $G \overset{\Gamma}{\supset} g$.

Denote by $z = \omega(\zeta)$ a conformal mapping of the half-disk $\mathbb{U}^+ := \{|\zeta| < 1, \operatorname{Im} > 0\}$ onto g, which takes the semicircle $\mathbb{T}^+ := \{|\zeta| = 1, \operatorname{Im} > 0\}$ to the arc Γ, and by Γ_r denote the image of the semicircle $\mathbb{T}^+_r := \{|\zeta| = r, \operatorname{Im} > 0\}$ under the mapping ω, i.e., $\Gamma_r = \omega(\mathbb{T}^+_r)$.

Following [89–91], introduce the Hardy-type space $e_2(g, \Gamma)$ of functions $\psi(z)$ harmonic in $g \in (\gamma, \Gamma)$, which belong to $C(g \cup \operatorname{int}\gamma)$ and satisfy on $\operatorname{int}\gamma$ condition $\psi(z) = 0$ and the following restriction:

$$\exists\, a \in (0, 1), \qquad \sup_{r\in(a,\,1)} \int\limits_{\Gamma_r} |\psi(z)|^2\, |dz| < \infty. \tag{2.1}$$

Note that $\psi(w)$ belongs to $e_2(g, \Gamma)$ independently of the choice of the mapping ω. Moreover, this space can be equivalently defined without using any mapping, similarly to the definition of the Smirnov space $E_p(\mathcal{B})$ in [47]: A function $\psi(z) \in C(g \cup \operatorname{int}\gamma)$ harmonic in g, which vanishes on $\operatorname{int}\gamma$, belongs to $e_2(g, \Gamma)$ if there exists a family of rectifiable arcs Γ_r, $\operatorname{int}\Gamma_r \subset g$, parametric with respect to $r \in (a, 1)$, $a \in (0, 1)$, such that the left and right end points of the arcs Γ_r approach z_0 and z_Q, respectively, along the points of γ, and the arcs Γ_r themselves tend to Γ as $r \to 1$ such that condition (2.1) holds.

The following theorem is an analogue of the Riesz theorem for classical Hardy spaces H_p of analytic functions in the unit disk $\mathbb{U}$; here we use the notation $z'_r = \omega\left(r\omega^{-1}(z')\right)$. This theorem, as well as other statements of this section, was formulated in [89–91].

Theorem 2.1. *If a function $\psi(z)$ belongs to $e_2(g, \Gamma)$, then it has a trace $\psi(z')$ on Γ that belongs to $L_2(\Gamma)$ and can be understood as nontangential limit values defined a.e. in Γ, and the following equalities hold*:

$$\lim_{r\to 1} \int\limits_{\Gamma} \left|\psi(z'_r) - \psi(z')\right|^2 |dz'| = 0, \qquad \lim_{r\to 1} \int\limits_{\Gamma_r} \left|\psi(z)\right|^2 |dz| = \int\limits_{\Gamma} \left|\psi(z')\right|^2 |dz'|.$$

Consider the following Dirichlet problem in the domain $g \in (\gamma, \Gamma)$:

$$\Delta\psi(z) = 0, \qquad z \in g, \tag{2.2}$$

$$\psi(z) = 0, \qquad z \in \gamma, \tag{2.3}$$

$$\psi(z') = h(z'), \qquad z' \in \Gamma, \tag{2.4}$$

where the function $h(z')$ belongs to $L_2(\Gamma)$ and $\psi(z')$ is the trace of the solution $\psi(z)$ understood in the sense defined above.

Theorem 2.2. *A solution of the Dirichlet problem* (2.2)–(2.4) *with an arbitrary function $h \in L_2(\Gamma)$ exists and is unique in the class $e_2(g, \Gamma)$; equality* (2.4) *holds a.e. on Γ and necessarily holds at the continuity points of the function $h(z')$.*

Theorems 2.1 and 2.2 imply the following

Proposition 2.1. *Operator $\mathbb{S}$ assigning to any function $\psi(z) \in e_2(g, \Gamma)$ its trace $\psi(z') \in L_2(\Gamma)$ establishes an isometrical isomorphism between spaces $e_2(g, \Gamma)$ and $L_2(\Gamma)$.*

Theorem 2.3. *Let $\psi(z)$ be the solution of problem* (2.2)–(2.4) *defined in Theorem* 2.2. *Then for $\psi(z)$ one has the estimate*

$$\max_{z\in\mathcal{E}} |\psi(z)| \le \frac{1}{\pi\delta} \left\|\frac{1}{\omega'}\right\|^{1/2}_{L_1(\mathbb{T}^+)} \|h\|_{L_2(\Gamma)}, \tag{2.5}$$

where $\mathcal{E}$ is a compact set in $g \cup \operatorname{int}\gamma$ and δ is the distance between $\omega^{-1}(\mathcal{E})$ and $\mathbb{T}^+$. For the gradient of the solution, the following estimate holds:

$$\max_{z\in\widetilde{\mathcal{E}}} |\operatorname{grad}\psi(z)| \le \frac{1}{\pi\delta^2} \max_{z\in\widetilde{\mathcal{E}}} \left|\omega^{-1\,\prime}(z)\right| \left\|\frac{1}{\omega'}\right\|^{1/2}_{L_1(\mathbb{T}^+)} \|h\|_{L_2(\Gamma)}; \tag{2.6}$$

here $\widetilde{\mathcal{E}}$ is a compact set in $g \cup \operatorname{int}\widetilde{\gamma}$, where $\widetilde{\gamma}$ is a Lyapunov arc in γ and δ is the distance between $\omega^{-1}(\widetilde{\mathcal{E}})$ and $\mathbb{T}^+$.

2.2. The multipole method for solving the Dirichlet problem. Let conditions $g \in (\gamma, \Gamma)$ and $G \overset{\Gamma}{\supset} g$ hold. Choose a point N on $\operatorname{int}\gamma$ and a point M on the complementary arc $\partial G \setminus \gamma$ and introduce a conformal mapping $Z = \Phi(z)$ of the extension G onto the upper half-plane $\mathbb{H}^+$ which satisfies the normalization condition

$$\Phi(N) = 0, \qquad \Phi(M) = \infty. \tag{2.7}$$

Introduce functions $\Omega_k(z)$, $k \in \mathbb{N}$, related to multipoles from the potential theory by the formula

$$\Omega_k(z) = \operatorname{Im}\big[\Phi(z)\big]^k, \qquad k \in \mathbb{N}. \tag{2.8}$$

Theorem 2.4. *System $\{\Omega_k(z)\}_{k\in\mathbb{N}}$ is complete and minimal in the space $e_2(g, \Gamma)$.*

According to this theorem, we search the solution of problem (2.2)–(2.4) as a limit of the sequence $\{\psi^N\}$ of approximate solutions

$$\psi(z) = \lim_{N\to\infty} \psi^N(z), \qquad \psi^N(z) := \sum_{k=1}^{N} a_k^N\, \Omega_n(z), \tag{2.9}$$

where the coefficients are determined from the condition of minimality of the difference $\psi^N - h$ in the norm $L_2(\Gamma)$, which results in the following system of linear equations:

$$\sum_{k=1}^{N} (\Omega_n, \Omega_k)\, a_k^N = (\Omega_n, h), \qquad n = \overline{1, N}, \tag{2.10}$$

where $(\cdot,\cdot)$ denotes the inner product in $L_2(\Gamma)$.

The convergence of this method is established by the following theorem, where $D^{(l,\,m)} = \dfrac{\partial^{l+m}}{\partial x^l\,\partial y^m}$ while l and m are nonnegative integers.

Theorem 2.5. *The following statements hold*:

(1) $\lim_{N\to\infty} \|\psi^N - h;\, L_2(\Gamma)\| = 0$;
(2) *the sequence $\big\{D^{(l,m)}\psi^N(z)\big\}_N$, where l and m are nonnegative integers, converges to $D^{(l,m)}\psi(z)$ as $N \to \infty$ uniformly on each compact set $\mathcal{E} \subset g$;*
(3) *the sequence $\big\{D^{(l,\,m)}\,\psi^N(z)\big\}_N$, where $l+m \le t$, converges to $D^{(l,\,m)}\,\psi(z)$ as $N \to \infty$ uniformly on each compact set $\widetilde{\mathcal{E}} \subset g \cup \operatorname{int}\widetilde{\gamma}$, where $\widetilde{\gamma}$ is an arc of class $C^{t,\alpha}$, $\alpha \in (0,1)$, in γ;*
(4) *for each k, there exist a limit $\lim_{N\to\infty} a_k^N =: a_k$ and a decomposition*

$$\psi(z) = \sum_{k=1}^{\infty} a_k\, \Omega_k(z), \tag{2.11}$$

which converges uniformly inside the disk $\Phi^{-1}(\mathbb{U}^+(R_0))$, where $R_0 := \min|\Phi(z')|$ while

$$\mathbb{U}^+(R_0) := \{|z| < R_0, \quad \operatorname{Im} z > 0\},$$

where the decomposition is infinitely differentiable.

Since the error $\psi(z) - \psi^N(z)$ evidently belongs to $e_2(g, \Gamma)$, inequalities (2.5) and (2.6) imply the following estimates in the norm $C(\mathcal{E})$ for the solution and its gradient via $L_2(\Gamma)$-norm of the error $h(z') - \psi^N(z')$ on Γ:

$$\max_{z\in\mathcal{E}} \left|\psi(z) - \psi^N(z)\right| \le \frac{1}{\pi\delta} \left\|\frac{1}{\omega'}\right\|_{L_1(\mathbb{T}^+)}^{1/2} \left\|h(z') - \psi^N(z')\right\|_{L_2(\Gamma)}, \tag{2.12}$$

$$\max_{z\in\widetilde{\mathcal{E}}} \left|\mathrm{grad}\psi(z) - \mathrm{grad}\psi^N(z)\right| \le \frac{1}{\pi\delta^2} \max_{z\in\widetilde{\mathcal{E}}} \left|\omega^{-1\,\prime}(z)\right| \left\|\frac{1}{\omega'}\right\|_{L_1(\mathbb{T}^+)}^{1/2} \left\|h(z') - \psi^N(z')\right\|_{L_2(\Gamma)}. \tag{2.13}$$

If the boundary function $h(z')$ in the boundary condition (2.4) is continuous, then the exact solution $\psi(z)$ of problem (2.2)–(2.4) and the approximate one $\psi^N(z)$ are harmonic in g and continuous in $\overline{g}$. Therefore, their difference has the same properties, and one can apply the maximum principle to it. Hence we obtain an a posteriori estimate of the approximate solution in the norm $C\left(\overline{g}\right)$:

$$\max_{z\in\overline{g}} \left|\psi(z) - \psi^N(z)\right| \le \max_{z'\in\Gamma} \left|h(z') - \psi^N(z')\right|. \tag{2.14}$$

Evidently, the right-hand side of this inequality can be easily calculated after obtaining the approximate solution.

Moreover, for each compact set $\mathcal{E} \subset g \cup \operatorname{int}\gamma$ there holds the estimate

$$\max_{z\in\mathcal{E}} \left|u(z) - u^N(z)\right| = \mathrm{O}\left(e^{-\lambda N}\right), \qquad N \to \infty, \qquad \lambda = \lambda(\mathcal{E}) > 0, \tag{2.15}$$

i.e., the multipole method converges exponentially in the domain g.

3. Solution of the Roache–Steinberg Problem by the Multipole Method

3.1. Formulation of auxiliary boundary-value problems. The formulation of the Roache–Steinberg problem of constructing a harmonic mapping (1.13) of a horseshoe-like domain $\mathcal{S}$ onto the square $\mathcal{Q}$ (see Fig. 7) was given in Sec. 1.7.

According to the general scheme of constructing a harmonic continuation of the boundary homeomorphism described in Sec. 1.3, the problem of construction of mapping (1.13) consists in solving two Dirichlet problems for the real part $u(x, y)$ and for the imaginary part $v(x, y)$ of the mapping function $F(z)$ separately, with boundary conditions corresponding to the boundary homeomorphism (1.16). Such Dirichlet problems for functions u and v have the following form:

$$\begin{aligned} &\Delta u(x,y) = 0, \qquad z \in \mathcal{S}; \\ &u(x,y) = x - 1, \quad z \in l_1; \qquad u(x,y) = 1, \quad z \in l_2; \\ &u(x,y) = -x - 1, \quad z \in l_3; \qquad u(x,y) = 0, \quad z \in l_4; \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\Delta v(x,y) = 0, \qquad z \in \mathcal{S}; \\ &v(x, y) = 0, \quad z \in l_1; \qquad v(x, y) = \mathcal{V}(z), \quad z \in l_2; \\ &v(x, y) = 1, \quad z \in l_3; \qquad v(x, y) = \pi^{-1} \arccos x, \quad z \in l_4. \end{aligned} \tag{3.2}$$

Here $\mathcal{V}(z)$ is determined by the formula

$$\mathcal{V}(z) = \begin{cases} \mathcal{E}\left(2^{-1}y/A, k\right), & z \in \ l_2 \cap \{\operatorname{Re} z \ge 0\}, \\ 1 - \mathcal{E}\left(2^{-1}y/A, k\right), & z \in \ l_2 \cap \{\operatorname{Re} z \le 0\}, \end{cases} \tag{3.3}$$

where $\mathcal{E}(\xi, k)$ is given by equality

$$\mathcal{E}(\xi, k) := 2^{-1}E(\xi, k)/E(k), \tag{3.4}$$

where $E(\xi,k)$ and $E(k)$ are, respectively, an incomplete and complete elliptic integrals of the second kind [8]

$$E\big(\xi,k\big):=\int\limits_0^{\xi}\sqrt{(1-k^2t^2)/(1-t^2)}\,dt,\qquad E(k):=E(1,k), \tag{3.5}$$

with absolute value $k=\sqrt{1-A^{-2}}$.

3.2. Derivation of boundary conditions on $\partial\mathcal{S}$. We give the derivation of the boundary condition (3.2) on the arc l_2. Other boundary conditions from (3.1) and (3.2) are derived in a similar way.

Setting $n=2$ in expression (1.9) for the boundary homeomorphism $B(z)$

$$B(z)=w_2-(i)^3\,\frac{s(z)-|l_1|}{|l_2|},\qquad z\in l_2,$$

and substituting here $w_2=1$, $|l_1|=1$, and $s(z)=|l_1|+s_2(z)$, where $s_2(z)$ is the arc length on $\partial\mathcal{S}$ measured from the point z_2 in the positive direction with respect to the domain $\mathcal{S}$ (i.e., the arc length on the curve l_2), we get the following expression for the function $B(z)$ on l_2:

$$B(z)\;=\;1+i\,\frac{s_2(z)}{|l_2|},\qquad z\in l_2. \tag{3.6}$$

One can easily see that for $s_2(z)$ there holds the equality

$$s_2(z)=\begin{cases}\lambda(y,A), & z\in\; l_2\cap\{\operatorname{Re}z\ge 0\},\\ |l_2|-\lambda(y,A), & z\in\; l_2\cap\{\operatorname{Re}z\le 0\},\end{cases} \tag{3.7}$$

where the function $\lambda(y,A)$ is determined by the formula

$$\lambda(y,A):=\int\limits_0^{y}\sqrt{1+\left[\frac{dx(\tau,A)}{d\tau}\right]^2}\,d\tau,\qquad x\,(y,A)=4\sqrt{1-\frac{y^2}{A^2}}, \tag{3.8}$$

and the length $|l_2|$ of the curve l_2 is expressed as a function of $\lambda(y,A)$ by the equality

$$|l_2|=2\,\lambda(1,A). \tag{3.9}$$

Changing variable $t=\tau/A$ in the integral (3.8), we obtain the following expression for $\lambda(y,A)$:

$$\lambda(y,A)=A\int\limits_0^{y/A}\sqrt{\frac{1-k^2t^2}{1-t^2}}\;dt, \tag{3.10}$$

where $k=\sqrt{1-A^{-2}}$. Rewriting equalities (3.9) and (3.10) by taking into account definition (3.5) of elliptic integrals $E(\xi,k)$ and $E(k)$ in the form

$$\lambda(y,A)\;=\;A\,E\Big(\frac{y}{A},k\Big),\qquad |l_2|=2\,A\,E(k), \tag{3.11}$$

and substituting (3.11) into the formula (3.7) for the arc length $s_2(z)$, we get

$$s_2(z)=\begin{cases}A\,E\big(y/A,\,k\big), & z\in\; l_2\cap\{\operatorname{Re}z\ge 0\},\\ 2AE(k)-A\,E\big(y/A,k\big), & z\in\; l_2\cap\{\operatorname{Re}z\le 0\}.\end{cases} \tag{3.12}$$

Substituting (3.11) and (3.12) into formula (3.6), we get

$$B(z)\;=\;1+i\,\mathcal{V}(z),\qquad z\in l_2, \tag{3.13}$$

where function $\mathcal{V}(z)$ is determined by (3.3). Taking into account that $B(z)=u(x,y)+iv(x,y)$ and separating the real and imaginary parts in (3.13), we arrive at boundary conditions (3.1) and (3.2) on l_2.

3.3. Reduction to a homogeneous boundary condition on a part of the boundary. In order to apply the multipole method described in Sec. 2 to solution of boundary-value problems (3.1) and (3.2) in domain $\mathcal{S}$, it is necessary to define boundary arcs γ and Γ ($\partial\mathcal{S} = \gamma \cup \Gamma$) and introduce an extension $\mathcal{G}$ of the domain $\mathcal{S}$ across the arc Γ. Also, one must construct auxiliary functions u_0 and v_0, which satisfy the original boundary conditions on γ, and pass from (3.1) and (3.2) to similar problems with a homogeneous Dirichlet condition on the arc γ with the help of these functions.

Introduce arcs γ and Γ by equalities

$$\gamma := l_3 \cup l_4 \cup l_1, \qquad \Gamma := l_2 \tag{3.14}$$

and define an extension $\mathcal{G} \supset \mathcal{S}$ of a horseshoe-like domain by the formula

$$\mathcal{G} := \big\{z : \ \operatorname{Im} z > 0, \ |z| > 1\big\}. \tag{3.15}$$

Note that the boundary $\partial\mathcal{G}$ consists of three parts: $\partial\mathcal{G} = l_3^+ \cup l_4 \cup l_1^+$, where l_4 is defined by formula (1.15), while $l_1^+ \supset l_1$ and $l_3^+ \supset l_3$ are set by formulas

$$l_1^+ := \big\{z : \ x \in [1, +\infty), \ y = 0\big\}, \qquad l_3^+ := \big\{z : \ x \in (-\infty, -1], \ y = 0\big\}. \tag{3.16}$$

Note that the arc $l_2 = \Gamma$ of the boundary $\partial\mathcal{S}$ (except for the endpoints) lies in the domain $\mathcal{G}$, and the arc γ is a part of the boundary $\partial\mathcal{S}$, so that $\partial\mathcal{S} = \gamma \cup \Gamma$. Thus, the domain $\mathcal{G}$ is an extension of the domain $\mathcal{S}$ across the arc Γ in the sense given in Sec. 2, i.e., $\mathcal{G} \overset{\Gamma}{\supset} \mathcal{S}$.

Define the function $u_0(x, y)$ as a solution of the boundary-value problem in the domain $\mathcal{G}$

$$\Delta\, u_0 = 0, \quad z \in \mathcal{G}; \tag{3.17}$$

$$u_0 = -x - 1, \quad z \in l_3^+; \qquad u_0 = 0, \quad z \in l_4; \qquad u_0 = x - 1, \quad z \in l_1^+; \tag{3.18}$$

with the growth condition at infinity

$$u_0(x, y) = \mathcal{O}\left(z \log z\right), \qquad \mathcal{G} \supset z \to \infty, \tag{3.19}$$

and the function $v_0(x, y)$ as a solution of the following boundary-value problem in the domain $\mathcal{G}$

$$\Delta\, v_0 = 0, \quad z \in \mathcal{G}; \tag{3.20}$$

$$v_0 = 1, \quad z \in l_3^+; \qquad v_0 = \pi^{-1} \arccos x, \quad z \in l_4; \qquad v_0 = 0, \quad z \in l_1^+; \tag{3.21}$$

$$v_0(x, y) = \mathcal{O}(1), \qquad \mathcal{G} \supset z \to \infty. \tag{3.22}$$

The solutions of these problems have the following form:

$$u_0(x, y) = 1 + \operatorname{Re}\left(\frac{2}{\pi i} \log \frac{z-1}{z+1} + \frac{z^2-1}{2z} - \frac{z^2+1}{\pi i z} \log \frac{z^2-1}{2z}\right), \tag{3.23}$$

$$v_0(x, y) = \pi^{-1} \arg z \tag{3.24}$$

found below in Secs. 3.4 and 3.5.

3.4. Construction of function u_0. We turn to the solution of problem (3.17)–(3.19) for $u_0(x, y)$ in domain $\mathcal{G}$. With the help of the Zhukovsky function

$$w = \mathfrak{H}(z) := \frac{1}{2}\Big(z + \frac{1}{z}\Big),$$

we take $\mathcal{G}$ to the upper half-plane $\mathbb{H}^+ := \{w = u + iv : v > 0\}$ and, substituting $\mathfrak{H}^{-1}(w)$ into relations (3.18), where $z = \mathfrak{H}^{-1}(w) = w + \sqrt{w^2 - 1}$ is the inverse Zhukovsky function, reduce (3.17)–(3.19) to a similar problem in $\mathbb{H}^+$ for the function

$$U_0\,(u, v) := u_0 \circ \mathfrak{H}^{-1}(w).$$

The formulation of this problem has the form

$$\Delta\, U_0 = 0, \qquad w \in \mathbb{H}^+; \qquad U_0 = -\frac{1}{2} + h(u), \qquad w \in \mathbb{R}; \tag{3.25}$$

$$|U_0(u,v)| = \mathcal{O}(w \log w), \qquad w \to \infty, \tag{3.26}$$

where the function $h(u)$ defined on the real axis $\mathbb{R} = \partial\mathbb{H}^+$ is given by the formula

$$h(u) \;=\; \begin{cases} -u - \sqrt{u^2-1} - 1, & u \in (-\infty,\, -1); \\ 0, & u \in (-1,\, 1); \\ u + \sqrt{u^2-1} - 1, & u \in (1,\, +\infty). \end{cases} \tag{3.27}$$

The solution of this (nonclassical) Dirichlet problem can be written in the following form, which follows from the results of [14, 15]:

$$U_0(u,v) \;=\; -\frac{1}{2} \,+\, \mathrm{Re}\big[\, i\,\alpha\, w + \mathcal{M}(w)\,\big], \tag{3.28}$$

where $\alpha \in \mathbb{R}$ is an arbitrary constant and $\mathcal{M}(w)$ is a modified Cauchy-type integral

$$\mathcal{M}(w) := \frac{w^2}{\pi i} \int\limits_{\mathbb{R}} \frac{h(t)dt}{t^2\,(t-w)}. \tag{3.29}$$

Calculating the integral $\mathcal{M}(w)$ by known substitutions, we write it as a combination of elementary functions. Substituting the expression obtained for $\mathcal{M}(w)$ into (3.28) and setting $\alpha = 0$, we get an expression for U_0. Returning to the plane z by means of the Zhukovsky function, we obtain the required solution $u_0(x,y) = U_0 \circ \mathfrak{H}(z)$ of problem (3.17)–(3.20) in form (3.23).

3.5. Construction of function v_0. Similarly to the argument from Sec. 3.4 for function u_0, we reduce the Dirichlet problem (3.20)–(3.22) for $v_0(x,y)$ in the domain $\mathcal{G}$ to a similar problem for function $V_0\,(u,v) := u_0 \circ \mathfrak{H}^{-1}(w)$ in $\mathbb{H}^+$, using the Zhukovsky function $w = \mathfrak{H}(z)$ as a substitution:

$$\Delta\, V_0 \;=0, \qquad w \in \mathbb{H}^+; \qquad\qquad V_0(u) \;= h(u), \qquad w \in \mathbb{R}; \tag{3.30}$$

$$\big|V_0(u,v)\big| = \mathcal{O}(1), \qquad w \to \infty, \tag{3.31}$$

where $h(u)$ is given by formula

$$h(u) \;=\; \begin{cases} 1, & u \in (-\infty,\, -1); \\ (1/\pi)\arccos u, & u \in (-1,\, 1); \\ 0, & u \in (1,\, +\infty). \end{cases} \tag{3.32}$$

Using the results of [14, 15] for the Riemann–Hilbert problem, we obtain the solution of problem (3.30)–(3.32) in the form

$$V_0(u,v) \;=\; \mathrm{Re}\left[\,\frac{1}{\pi i}\log(1+w) \,+\, \Xi\,(w)\right], \tag{3.33}$$

where $\Xi(w)$ is a Cauchy-type integral defined by the equality

$$\Xi(w) := \frac{1}{\pi^2 i} \int\limits_{-1}^{1} \frac{\arccos t\, dt}{t-w}. \tag{3.34}$$

Making substitutions

$$\arccos t = -i\log u, \qquad u := t + \sqrt{t^2-1}$$

in this integral and using the relation $\log u = \lim\limits_{\alpha\to 0} \frac{du^\alpha}{d\alpha}$, we rewrite integral (3.34) as the following limit:

$$\Xi\,(w) = -\frac{1}{\pi^2}\lim_{\alpha\to 0}\frac{d}{d\alpha}\,\Xi(\alpha,w), \qquad\qquad \Xi(\alpha,w) := \int\limits_{\mathbb{T}^+} \frac{u^{\alpha-1}\,(u^2-1)\,du}{u^2 - 2\,w\,u + 1}, \tag{3.35}$$

where $\mathbb{T}^+ = \{w : |w| = 1, \arg w \in (0, \pi)\}$ is the unit semicircle. (Note that the method of calculation of an integral by replacing a logarithm with an "infinitesimal power" can be found in [90].) Representing the integrand from (3.35) as a sum of elementary fractions, we find that

$$\Xi(\alpha, w) = I_0(\alpha, w) + I_1(\alpha, w) + I_2(\alpha, w), \tag{3.36}$$

$$I_0 := \int\limits_{\mathbb{T}^+} u^{\alpha-1} du, \qquad I_1 := u_1 \int\limits_{\mathbb{T}^+} \frac{u^{\alpha-1}\, du}{u - u_1}, \qquad I_2 := u_2 \int\limits_{\mathbb{T}^+} \frac{u^{\alpha-1}\, du}{u - u_2},$$

where $u_1 := w - \sqrt{w^2 - 1}$ and $u_2 := w + \sqrt{w^2 - 1}$. Calculating I_j, we find

$$\begin{aligned} I_0 = \frac{1}{\alpha}\,(1 - e^{i\pi\alpha}), \qquad I_1 = \frac{u_1}{\alpha - 1}\Big[e^{i\pi\alpha}\, F(1, 1-\alpha; 2-\alpha; -u_1) + F(1, 1-\alpha; 2-\alpha; u_1)\Big], \\ I_2 = e^{i\pi\alpha}\, \alpha^{-1}\, F\big(1, \alpha; 1+\alpha; -1/u_2\big) + \alpha^{-1}\, F\big(1, \alpha; 1+\alpha; 1/u_2\big). \end{aligned} \tag{3.37}$$

Here $F(a, b; c; \zeta)$ is the hypergeometric Gauss function [9], defined by the series converging in the unit disk:

$$F\,(a,\, b;\, c;\, \zeta) \,=\, \sum_{n=0}^{\infty} \frac{(a)_n\, (b)_n}{(c)_n\, n!}\, \zeta^n, \qquad |\zeta| < 1, \tag{3.38}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol and $\Gamma(s)$ is the gamma function. To derive formulas for I_1 and I_2, we use the Euler representation for the function $F(a, b; c; \zeta)$, and for the derivation of I_1 we additionally use the formula of analytical continuation to the point $\zeta = \infty$.

Differentiating equalities (3.37) with respect to α, taking into account (3.38), and passing to the limit as $\alpha \to 0$, we get

$$\begin{aligned} \lim_{\alpha\to 0} \frac{d}{d\alpha}\, I_0 \,=\, \frac{\pi^2}{2}, \qquad \lim_{\alpha\to 0} \frac{d}{d\alpha}\, I_1 \,=\, i\,\pi \sum_{n=1}^{\infty} \frac{(-u_1)^n}{n} \,-\, 2 \sum_{n=1}^{\infty} \frac{u_1^{2n-1}}{(2n-1)^2}, \\ \lim_{\alpha\to 0} \frac{d}{d\alpha}\, I_2 \,=\, -\frac{\pi^2}{2} \,+\, i\,\pi \sum_{n=1}^{\infty} \frac{(-1/u_2)^n}{n} \,+\, 2 \sum_{n=1}^{\infty} \frac{(1/u_2)^{2n-1}}{(2n-1)^2}. \end{aligned} \tag{3.39}$$

Summing up the right-hand sides of Eqs. (3.39), dividing the sum by $(-\pi^2)$, and substituting $u_1 = 1/u_2$, we find an expression for $\Xi(w)$:

$$\Xi(w) = -\frac{2i}{\pi} \sum_{n=1}^{\infty} \frac{(-1/u_2)^n}{n} \,=\, \frac{2i}{\pi} \log \frac{1 + u_2}{u_2}. \tag{3.40}$$

Substituting (3.40) into (3.33) and returning to the plain z by the Zhukovsky function $w = \mathfrak{H}(z)$, we obtain the required solution $v_0(x, y) = V_0 \circ \mathfrak{H}(z)$ of problem (3.20)–(3.22) in form (3.24).

3.6. Application of the multipole method. Representing functions $u(x, y)$ and $v(x, y)$ in the form

$$u(x, y) = u_0(x, y) \,+\, U(x, y), \qquad v(x, y) = v_0(x, y) \,+\, V(x, y), \tag{3.41}$$

due to (3.1), (3.2), and (3.17)–(3.22), we get the following formulations of problems for functions U and V:

$$\Delta\, U = 0, \quad z \in \mathcal{S}; \qquad U = 0, \quad z \in \gamma; \qquad U = 1 - u_0, \quad z \in \Gamma; \tag{3.42}$$

$$\Delta\, V = 0, \quad z \in \mathcal{S}; \qquad V = 0, \quad z \in \gamma; \qquad V = \mathcal{V} - v_0, \quad z \in \Gamma, \tag{3.43}$$

where the arcs γ and Γ are defined by relations (3.14), function $\mathcal{V}$ is given by Eq. (3.3), and functions u_0 and v_0 are given by formulas (3.23) and (3.24).

We construct the solutions of problems (3.42) and (3.43) by the multipole method described in Sec. 2, which uses in this case a system of approximative functions $\Omega_k(x, y)$, defined by the formula

$$\Omega_k(x, y) := \operatorname{Im}\big(z^{\,k} + z^{\,-k}\big), \qquad k \in \mathbb{N}. \tag{3.44}$$

Using results from Sec. 2, we see that the system of these functions is complete and minimal in the space $e_2(\mathcal{S},\Gamma)$. Then the solutions U and V of boundary-value problems (3.42) and (3.43) have the form

$$U(x,y) = \lim_{N\to\infty} U^N(x,y), \qquad U^N(x,y) := \sum\nolimits_{k=1}^{N} a_k^N \, \Omega_k(x,y), \tag{3.45}$$

$$V(x,y) = \lim_{N\to\infty} V^N(x,y), \qquad V^N(x,y) := \sum\nolimits_{k=1}^{N} b_k^N \, \Omega_k(x,y), \tag{3.46}$$

where coefficients a_k^N and b_k^N can be found from the system of linear equations

$$\sum\nolimits_{k=1}^{N} (\Omega_n,\, \Omega_k)\, a_k^N = (\Omega_n, 1-u_0), \qquad \sum\nolimits_{k=1}^{N} (\Omega_n,\, \Omega_k)\, b_k^N = (\Omega_n, \mathcal{V}-v_0), \quad n=\overline{1,N}. \tag{3.47}$$

The sequences $\{U^N\}$ and $\{V^N\}$ are infinitely differentiable on the set $\mathcal{S}\cup\widetilde{\gamma}$, where $\widetilde{\gamma}$ is the arc γ without the points $z=-2$, $z=-1$, $z=1$, and $z=2$. Then, setting

$$u^N := u_0 + U^N, \qquad v^N := v_0 + V^N, \tag{3.48}$$

we get the relations

$$\max_{\mathcal{E}} \big|\, D^{(l,t)}\,(u^N - u)\,\big| \to 0, \qquad \max_{\mathcal{E}} \big|\, D^{(l,t)}\,(v^N - v)\,\big| \to 0, \qquad N\to\infty$$

for any compact set $\mathcal{E} \subset \mathcal{S}\cup\widetilde{\gamma}$.

Using the error estimates for u and v in the $C(\overline{\mathcal{S}})$ norm implied by the general formula (2.14)

$$\max_{\overline{\mathcal{S}}} |u-u^N| \le \max_{\Gamma} |1-u^N|, \qquad \max_{\overline{\mathcal{S}}} |v-v^N| \le \max_{\Gamma} |\mathcal{V}-v^N|, \tag{3.49}$$

we obtain an a posteriori error estimate for the mapping function in the same norm:

$$\max_{\overline{\mathcal{S}}} |F-F^N| \le \max_{\Gamma} |1-u^N| + \max_{\Gamma} |\mathcal{V}-v^N|, \tag{3.50}$$

where the right-hand side of the inequality can be easily found from the numerical solution of the problem.

4. Analysis of the Effect of Irremovable Error

4.1. Difference solutions of the Roache–Steinberg problem. Consider papers [6, 44, 49, 71] devoted to solution of the Roache–Steinberg problem (see Sec. 1.7) on calculation of the mapping $F(z)$ and generation of the harmonic grid $\mathfrak{S}_h$ in the domain $\mathcal{S}$ by difference methods.

Numerical experiments carried out by the authors of [49, 71] showed that for sufficiently small ratio A of half-axes of the elliptic arc l_2, the obtained grid $\mathfrak{S}_h$ was quite satisfactory. An example of such a grid constructed for $A=2$ can be seen in Fig. 8. The step of the preimage grid $\mathfrak{Q}_h$ was $h=1/32$; thus, the grid $\mathfrak{S}_h$ in the domain $\mathcal{S}$ consisted of intersections of 33 coordinate lines in one direction and 33 coordinate lines in the other one. But for values $A>3.8$, not only was the quality of the grid worse, but it also had overlaps and was partially situated outside of the domain $\mathcal{S}$, i.e., the homeomorphism property of the numerically obtained discretized mapping F_h^{-1} was lost. These phenomena are demonstrated on the example of the grid $\mathfrak{S}_h$ calculated for $A=5$ and depicted in Fig. 9.

In [6], Azarenok reproduced these results and, moreover, studied the problem numerically in more detail. His detailed depictions of the grid $\mathfrak{S}_h$ in the points of overlapping and in those situated outside of the calculation domain in gradually refined scale are consecutively given on Fig. 10, 11, 12. These numerical results are obtained by using the second-order difference scheme [32] on a square grid $\mathfrak{Q}_h$ for $h=1/20$; thus, the grid $\mathfrak{S}_h$ corresponded to partition 21×21.

Further, it was shown in [6] that for $A=5$, the 5-times refinement of the grid $\mathfrak{Q}_h$ has no substantial influence on the quality of the grid $\mathfrak{S}_h$ and preserves overlaps. Only for the 501×501 grid do the overlaps and points outside the domain $\mathcal{S}$ disappear, i.e., for such refinement of the grid $\mathfrak{Q}_h$, the

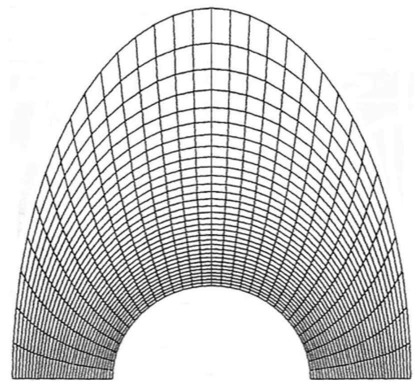

Fig. 8

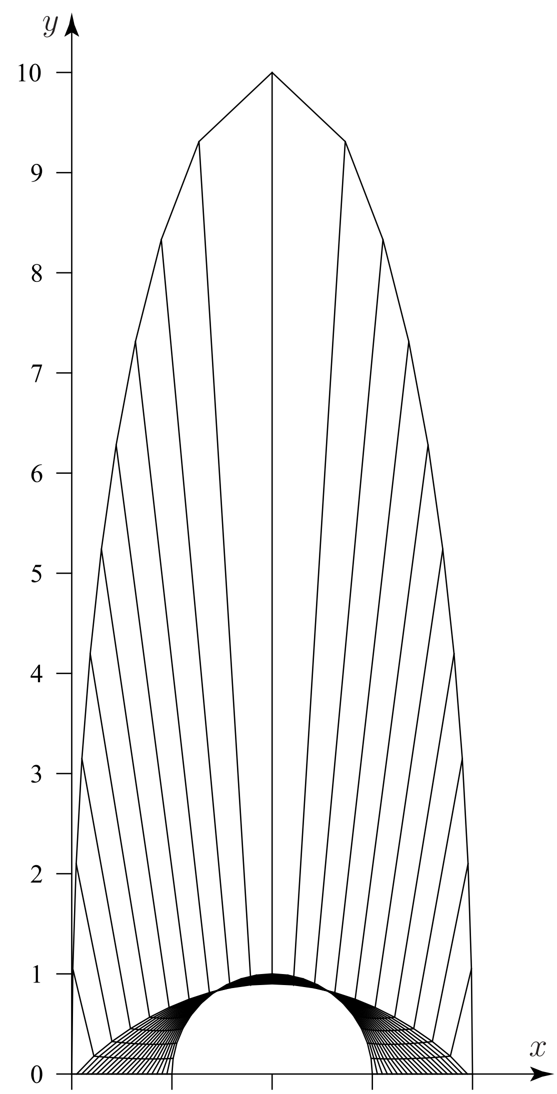

Fig. 9

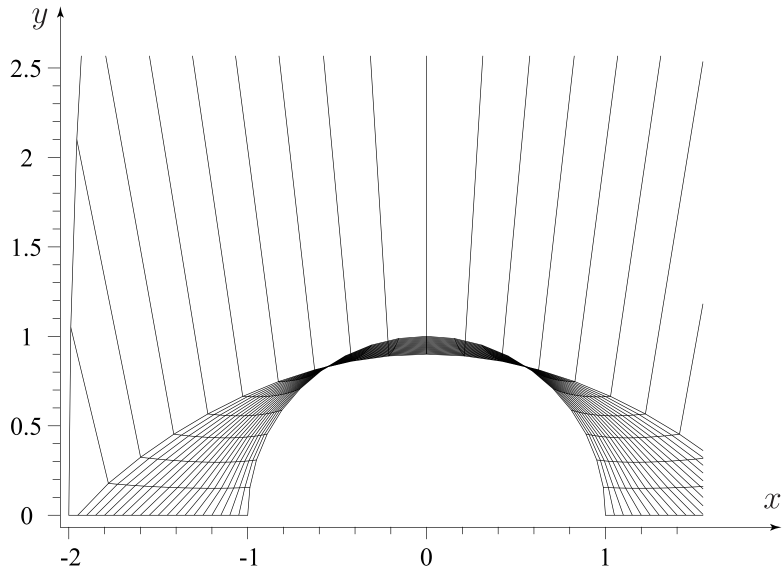

Fig. 10

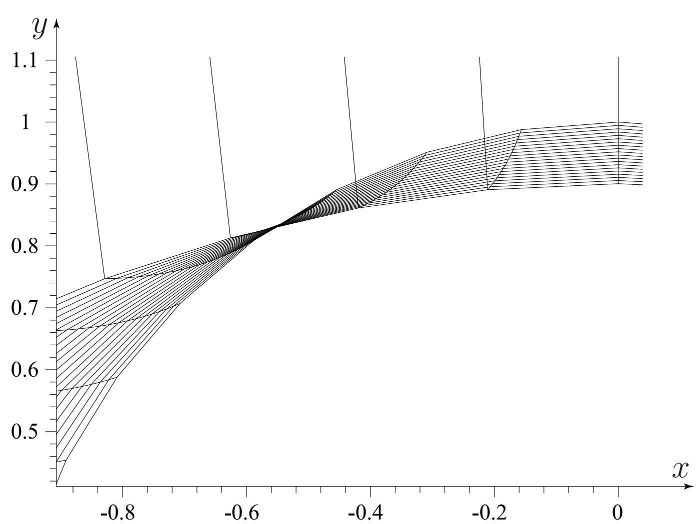

Fig. 11

mapping F_h^{-1} is a homeomorphism $\mathcal{Q} \overset{Hom}{\longrightarrow} \mathcal{S}$, but the quality of the obtained grid $\mathfrak{S}_h$ is not satisfactory; the calculation time on a 2.4 GHz PC in this case was 2 hours.

Finally, in paper [6], it was shown that neither further refinement of the grid $\mathfrak{Q}_h$ nor increase in accuracy of the difference scheme (up to the fourth order with a 25-point stencil and more), nor minimization of the functional (1.12) using method [43] (in this case the calculation time on the same computer was 16 hours) instead of solving differential equations (1.10) result in substantial change of the knots of the grid, namely, their coordinates remained unchanged with relative accuracy 10^{-6}.

Thus all the mentioned ways of calculation, starting from 501×501 partition, give practically one and the same "unimprovable" grid. Denoting by $F_{h,\,ext}\,(z)$ the discretized harmonic mapping

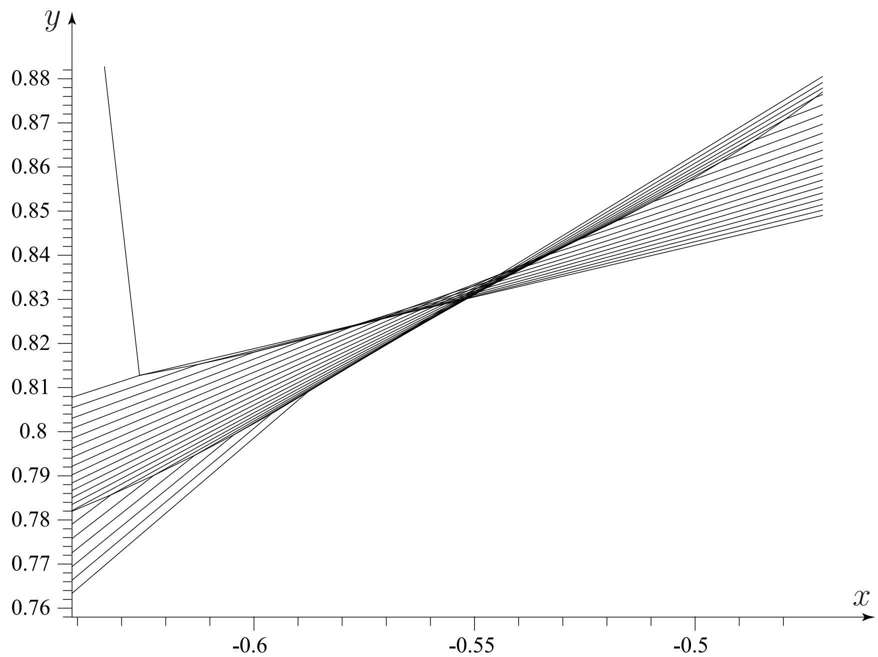

Fig. 12

corresponding to this grid and by $F(z)$ the exact mapping, we see that the relative error

$$\delta(z) := \frac{\max\limits_{\overline{\mathcal{S}}} \left|F(z) - F_{h,\,ext}(z)\right|}{\max\limits_{\overline{\mathcal{S}}} \left|F(z)\right|} \tag{4.1}$$

is "unimprovable" in the sense explained above.

4.2. Comparison with a high-precision solution. In order to study the error $\delta(z)$, we implemented the numerical solution of the Roache–Steinberg problem obtained in Sec. 3. The number of terms N in representations (3.45) and (3.46) was taken equal to 50. In this case, the relative error ε of the obtained solution $F^N(z)$ defined by the formula

$$\varepsilon := \frac{\max\limits_{\overline{\mathcal{S}}} \left|F(z) - F^N(z)\right|}{\max\limits_{\overline{\mathcal{S}}} \left|F(z)\right|}$$

turned out to be less than 10^{-9}, and the calculation time on a 2.4 GHz PC did not exceed 1 sec. Thus, for the purposes of this work, the obtained mapping can be considered exact, i.e., coinciding with $F(z)$.

Using this mapping and the mapping $F_{h,\,ext}(z)$, based on the numerical results from [6], we find the distribution of the error $\delta(z)$ over the domain $\mathcal{S}$. The graph of this quantity as a surface over the domain $\mathcal{S}$ is shown in Fig. 14, and the projection of this surface on the plane (y, δ) is represented in Fig. 15. From the last figure we see that the maximum of the relative error $\delta(z)$ is about 60% and over the greater part of the domain $\mathcal{S}$ it is 50%. Such a high unimprovable error shows substantial calculation difficulties in the use of Winslow's method combined with typical difference schemes used in papers [6, 44, 49, 71].

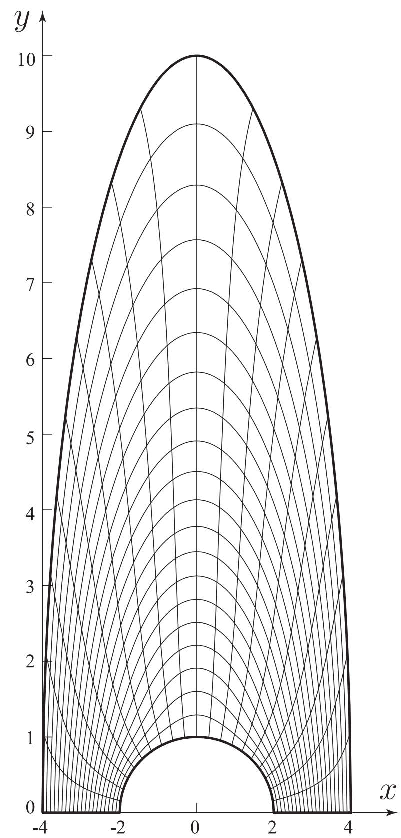

Fig. 13

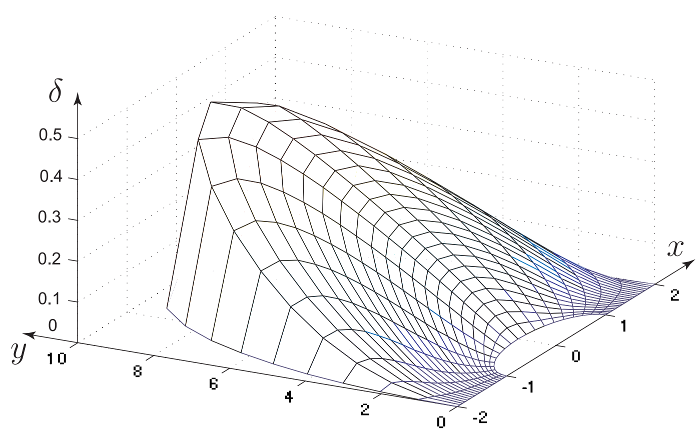

Fig. 14

Thus, construction of high-precision harmonic mappings of complex domains, despite substantial progress in this field, remains a very actual problem. As one of the ways to overcome these difficulties, the multipole method can be suggested.

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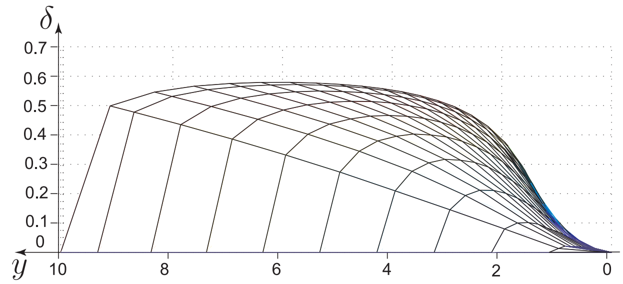

Fig. 15

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S. I. Bezrodnykh

Dorodnitsyn Computing Center of the Russian Academy of Sciences, 40 Vavilova str., Moscow, Russia

Sternberg Astronomical Institute of the Lomonosov Moscow State University, 13 Universitetsky prospect, Moscow, Russia

E-mail: `sergeyib@pochta.ru`

V. I. Vlasov

Dorodnitsyn Computing Center of the Russian Academy of Sciences, 40 Vavilova str., Moscow, Russia

E-mail: `vlasov@ccas.ru`