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OSCILLATION OF FOURTH-ORDER DELAY DIFFERENTIAL EQUATIONS

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This article deals with the oscillation of a certain class of fourth-order delay differential equations. Some new oscillation criteria (including Hille- and Nehari-type criteria) are presented. The results obtained in the paper improve some results from [C. Zhang, T. Li, B. Sun, and E. Thandapani, *Appl. Math. Lett.,* **24**, 1618 (2011)]. Two examples are presented to illustrate our main results.

1. Introduction

In the present paper, we consider the oscillation of a fourth-order quasilinear delay differential equation

$$\left(r(t)\left(x'''(t)\right)^{\alpha}\right)' + q(t)x^{\alpha}(\tau(t)) = 0, \quad \text{for} \quad t \geq t_0. \tag{1.1}$$

We assume that the following assumptions hold:

(H_1) α is a quotient of odd positive integers;

(H_2) $r \in \mathrm{C}^1[t_0,\infty)$, $r'(t) \geq 0$, $r(t) > 0$, q, $\tau \in \mathrm{C}[t_0,\infty)$, $q(t) \geq 0$, $\tau(t) \leq t$, and $\lim_{t\to\infty}\tau(t) = \infty$.

We say that a function $x \in \mathrm{C}^3[T_x,\infty)$, $T_x \geq t_0$, is a solution of Eq. (1.1) if it has the property $r(x''')^{\alpha} \in \mathrm{C}^1[T_x,\infty)$ and satisfies Eq. (1.1) on $[T_x,\infty)$. We restrict ourselves to the analysis of solutions x of Eq. (1.1) with the following property: $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. It is assumed that Eq. (1.1) possesses solutions of this kind. A solution of Eq. (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$; otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

In recent decades, the oscillation of second- and third-order differential equations have been deeply studied in the literature. We refer the reader to the related books [1, 3–5, 13, 15, 21] and to the papers [2, 6–12, 14, 16–20, 22]. In what follows, we present some related results that served as a motivation for the contents of this paper. Agarwal, et al. [2], Kamo and Usami [11, 12], and Kusano, et al. [14] studied the oscillation of the fourth-order nonlinear differential equation

$$\left(r(t)\left(x''(t)\right)^{\alpha}\right)'' + q(t)x^{\beta}(t) = 0.$$

Grace, et al. [10] examined the oscillation behavior of the fourth-order nonlinear differential equation

$$\left(r(t)\left(x'(t)\right)^{\alpha}\right)''' + q(t)f(x(g(t))) = 0.$$

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Agarwal, et al. [7] and Zhang, et al. [22] studied the oscillatory properties of the higher-order differential equation

$$\left(r(t)\left(x^{(n-1)}(t)\right)^{\alpha}\right)' + q(t)x^{\beta}(\tau(t)) = 0, \tag{1.2}$$

under the conditions

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)}\, dt = \infty,$$

and

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)}\, dt < \infty. \tag{1.3}$$

Zhang, et al. [22] obtained some results which ensure that every solution x of (1.2) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$ in the case where (1.3) holds. For a special case $n = 4,$ they proved the following result: Let $(H_1),$ $(H_2),$ and (1.3) hold and let $\tau(t) < t.$ Further, assume that, for some constant $\lambda_0 \in (0, 1),$ the delay differential equation

$$y'(t) + q(t)\left(\frac{\lambda_0 \tau^3(t)}{6r^{1/\alpha}(\tau(t))}\right)^{\alpha} y(\tau(t)) = 0 \tag{1.4}$$

is oscillatory. If

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[q(s)\left(\frac{\lambda_1}{2}\tau^2(s)\right)^{\alpha} \delta^{\alpha}(s) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{1}{\delta(s) r^{1/\alpha}(s)}\right] ds = \infty \tag{1.5}$$

for some constant $\lambda_1 \in (0, 1),$ where

$$\delta(t) := \int_t^{\infty} r^{-(1/\alpha)}(s)\, ds,$$

then every solution of (1.1) is either oscillatory or converges to zero as $t \to \infty$.

Our aim in the present paper is to employ the Riccatti technique to establish some new conditions for the oscillation of all solutions of (1.1). The results not only differ from the results obtained in [22] but also improve some of them. Some examples are considered to illustrate the main results.

2. Main results

In this section, we derive some new criteria for the oscillation of (1.1). To prove our main results, we need the following lemma:

Lemma 2.1 (see [3], Lemma 2.2.3)**.** *Let* $f \in C^n([t_0,\infty),\mathbb{R}^+)$*. Assume that* $f^{(n)}(t)$ *is of fixed sign and not identically zero on* $[t_0,\infty)$ *and that there exists* $t_1 \ge t_0$ *such that* $f^{(n-1)}(t)f^{(n)}(t) \le 0$ *for all* $t \ge t_1$*. If* $\lim_{t\to\infty} f(t) \ne 0$*, then, for every* $k \in (0,1)$*, there exists* $t_k \in [t_1,\infty)$ *such that*

$$f(t) \ge \frac{k}{(n-1)!}t^{n-1}|f^{(n-1)}(t)|, \quad \text{for} \quad t \in [t_k,\infty).$$

We are now ready to state and prove the main results. For convenience, we denote

$$R(t) := \int_t^\infty \frac{1}{r^{\frac{1}{\alpha}}(s)}\,ds, \quad \rho'_+(t) := \max\{0,\rho'(t)\}, \quad \text{and} \quad \theta'_+(t) := \max\{0,\theta'(t)\}.$$

In the sequel, all functional inequalities encountered in the present section are assumed to be eventually true. This means that they are satisfied for all sufficiently large t.

Theorem 2.1. *Let* (H_1)*,* (H_2)*, and (1.3) hold. Assume that there exists a positive function* $\rho \in C^1[t_0,\infty)$ *such that*

$$\int_{t_0}^\infty \left[q(s)\left(\frac{\tau^2(s)}{s^2}\right)^\alpha \rho(s) - \frac{2^\alpha}{(\alpha+1)^{\alpha+1}}\frac{r(s)(\rho'_+(s))^{\alpha+1}}{(k_1\rho(s)s^2)^\alpha}\right]ds = \infty, \tag{2.1}$$

for some constant $k_1 \in (0,1)$*. Assume further that there exists a positive function* $\theta \in C^1[t_0,\infty)$ *such that*

$$\int_{t_0}^\infty \left[\theta(s)\int_s^\infty\left[\frac{1}{r(\vartheta)}\int_\vartheta^\infty q(\varsigma)\left(\frac{\tau^2(\varsigma)}{\varsigma^2}\right)^\alpha d\varsigma\right]^{\frac{1}{\alpha}} d\vartheta - \frac{(\theta'_+(s))^2}{4\theta(s)}\right]ds = \infty. \tag{2.2}$$

If

$$\int_{t_0}^\infty \left[q(s)\left(\int_s^\infty\int_u^\infty R(v)dvdu\right)^\alpha - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{\int_s^\infty R(v)dv}{\int_s^\infty\int_u^\infty R(v)dvdu}\right]ds = \infty \tag{2.3}$$

and

$$\int_{t_0}^\infty \left[q(s)\left(\frac{k_2}{2}\tau^2(s)\right)^\alpha R^\alpha(s) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}R(s)r^{1/\alpha}(s)}\right]ds = \infty \tag{2.4}$$

for some constant $k_2 \in (0,1)$*, then every solution of (1.1) is oscillatory.*

Proof. Assume that (1.1) has a nonoscillatory solution x. Without loss of generality, we can assume that x is eventually positive. It follows from (1.1) that there exist four possible cases for $t \ge t_1$, where $t_1 \ge t_0$ is sufficiently large:

$$Case\ 1:\ x(t) > 0, x'(t) > 0, x''(t) > 0, x'''(t) > 0, x^{(4)}(t) \le 0, (r(x''')^\alpha)'(t) \le 0.$$

$$Case\ 2:\ x(t) > 0, x^{'}(t) > 0, x^{''}(t) < 0, x^{'''}(t) > 0, x^{(4)}(t) \le 0, (r(x^{'''})^{\alpha})^{'}(t) \le 0.$$

$$Case\ 3:\ x(t) > 0, x^{'}(t) < 0, x^{''}(t) > 0, x^{'''}(t) < 0, (r(x^{'''})^{\alpha})^{'}(t) \le 0.$$

$$Case\ 4:\ x(t) > 0, x^{'}(t) > 0, x^{''}(t) > 0, x^{'''}(t) < 0, (r(x^{'''})^{\alpha})^{'}(t) \le 0.$$

Assume that we have Case 1. By the Kiguradze Lemma [13], we find $x(t) \ge (t/2)x^{'}(t)$ and, hence,

$$\frac{x(\tau(t))}{x(t)} \ge \frac{\tau^2(t)}{t^2}. \tag{2.5}$$

Define

$$\omega(t) := \rho(t)\,\frac{r(t)(x^{'''})^{\alpha}(t)}{x^{\alpha}(t)}, \quad t \ge t_1. \tag{2.6}$$

Then $\omega(t) > 0$ for $t \ge t_1$ and

$$\omega^{'}(t) = \rho^{'}(t)\,\frac{r(t)(x^{'''})^{\alpha}(t)}{x^{\alpha}(t)} + \rho(t)\frac{(r(x^{'''})^{\alpha})^{'}(t)}{x^{\alpha}(t)} - \alpha\rho(t)\,\frac{x^{\alpha-1}(t)x^{'}(t)r(t)(x^{'''})^{\alpha}(t)}{x^{2\alpha}(t)}. \tag{2.7}$$

It follows from Lemma 2.1 that

$$x^{'}(t) \ge \frac{k}{2}t^2 x^{'''}(t) \tag{2.8}$$

for every $k \in (0,1)$ and all sufficiently large t. Hence, by (2.7) and (2.8), we obtain

$$\omega^{'}(t) \le \rho^{'}(t)\,\frac{r(t)(x^{'''})^{\alpha}(t)}{x^{\alpha}(t)} + \rho(t)\,\frac{(r(x^{'''})^{\alpha})^{'}(t)}{x^{\alpha}(t)} - \frac{\alpha k}{2}\,t^2\rho(t)\,\frac{x^{'''}(t)r(t)(x^{'''})^{\alpha}(t)}{x^{\alpha+1}(t)}.$$

Therefore, in view of (1.1), we get

$$\omega^{'}(t) \le -q(t)\left(\frac{\tau^2(t)}{t^2}\right)^{\alpha}\rho(t) + \frac{\rho_{+}^{'}(t)}{\rho(t)}\,\omega(t) - \frac{\alpha k}{2}\,\frac{t^2}{(r(t)\rho(t))^{\frac{1}{\alpha}}}\omega^{\frac{\alpha+1}{\alpha}}(t). \tag{2.9}$$

We set

$$A := \frac{\alpha k t^2}{2(r(t)\rho(t))^{\frac{1}{\alpha}}}, \quad B := \frac{\rho_{+}^{'}(t)}{\rho(t)}, \quad y := \omega(t).$$

Using the inequality

$$By - Ay^{\frac{\alpha+1}{\alpha}} \le \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\,\frac{B^{\alpha+1}}{A^{\alpha}}, \quad A, B > 0,$$

we find

$$\frac{\rho_+'(t)}{\rho(t)}\omega(t) - \frac{\alpha k t^2}{2(r(t)\rho(t))^{\frac{1}{\alpha}}}\omega^{\frac{\alpha+1}{\alpha}}(t) \le \frac{2^\alpha}{(\alpha+1)^{\alpha+1}} \frac{r(t)(\rho_+'(t))^{\alpha+1}}{(k\rho(t)t^2)^\alpha}.$$

Hence, we obtain

$$\omega'(t) \le -q(t)\left(\frac{\tau^2(t)}{t^2}\right)^\alpha \rho(t) + \frac{2^\alpha}{(\alpha+1)^{\alpha+1}} \frac{r(t)(\rho_+'(t))^{\alpha+1}}{(k\rho(t)t^2)^\alpha},$$

which implies that

$$\int_{t_1}^{t}\left[q(s)\left(\frac{\tau^2(s)}{s^2}\right)^\alpha \rho(s) - \frac{2^\alpha}{(\alpha+1)^{\alpha+1}} \frac{r(s)(\rho_+'(s))^{\alpha+1}}{(k\rho(s)s^2)^\alpha}\right] ds \le \omega(t_1),$$

for every $k \in (0,1)$ and all sufficiently large t but this contradicts (2.1).

Assume that we have Case 2. Integrating (1.1) from t to l, we find

$$r(l)(x''')^\alpha(l) - r(t)(x''')^\alpha(t) + \int_t^l q(s)x^\alpha(\tau(s))\, ds = 0.$$

By virtue of $x > 0$, $x' > 0$, and $x'' < 0$, we get $x(t) \ge (t/2)x'(t)$. Therefore, (2.5) holds. Thus, by (2.5), we obtain

$$r(l)(x''')^\alpha(l) - r(t)(x''')^\alpha(t) + \int_t^l q(s)\left(\frac{\tau^2(s)}{s^2}\right)^\alpha x^\alpha(s)\, ds \le 0,$$

whence it follows, by $x' > 0$, that

$$r(l)(x''')^\alpha(l) - r(t)(x''')^\alpha(t) + x^\alpha(t)\int_t^l q(s)\left(\frac{\tau^2(s)}{s^2}\right)^\alpha ds \le 0.$$

Letting $l \to \infty$, we arrive at the inequality

$$-r(t)(x''')^\alpha(t) + x^\alpha(t)\int_t^\infty q(s)\left(\frac{\tau^2(s)}{s^2}\right)^\alpha ds \le 0,$$

i.e.,

$$-x'''(t) + x(t)\left[\frac{1}{r(t)}\int_t^\infty q(s)\left(\frac{\tau^2(s)}{s^2}\right)^\alpha ds\right]^{\frac{1}{\alpha}} \le 0.$$

Integrating again from t to ∞, we get

$$x''(t) + x(t) \int_t^\infty \left[\frac{1}{r(\vartheta)} \int_\vartheta^\infty q(s) \left(\frac{\tau^2(s)}{s^2} \right)^\alpha ds \right]^{\frac{1}{\alpha}} d\vartheta \leq 0. \tag{2.10}$$

Define

$$\xi(t) := \theta(t) \frac{x'(t)}{x(t)}, \quad t \geq t_1.$$

Then $\xi(t) > 0$ for $t \geq t_1$ and

$$\xi'(t) = \theta'(t) \frac{x'(t)}{x(t)} + \theta(t) \frac{x''(t)x(t) - (x')^2(t)}{x^2(t)}$$

$$= \theta(t) \frac{x''(t)}{x(t)} + \frac{\theta'(t)}{\theta(t)} \xi(t) - \frac{\xi^2(t)}{\theta(t)}.$$

Hence, by (2.10), we find

$$\xi'(t) \leq -\theta(t) \int_t^\infty \left[\frac{1}{r(\vartheta)} \int_\vartheta^\infty q(s) \left(\frac{\tau^2(s)}{s^2} \right)^\alpha ds \right]^{\frac{1}{\alpha}} d\vartheta + \frac{\theta'_+(t)}{\theta(t)} \xi(t) - \frac{\xi^2(t)}{\theta(t)}. \tag{2.11}$$

Thus, we have

$$\xi'(t) \leq -\theta(t) \int_t^\infty \left[\frac{1}{r(\vartheta)} \int_\vartheta^\infty q(s) \left(\frac{\tau^2(s)}{s^2} \right)^\alpha ds \right]^{\frac{1}{\alpha}} d\vartheta + \frac{(\theta'_+(t))^2}{4\theta(t)}.$$

This yields

$$\int_{t_1}^t \left[\theta(s) \int_s^\infty \left[\frac{1}{r(\vartheta)} \int_\vartheta^\infty q(\varsigma) \left(\frac{\tau^2(\varsigma)}{\varsigma^2} \right)^\alpha d\varsigma \right]^{\frac{1}{\alpha}} d\vartheta - \frac{(\theta'_+(s))^2}{4\theta(s)} \right] ds \leq \xi(t_1),$$

which contradicts (2.2).

Assume that we have Case 3. Recalling that $r(x''')^\alpha$ is nonincreasing, we obtain

$$r^{1/\alpha}(s) x'''(s) \leq r^{1/\alpha}(t) x'''(t), \quad s \geq t \geq t_1.$$

Dividing this inequality by $r^{1/\alpha}(s)$ and integrating the inequality obtained as a result from t to l, we get

$$x''(l) \leq x''(t) + r^{1/\alpha}(t) x'''(t) \int_t^l r^{-1/\alpha}(s)\, ds.$$

Letting $l \to \infty,$ we arrive at the inequality

$$x''(t) \geq -r^{1/\alpha}(t)x'''(t)R(t). \tag{2.12}$$

Integrating (2.12) from t to $\infty,$ we obtain

$$-x'(t) \geq \int_t^\infty -r^{1/\alpha}(s)x'''(s)R(s)\,ds \geq -r^{1/\alpha}(t)x'''(t)\int_t^\infty R(s)\,ds. \tag{2.13}$$

Integrating (2.13) from t to $\infty,$ we get

$$x(t) \geq \int_t^\infty -r^{1/\alpha}(u)x'''(u)\int_u^\infty R(s)\,ds\,du \geq -r^{1/\alpha}(t)x'''(t)\int_t^\infty\int_u^\infty R(s)\,ds\,du. \tag{2.14}$$

We now define

$$\varphi(t) := \frac{r(t)(x''')^\alpha(t)}{x^\alpha(t)}, \quad t \geq t_1. \tag{2.15}$$

Thus, $\varphi(t) < 0$ for $t \geq t_1,$ and, by (2.13), we conclude that

$$\begin{aligned}\varphi'(t) &= \frac{(r(x''')^\alpha)'(t)}{x^\alpha(t)} - \alpha\frac{r(t)(x''')^\alpha(t)x'(t)}{x^{\alpha+1}(t)}\\ &\leq -q(t)\frac{x^\alpha(\tau(t))}{x^\alpha(t)} - \alpha\frac{r^{\frac{\alpha+1}{\alpha}}(t)(x''')^{\alpha+1}(t)}{x^{\alpha+1}(t)}\int_t^\infty R(s)\,ds.\end{aligned} \tag{2.16}$$

Hence, by (2.15) and (2.16), we obtain

$$\varphi'(t) \leq -q(t) - \alpha\varphi^{\frac{\alpha+1}{\alpha}}(t)\int_t^\infty R(s)\,ds. \tag{2.17}$$

From (2.14), we get

$$\varphi(t)\left(\int_t^\infty\int_u^\infty R(s)\,ds\,du\right)^\alpha \geq -1. \tag{2.18}$$

Multiplying (2.17) by $\left(\int_t^\infty\int_u^\infty R(s)\,ds\,du\right)^\alpha$ and integrating the resulting inequality from t_1 to $t,$ we find

$$\left(\int_t^\infty\int_u^\infty R(s)\,ds\,du\right)^\alpha\varphi(t) - \left(\int_{t_1}^\infty\int_u^\infty R(s)\,ds\,du\right)^\alpha\varphi(t_1)$$

$$+\alpha\int_{t_1}^{t}\int_{s}^{\infty}R(v)\,dv\left(\int_{s}^{\infty}\int_{u}^{\infty}R(v)\,dv\,du\right)^{\alpha-1}\varphi(s)\,ds$$

$$+\int_{t_1}^{t}q(s)\left(\int_{s}^{\infty}\int_{u}^{\infty}R(v)\,dv\,du\right)^{\alpha}ds$$

$$+\alpha\int_{t_1}^{t}\varphi^{\frac{\alpha+1}{\alpha}}(s)\left(\int_{s}^{\infty}\int_{u}^{\infty}R(v)\,dv\,du\right)^{\alpha}\int_{s}^{\infty}R(v)\,dv\,ds\le 0.$$

We set

$$B:=\int_{s}^{\infty}R(v)\,dv\left(\int_{s}^{\infty}\int_{u}^{\infty}R(v)\,dv\,du\right)^{\alpha-1},$$

and

$$A:=\left(\int_{s}^{\infty}\int_{u}^{\infty}R(v)\,dv\,du\right)^{\alpha}\int_{s}^{\infty}R(v)\,dv,\quad y:=-\varphi(s).$$

Using the inequality

$$-By+Ay^{\frac{\alpha+1}{\alpha}}\ge-\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}},\quad A,B>0,\tag{2.19}$$

we conclude that

$$\int_{s}^{\infty}R(v)\,dv\left(\int_{s}^{\infty}\int_{u}^{\infty}R(v)\,dv\,du\right)^{\alpha-1}\varphi(s)+\varphi^{\frac{\alpha+1}{\alpha}}(s)\left(\int_{s}^{\infty}\int_{u}^{\infty}R(v)\,dv\,du\right)^{\alpha}\int_{s}^{\infty}R(v)\,dv$$
$$\ge-\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{\int_s^{\infty}R(v)\,dv}{\int_s^{\infty}\int_u^{\infty}R(v)\,dv\,du}.$$

Hence, by (2.18), we obtain

$$\int_{t_1}^{t}\left[q(s)\left(\int_{s}^{\infty}\int_{u}^{\infty}R(v)\,dv\,du\right)^{\alpha}-\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{\int_s^{\infty}R(v)\,dv}{\int_s^{\infty}\int_u^{\infty}R(v)\,dv\,du}\right]ds$$
$$\le\left(\int_{t_1}^{\infty}\int_{u}^{\infty}R(s)\,ds\,du\right)^{\alpha}\varphi(t_1)+1$$

but this contradicts (2.3).

Assume that we have Case 4. In view of the proof of Case 3, we have (2.12). On the other hand, by Lemma 2.1, we get

$$x(t) \geq \frac{k}{2} t^2 x''(t) \tag{2.20}$$

for every $k \in (0,1)$ and all sufficiently large t. We now define

$$\phi(t) := \frac{r(t)(x''')^{\alpha}(t)}{(x'')^{\alpha}(t)}, \quad t \geq t_1. \tag{2.21}$$

Then $\phi(t) < 0$ for $t \geq t_1$ and, by virtue of (2.20) and (2.21), we conclude that

$$\phi'(t) = -q(t)\frac{x^{\alpha}(\tau(t))}{(x''(\tau(t)))^{\alpha}} \frac{(x''(\tau(t)))^{\alpha}}{(x'')^{\alpha}(t)} - \alpha \frac{\phi^{\frac{\alpha+1}{\alpha}}(t)}{r^{\frac{1}{\alpha}}(t)} \leq -q(t)\left(\frac{k}{2}\tau^2(t)\right)^{\alpha} - \alpha \frac{\phi^{\frac{\alpha+1}{\alpha}}(t)}{r^{\frac{1}{\alpha}}(t)}. \tag{2.22}$$

Multiplying this inequality by $R^{\alpha}(t)$ and integrating the resulting inequality from t_1 to t, we find

$$R^{\alpha}(t)\phi(t) - R^{\alpha}(t_1)\phi(t_1) + \alpha \int_{t_1}^{t} r^{-1/\alpha}(s) R^{\alpha-1}(s)\phi(s)\,ds$$

$$\leq -\int_{t_1}^{t} q(s)\left(\frac{k}{2}\tau^2(s)\right)^{\alpha} R^{\alpha}(s)\,ds - \alpha \int_{t_1}^{t} \frac{\phi^{(\alpha+1)/\alpha}(s)}{r^{1/\alpha}(s)} R^{\alpha}(s)\,ds.$$

We set $B := r^{-1/\alpha}(s)R^{\alpha-1}(s)$, $A := R^{\alpha}(s)/r^{1/\alpha}(s)$, and $y := -\phi(s)$. In view of inequalities (2.19) and (2.12), for every $k \in (0,1)$ and all sufficiently large t, we obtain

$$\int_{t_1}^{t} \left[q(s)\left(\frac{k}{2}\tau^2(s)\right)^{\alpha} R^{\alpha}(s) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{1}{R(s) r^{1/\alpha}(s)} \right] ds \leq R^{\alpha}(t_1)\phi(t_1) + 1$$

but this contradicts (2.4).

Theorem 2.1 is proved.

It is well known (see [8]) that the differential equation

$$(a(t)(x'(t))^{\alpha})' + q(t)x^{\alpha}(t) = 0, \tag{2.23}$$

where $\alpha > 0$ is the ratio of odd positive integers, $a, q \in \mathrm{C}([t_0,\infty),\mathbb{R}^+)$, is nonoscillatory if and only if there exist a number $T \geq t_0$ and a function $v \in \mathrm{C}^1([T,\infty),\mathbb{R})$ satisfying the inequality

$$v'(t) + \alpha a^{-1/\alpha}(t)(v(t))^{(1+\alpha)/\alpha} + q(t) \leq 0, \quad \text{on} \quad [T,\infty).$$

In what follows, we compare the oscillatory behavior of Eq. (1.1) with the second-order half-linear equations of type (2.23). There are numerous results concerning the oscillation of Eq. (2.23) (see, e.g., [1, 3–5, 17, 18, 20, 21]), which include Hille and Nehari types, Philos type, etc.

Theorem 2.2. *Let* (H_1), (H_2), *and (1.3) hold. Assume that the equation*

$$\left(\frac{r(t)}{t^{2\alpha}}(x'(t))^{\alpha}\right)' + q(t)\left(\frac{k_1\tau^2(t)}{2t^2}\right)^{\alpha} x^{\alpha}(t) = 0 \tag{2.24}$$

is oscillatory for some constant $k_1 \in (0,1)$, *the equation*

$$x''(t) + x(t)\int_t^{\infty}\left[\frac{1}{r(\vartheta)}\int_{\vartheta}^{\infty} q(s)\left(\frac{\tau^2(s)}{s^2}\right)^{\alpha} ds\right]^{\frac{1}{\alpha}} d\vartheta = 0 \tag{2.25}$$

is oscillatory, the equation

$$\left(\left(\int_t^{\infty} R(s)\,ds\right)^{-\alpha}(x'(t))^{\alpha}\right)' + q(t)x^{\alpha}(t) = 0 \tag{2.26}$$

is oscillatory, and the equation

$$\left(r(t)(x'(t))^{\alpha}\right)' + q(t)\left(\frac{k_2}{2}\tau^2(t)\right)^{\alpha} x^{\alpha}(t) = 0 \tag{2.27}$$

is oscillatory for some constant $k_2 \in (0,1)$. *Then every solution of (1.1) is oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.1, we get (2.9), (2.11), (2.17), and (2.22). If we set $\rho(t) = 1$ in (2.9), then we get

$$\omega'(t) + \frac{\alpha k t^2}{2(r(t))^{\frac{1}{\alpha}}}\,\omega^{\frac{\alpha+1}{\alpha}}(t) + q(t)\left(\frac{\tau^2(t)}{t^2}\right)^{\alpha} \le 0$$

for every constant $k \in (0,1)$. Thus, we can see that equation (2.24) is nonoscillatory for every constant $k_1 \in (0,1)$, which is a contradiction. If we now set $\theta(t) = 1$ in (2.11), then we find

$$\xi'(t) + \xi^2(t) + \int_t^{\infty}\left[\frac{1}{r(\vartheta)}\int_{\vartheta}^{\infty} q(s)\left(\frac{\tau^2(s)}{s^2}\right)^{\alpha} ds\right]^{\frac{1}{\alpha}} d\vartheta \le 0.$$

Hence, equation (2.25) is nonoscillatory, which is a contradiction. Thus, it follows from (2.17) that

$$\varphi'(t) + \alpha\varphi^{\frac{\alpha+1}{\alpha}}(t)\int_t^{\infty} R(s)\,ds + q(t) \le 0.$$

Therefore, we conclude that equation (2.26) is nonoscillatory, which is a contradiction. From (2.22), we get

$$\phi'(t) + \alpha \frac{\phi^{\frac{\alpha+1}{\alpha}}(t)}{r^{\frac{1}{\alpha}}(t)} + q(t)\left(\frac{k}{2}\tau^2(t)\right)^{\alpha} \le 0$$

for every constant $k \in (0,1)$. Hence, we can see that equation (2.27) is nonoscillatory for every constant $k_2 \in (0,1)$, which is a contradiction.

Theorem 2.2 is proved.

It is well known (see [18]) that if

$$\int_{t_0}^{\infty} \frac{1}{a(t)}\,dt = \infty, \quad \text{and} \quad \liminf_{t\to\infty}\left(\int_{t_0}^{t} \frac{1}{a(s)}\,ds\right)\int_{t}^{\infty} q(s)\,ds > \frac{1}{4},$$

then equation (2.23) with $\alpha = 1$ is oscillatory. It is also well known (see [20], Theorem 3.3) that if

$$\int_{t_0}^{\infty} \frac{1}{a(t)}\,dt < \infty, \quad \text{and} \quad \liminf_{t\to\infty}\left(\int_{t}^{\infty} \frac{1}{a(s)}\,ds\right)^{-1}\int_{t}^{\infty}\left(\int_{s}^{\infty} \frac{1}{a(v)}\,dv\right)^2 q(s)\,ds > \frac{1}{4},$$

then equation (2.23) with $\alpha = 1$ is oscillatory.

Based on the above results and Theorem 2.2, we can easily obtain the following Hille and Nehari type oscillation criteria for (1.1) with $\alpha = 1$.

Theorem 2.3. *Let $\alpha = 1$, (H_1), (H_2), and (1.3) hold. Assume that*

$$\int_{t_0}^{\infty} \frac{t^2}{r(t)}\,dt = \infty, \quad \textit{and} \quad \liminf_{t\to\infty}\left(\int_{t_0}^{t} \frac{s^2}{r(s)}\,ds\right)\int_{t}^{\infty} q(s)\frac{\tau^2(s)}{s^2}\,ds > \frac{1}{2k_1}$$

for some constant $k_1 \in (0,1)$,

$$\liminf_{t\to\infty} t\int_{t}^{\infty}\int_{\eta}^{\infty} \frac{1}{r(\vartheta)}\int_{\vartheta}^{\infty} q(s)\frac{\tau^2(s)}{s^2}\,ds\,d\vartheta\,d\eta > \frac{1}{4}, \tag{2.28}$$

$$\int_{t_0}^{\infty}\int_{t}^{\infty} R(s)\,ds\,dt = \infty, \quad \textit{and} \quad \liminf_{t\to\infty}\left(\int_{t_0}^{t}\int_{s}^{\infty} R(v)\,dv\,ds\right)\int_{t}^{\infty} q(s)\,ds > \frac{1}{4},$$

and

$$\liminf_{t\to\infty}\left(\int_{t}^{\infty} \frac{1}{r(s)}\,ds\right)^{-1}\int_{t}^{\infty}\left(\int_{s}^{\infty} \frac{1}{r(v)}\,dv\right)^2 q(s)\tau^2(s)\,ds > \frac{1}{2k_2} \tag{2.29}$$

for some constant $k_2 \in (0,1)$. *Then every solution of Eq. (1.1) with* $\alpha = 1$ *is oscillatory.*

Theorem 2.4. *Let* $\alpha = 1$, (H_1), (H_2), *(1.3), and (2.28) hold. Assume that*

$$\int_{t_0}^{\infty} \frac{t^2}{r(t)}\, dt < \infty,$$

$$\liminf_{t\to\infty} \left(\int_t^{\infty} \frac{s^2}{r(s)}\, ds \right)^{-1} \int_t^{\infty} \left(\int_s^{\infty} \frac{v^2}{r(v)}\, dv \right)^2 q(s) \frac{\tau^2(s)}{s^2}\, ds > \frac{1}{2k_1}$$

for some constant $k_1 \in (0,1)$,

$$\int_{t_0}^{\infty} \int_t^{\infty} R(s)\, ds\, dt < \infty,$$

$$\liminf_{t\to\infty} \left(\int_t^{\infty} \int_s^{\infty} R(v)\, dv\, ds \right)^{-1} \int_t^{\infty} \left(\int_s^{\infty} \int_u^{\infty} R(v)\, dv\, du \right)^2 q(s)\, ds > \frac{1}{4},$$

and (2.29) holds for some constant $k_2 \in (0,1)$. *Then every solution of (1.1) with* $\alpha = 1$ *is oscillatory.*

3. Examples

In this section, we present two examples to illustrate our main results.

Example 3.1. Consider a differential equation

$$\left(t^5 x'''(t)\right)' + \beta t x(t) = 0, \quad t \ge 1, \tag{3.1}$$

where $\beta > 0$ is a constant. Let

$$\alpha = 1, \quad r(t) = t^5, \quad q(t) = \beta t, \quad \tau(t) = t.$$

Thus, we get

$$R(t) = \frac{1}{4t^4}, \quad \int_s^{\infty} R(v)\, dv = \frac{1}{12s^3}, \quad \int_s^{\infty}\int_u^{\infty} R(v)\, dv\, du = \frac{1}{24s^2}.$$

If we now set $\rho(t) = \theta(t) = 1$, then we conclude that (2.1) and (2.2) are satisfied. As a result of calculations, we see that (2.3) and (2.4) hold for $\beta > 12$. Hence, by Theorem 2.1, every solution of Eq. (3.1) is oscillatory for $\beta > 12$. However, the results of [22] cannot confirm this conclusion.

Example 3.2. Consider a delay differential equation

$$\left(e^t x'''(t)\right)' + 2\sqrt{10}\, e^{t+\arcsin\frac{\sqrt{10}}{10}} x\left(t - \arcsin\frac{\sqrt{10}}{10}\right) = 0, \quad t \geq 1. \tag{3.2}$$

It is easy to see that every solution of (3.2) is oscillatory due to Theorem 2.1. One of these solutions is $x(t) = e^t \sin t$. However, it follows from [22] (Corollary 2.1) that (3.2) may have nonoscillatory solutions x satisfying $\lim_{t\to\infty} x(t) = 0$. Hence, our results supplement and improve the results obtained in [22].

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