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RISK MODELS WITH STOCHASTIC PREMIUM AND RUIN PROBABILITY ESTIMATION

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We investigate the risk model called the *random premiums model* that generalizes the classical risk process. Within this model, the total claim amount process is the same as in the classical model while the premium income, unlike the classical case, is considered to be a stochastic process. A representation of the ruin probability for the random premiums risk process (i.e. the analog of the Beekman convolution formula) is derived. Some aspects of numerical estimation of the ruin probability are investigated. The Cramér–Lundberg theory is generalized for the random premiums model and results obtained by the other authors are surveyed. Prospects for application of the investigated model in practical problems of financial mathematics are discussed.

1. Introduction

1.1. Basic notation

In this paper we deal with some mathematical models of *collective risk theory.* A typical model describes the dynamics of the *reserve* (or *surplus*) of an insurance company. Though, as we will discuss further, some of the models considered and investigated here can be used for the description of some other financial structures (apart from insurance business), we will, for uniformity, reason in terms of insurance.

Any insurance company is established in order to decrease the risk of its clients and to help them to avoid inconvenient consequences from various accidents occurring at random times by means of covering their *claims.* In turn, clients should pay *premiums* to the insurance company in order to cover its liability and create the desired reserve.

Suppose an insurance company is established with a certain *initial capital* $z \geqslant 0$, the total *premium* arriving from its clients over the time interval $[0, t]$ is equal to $I(t)$ and the total amount of money paid by the company to its clients until t is $D(t)$. Then the *surplus process* can be defined as

$$R(t) = z + I(t) - D(t). \tag{1}$$

Everywhere in this paper we will assume that $D(t)$ is a random process; if $N(t)$ is the total number of claims occurred during $[0, t]$ and the size of ith claim is Z_i then $D(t) = \sum_{i \leqslant N(t)} Z_i$.

The difference $Q(t) = I(t) - D(t)$ is called the random *safety loading* of the company. Let us define the *relative safety loading* as follows:

$$\rho = \lim_{t\to\infty} \frac{\mathsf{E}[I(t) - D(t)]}{\mathsf{E}D(t)}, \tag{2}$$

provided that the limit above exists.

We will say that the risk process $R(t)$ satisfies the *positive safety loading* (PSL) condition, if $\rho > 0$. Unless otherwise specified, everywhere we assume that the PSL condition holds.

Let us define the *ruin probability* as a function of initial capital z:

$$\Psi(z) = \mathsf{P}\left(\inf_{t>0} R(t) < 0 \,|\, R(0) = z\right),$$

and the *ruin* of the insurance company is an event that occurs when the safety loading $Q(t)$ becomes less then $(-z)$. The *survival probability* is then

$$\Phi(z) = 1 - \Psi(z).$$

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1.2. The classical risk process

The classical risk process that we consider first is one of the most commonly used and well-studied models in the collective risk theory. It can be defined by the following balance equation for the surplus process

$$R_{\mathrm{cl}}(t) = z + ct - \sum_{j=1}^{N_1(t)} X_j, \quad t \geqslant 0, \tag{3}$$

where $c > 0$ is the constant premium rate; $\{X_j\}_{j\geqslant 1}$ is a sequence of independent identically distributed random variables (i.i.d.r.v.) having the common distribution function (d.f.) $F_X(x)$ with the expectation $\mathsf{E}(X_j) = a_X$ and defining the sizes of claims; $N_1(t)$ is a homogeneous Poisson process of intensity λ_1 independent of r.v.'s $\{X_j\}_{j\geqslant 1}$ that determines a number of claims that occur within $[0, t]$.

The relative safety loading takes the form

$$\rho_{\mathrm{cl}} = \frac{c}{\lambda_1 a_X} - 1. \tag{4}$$

It is assumed that the classical risk process (3) satisfies the PSL condition $\rho_{\mathrm{cl}} > 0$, so that $\lambda_1 a_X < c$.

It is known that for the classical risk process the *Beekman convolution formula* for the ruin probability holds (its deduction can be found in many sources, e.g., [17]):

$$\Psi_{\mathrm{cl}}(z) = \left(1 - \frac{\lambda_1 a_X}{c}\right) \sum_{k=0}^{\infty} \left(\frac{\lambda_1 a_X}{c}\right)^k (1 - \widetilde{F}_X^{*k}(z)), \tag{5}$$

where

$$\widetilde{F}_X(x) = \frac{1}{a_X} \int_0^x (1 - F_X(y))dy, \quad x \geqslant 0, \tag{6}$$

and $\widetilde{F}_X(x) = 0$ if $x < 0$; $\widetilde{F}_X^{*k}(x)$ stands for the k-fold Stieltjes convolution of the d.f. $\widetilde{F}_X$: $\widetilde{F}_X^{*k}(x) = (\widetilde{F}_X^{*(k-1)} * * \widetilde{F}_X)(x)$, $k \geqslant 1$; $\widetilde{F}_X^{*0}$ is a d.f. with a single step at 0. Note that $\widetilde{F}_X(0) = 0$ and

$$\Psi_{\mathrm{cl}}(0) = \frac{\lambda_1 a_X}{c}. \tag{7}$$

2. Risk process with random premium

2.1. Risk process with random premium as a generalization of the classical model

In this section we proceed to the description of the model that can be considered as a radical extension of the classical risk model. We describe the basic results obtained in [24].

Assume that the premium is no longer a linear function but a stochastic process in the form of the random sum. So the balance equation for a surplus process for such a model can be written as follows:

$$R(t) = z + \sum_{i=1}^{N_2(t)} Y_i - \sum_{j=1}^{N_1(t)} X_j, \quad t \geqslant 0. \tag{8}$$

The i.i.d.r.v.'s $\{X_j\}_{j\geqslant 1}$ having the common d.f. F_X and $\mathsf{E}(X_j) = a_X$ are the claim sizes; the standard Poisson process $N_1(t)$ with intensity λ_1 defines the number of claims. The sequence $\{Y_i\}_{i\geqslant 1}$ of i.i.d.r.v.'s, independent of $\{X_j\}_{j\geqslant 1}$ and $N_1(t)$ and having the common d.f. G_Y with the expectation $\mathsf{E}(Y_i) = b_Y$, are the sizes of premium payments; the standard Poisson process $N_2(t)$ of intensity λ_2, also independent of the r.v.'s $\{X_j\}_{j\geqslant 1}$ and of the process $N_1(t)$, defines the number of payments within $[0, t]$.

Let us call the process defined by the balance equation (8) *the risk process with random premium* or just *the random premium process*. First of all, we were interested in obtaining the analytical representation for the ruin probability as a function of initial capital, i.e. the analog of the Beekman convolution formula.

Let us note that the random premium process was considered by other authors before, namely in [5]. The investigations in [5] concerned mainly the approximations of the ruin probability for the random premium process as an extension of the Cramér–Lundberg theory. In [25] we focused on the analytical ruin probability representation for the common case of random premium process, its numerical estimation and modeling. In this section we describe mainly the results that reflect the explicit ruin probability representation valid for any z, and in Section 4 we review the Cramér–Lundberg theory for the process (8), including the results of [5].

2.2. Representation for the ruin probability

The risk process with random premium as well as the classical risk process can be written in the form of a sequence of random sums of i.i.d.r.v.'s. Namely, let S_n be the surplus of the company at the moment of occurrence of nth claim. Then the following recursion relation holds:

$$S_0 = z, \quad S_{n+1} := S_n + \mu_{n+1} - X_{n+1}, \quad n \geqslant 0, \tag{9}$$

where $\{\mu_n\}$ is the total amount of premiums arrived between the $(n-1)$-th and the n-th claims. For the classical risk process $\mu_n = c\theta_n$, where i.i.d.r.v.'s $\{\theta_n\}$ are interoccurrence times. For the random premium process, $\{\mu_n\}$ can be defined by the relation

$$\mu_n = \sum_{i=0}^{N_2(l_n^{(1)})} Y_i - \sum_{i=0}^{N_2(l_{n-1}^{(1)})} Y_i = \sum_{i\,:\,l_n^{(1)} < l_i^{(2)} \leqslant l_{n-1}^{(1)}} Y_i, \qquad n \geqslant 1, \tag{10}$$

where $\{l_n^{(1)}\}$ and $\{l_i^{(2)}\}$ are the moments of the claims and payments respectively, $l_0^{(1)} := l_0^{(2)} := 0$. Note that for the random premium process (as in the classical case) the r.v.'s $\{\mu_n\}$ are independent and identically distributed.

The PSL condition for the random premium process leads to the inequality $a_X < \mathsf{E}(\mu_1)$.

Let us introduce some additional variables that will be helpful for the interpretation of the analytical representation of the ruin probability. We do not state the whole deduction of this analytical representation here. It can be found in [24] and [26].

Consider the following *random walk*:

$$h_0 = 0, \quad h_n = \xi_1 + \cdots + \xi_n, \quad n \geqslant 1, \tag{11}$$

where

$$\xi_n = X_n - \mu_n, \quad n \geqslant 1. \tag{12}$$

Denote by $W(x)$ the d.f. corresponding to r.v. ξ_1.

Due to the PSL condition and by the strong law of large numbers,

$$\mathsf{P}\left(\lim_{n\longrightarrow\infty} h_n = -\infty\right) = 1.$$

Then *total maximum* of the random walk is

$$M = \sup_{n\geqslant 0} h_n, \quad M \geqslant 0.$$

Note that

$$\Phi(x) = \mathsf{P}(\sup_{n\geqslant 0} h_n \leqslant x) = \mathsf{P}(M \leqslant x), \tag{13}$$

where $\Phi(x)$ is the survival probability for the process (8).

The random variable

$$\tau = \inf\{n : h_n > 0\} \tag{14}$$

is called the *first strict ladder epoch* (or simply *ladder epoch*) for the random walk (11) and h_τ is called the *first strict ladder height* (or simply *ladder height*). As $h_n \to -\infty$, it is possible that

$$q = \mathsf{P}(\tau = \infty) > 0, \tag{15}$$

that is, τ is a *non-proper* r.v. Denote

$$F_h(x) = \mathsf{P}(h_\tau \leqslant x \mid \tau < \infty). \tag{16}$$

Then, as shown in [24], we have a chain of three expressions for the ruin probability for the random premium process:

$$\Psi(z) = 1 - \Phi(z) = q\sum_{k=0}^{\infty}(1-q)^k(1 - F_h^{*k}(z)), \tag{17}$$

where

$$\ln \frac{1}{1-(1-q)\widehat{f}_h(s)} = \sum_{n=1}^{\infty} \frac{1}{n} \int\limits_{0+}^{\infty} e^{isx} dW^{n*}(x), \tag{18}$$

and

$$W(x) = \frac{1}{1+\lambda_2/\lambda_1} \sum_{k=0}^{\infty} \left(\frac{\lambda_2/\lambda_1}{1+\lambda_2/\lambda_1} \right)^k F_X(x) * \overline{G}_Y^{*k}(x),\ \overline{G}_Y(x) = 1 - G_Y(-x-0). \tag{19}$$

In (18) $\widehat{f}(s)$ denotes a *Fourier–Stieltjes transform* for the probabilistic distribution function $F_h(x)$:

$$\widehat{f}_h(s) = \int\limits_{-\infty}^{\infty} e^{ixs} dF_h(x), \tag{20}$$

i.e., $\widehat{f}_h(s)$ is a characteristic function (ch.f.) for a d.f. $F_h(x)$. (Further we shall use the symbol $\widehat{\ }$ to denote characteristic functions.)

The following equalities, corresponding to the r.v. μ_n (given by (10)), will be used in what follows:

$$\mathsf{E}(\mu_1) = \mathsf{E}(Y_1)\lambda_2/\lambda_1, \tag{21}$$

$$\mathsf{E}(\mu_1^2) = (\lambda_2/\lambda_1)\,\mathsf{E}(Y_1^2) + 2\,(\lambda_2/\lambda_1)^2\,(\mathsf{E}(Y_1))^2, \tag{22}$$

$$\mathsf{D}(\mu_1) = (\lambda_2/\lambda_1)\,\mathsf{E}(Y_1^2) + (\lambda_2/\lambda_1)^2\,(\mathsf{E}(Y_1))^2. \tag{23}$$

For the classical model:

$$\mathsf{E}(\mu_1^{\mathrm{cl}}) = c/\lambda_1, \tag{24}$$

$$\mathsf{E}((\mu_1^{\mathrm{cl}})^2) = 2\,(c/\lambda_1)^2\,. \tag{25}$$

The PSL condition for the random premium process can then be written in the form

$$a_X\lambda_1 < b_Y\lambda_2, \tag{26}$$

and the relative safety loading is

$$\rho = \frac{\lambda_2 b_Y}{\lambda_1 a_X} - 1. \tag{27}$$

3. Algorithms for the numerical estimation of the ruin probability

3.1. Ruin probability for the classical risk process: description of the calculation scheme

Let us turn again to the classical risk process and discuss in brief aspects of numerical estimation of the ruin probability within the framework of the classical model. Developing an applicable algorithm for numerical estimation of the ruin probability is an important task in applications of collective risk theory.

A number of works have studied the problem of estimating the ruin probability for the process (3). Namely, in [8] there was proposed a nonparametric estimate for the classical risk process ruin probability built on an sample of the claim size values with the assumption that the sample consists of a nonrandom number of values. In [4] a method for calculating of the ruin probability with the use of a sample of random number of claim values was proposed and the asymptotic properties of the proposed estimate were investigated.

Here we present a method allowing us to obtain an estimate of the ruin probability and requiring a minimal number of calculation operations.

Calculation of the ruin probability by the direct usage of the Beekman formula (5) is quite a laborious procedure due to the presence of the k-fold convolution function in the right part of (5). As far as we know, all the schemes for estimation of the ruin probabilities described in famous works use the direct calculation of the k-fold convolution of the claims distribution function. The algorithm that we describe below is based on the Fourier method that allows us to simplify substantially the calculation as the direct evaluation of the k-fold convolution is replaced by far more simple calculations.

Suppose that the parameters c, a_X and λ_1 are known. Assume that one has a random sample $X_1, X_2, \dots, X_N$, where $N \geqslant 1$ is a number that is actually a realization of the Poisson process $N_1(t)$. Due to the Beekman formula (5),

$$\Psi_{\text{cl}}(z) = 1 - \left(1 - \frac{\lambda_1 a_X}{c}\right) \overline{\Psi}(z), \tag{28}$$

where

$$\overline{\Psi}(z) = \sum_{k=0}^{\infty} \left(\frac{\lambda_1 a_X}{c}\right)^k \widetilde{F}_X^{*k}(z), \tag{29}$$

and hence estimation of the ruin probability is reduced to the calculation of the function $\overline{\Psi}(z)$.

The basic principle of the proposed algorithm is in the application of the Fourier transformation to both parts of relation (29), which allows to reduce the whole procedure of evaluation of the function $\Psi(z)$ to the calculation of the expression

$$\sum_{k=0}^{\infty} \left(\frac{\lambda_1 a_X}{c}\right)^k \widehat{\widetilde{f}}{}_X^k(s). \tag{30}$$

After applying the inverse Fourier transformation to (30), we will have the desired estimate of the function $\overline{\Psi}(z)$. Thus, the expression to be evaluated is

$$\overline{\Psi}(z) = \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} \left(\frac{\lambda_1 a_X}{c}\right)^k e^{-izt} \left[\int_0^{\infty} e^{itu} d\widetilde{F}_X(u)\right]^k dt. \tag{31}$$

Considering this calculational scheme we believe that it must be applicable to cases where the only information available to an analyst is the random sample of claim size values (and the analytical explicit form of the function F_X might be unknown).

So assume that we have a sample $X_1, X_2, \dots, X_N$ obtained from the observation. Using it and operating with some numerical methods we can simulate the random sample $\widetilde{X}_1, \widetilde{X}_2, \dots, \widetilde{X}_N$ of the values having the d.f. $\widetilde{F}_X(x) = \frac{1}{a_X}\int_0^x (1 - F_X(x))dx$.

Then, using the operation of filtration, the necessity of which will be made obvious below, we can obtain the smoothed sample $Z_1, Z_2, \dots, Z_N$ in such a way that the generated random variables Z_i have the d.f.

$$F_H(x) = \int_0^{\infty} H(x-u) d\widetilde{F}_X(u) = (\widetilde{F}_X * H)(x),$$

where $H(x)$ is a *smoothing function.* For the modeling we used the d.f. with the density

$$h(x) = \frac{1}{2\pi\sigma} \frac{\sin^2(x/2\sigma)}{(x/2\sigma)^2} \quad (\sigma \text{ is a } \textit{smoothing parameter}). \tag{32}$$

The use of the smoothing procedure has a practical meaning for the evaluation of the inverse Fourier transformation to the function (30). The Fourier image of a function (32) is equal to 0 outside the limited interval (see, e.g., [20] or [10]), so that

$$\widehat{h}(s) = \begin{cases} 1 - \sigma \cdot |s|, & \text{for } |s| < 1/\sigma, \\ 0, & \text{for } |s| \geqslant 1/\sigma. \end{cases} \tag{33}$$

This allows us to bound the arguments domain of the function

$$\widehat{f}_H(s) = \widehat{\widetilde{f}}_X(s) \cdot \widehat{h}(s). \tag{34}$$

Therefore, the further calculation of the inverse Fourier image of a bounded function is a correct procedure. Note that the function $F_H(x)$ comes closer to $\widetilde{F}_X(x)$ as the parameter σ decreases. As we need to estimate the accuracy of the ruin probability calculation, we are interested in the dependence of the distance between F_H and $\widetilde{F}_X$ on the parameter σ. Using the well-known probabilistic *Lévy metric* [19], we find the following inequality (see [25] for details):

$$\sup_x |F_H(x) - \widetilde{F}_X(x)| \leqslant \sqrt{\sigma}\left(\frac{1}{\pi} + 2e \ln \frac{1}{\sqrt{\sigma}}\right). \tag{35}$$

Thus, one can state that the choice of the parameter σ has to be made with respect to two opposite conditions. The first is that the larger is σ, the narrower is the arguments domain for the function $\widehat{f}_H$. On the other hand, as σ increases, the distance between $\widetilde{F}_X(x)$ and $F_H(x)$ increases. The first condition determines the correctness of our computations (the accuracy of inverse Fourier transformation of the $\widehat{f}_H(s)$); the second limits the correctness of substitution of the function $\widetilde{F}_X$ by a close smoothed function. Applying the proposed algorithm to the random samples modeled in the described way, we were able to trace the change of accuracy of calculations with the change in the parameter σ.

A more detailed description of the computational scheme can be found in [25]. That work also contains a survey of some numerical results of calculation of the probability of ruin for the classical risk process.

3.2. Analyzing the accuracy of calculations

The total error of the calculation of ruin probability with the algorithm outlined above consists of errors arising at the following stages:

- the construction of the empirical distribution function F_H based on the initial random sample of the claim values $X_1, \dots, X_N$;
- evaluation of the Fourier image of F_H with the FFT procedure;
- calculation of $[\widehat{\widehat{\psi}}_n]$ and evaluation of the inverse Fourier transform from $[\widehat{\widehat{\psi}}_n]$ (also with FFT) and the ruin probability $[\Psi_k]$.

Moreover, the deviation between $\widetilde{F}_X$ and $F_H = \widetilde{F}_X * H$ also contributes to the total error.

If we analyze carefully the errors that arise at each stage of the calculation scheme, then we finally arrive at the following inequality for the total error $R_{\overline{\Psi}}$ of the ruin probability computation:

$$\begin{aligned} R_{\overline{\Psi}} \quad &\leqslant \quad \frac{1}{\pi}\left[X_{\max}\left(\frac{C_1}{(N-1)^2}+\frac{C_2}{(K-1)!}\right)+\frac{C_3}{(K-1)^2}+\frac{1}{K}\right]\cdot \ln\frac{1}{\sigma\sqrt{\sigma}}+ \\ &+ \quad C_4\sqrt{\sigma}+\sqrt{\sigma}\left(\frac{1}{\pi}+2e\ln\frac{1}{\sqrt{\sigma}}\right)+\frac{C_5}{(K-1)^2}, \end{aligned} \tag{36}$$

where $X_{\max}$ is the largest element of the initial random sample $X_1, X_2, \dots, X_N$, N is the volume of this random sample, and K is the number of dots in the Fourier transform operation, which we apply in the algorithm calculating values (34).

3.3. Ruin probability for the random premium process

The problem of numerical estimation of the ruin probability for the random premium process is also of sufficient importance; moreover, in this case it becomes far more difficult that in the classical one. Like the problem of evaluation of the ruin probability for the classical risk process, in this case one has to deal with the k-fold convolution of the d.f. series, and here we have the chain of three formulas with the convolutions — (17), (18), and (19) — unlike the classical case. To reduce the computation procedure, we used again the method based on the Fourier transform. Say, instead of direct calculation of the expression (18) for the computation of W, it is appropriate to calculate at first its ch.f.

$$\widehat{w}(s) = \frac{\widehat{f}_X(s)}{1+\frac{\lambda_2}{\lambda_1}(1-\widehat{g}_Y(-s))}, \tag{37}$$

and then to get $W(x)$ by using the inverse Fourier transformation.

In the same way, applying the Fourier transform to (17), we get

$$\widehat{\phi}_n = \frac{q}{1-(1-q)\widehat{f}_h^{(n)}}. \tag{38}$$

Let us outline briefly the basic stages of the whole calculation scheme for the ruin probability:

- calculation of the arrays $[\widehat{f}_n]$ and $[\widehat{g}_n]$ by the FFT procedure;
- evaluation of $[\widehat{w}_n^r]$ and the application of the inverse FFT procedure for the calculation of the values of W and W^{*r};
- estimation of the series (18) and calculation of the values of ch.f. $\widehat{f}_h$;

- Computation of the survival estimate array $[\widetilde{\Phi}_k]$ (and the corresponding ruin probability $[\widetilde{\Psi}_k]$) by application of the inverse FFT after the evaluation of $[\phi_n]$ with (38).

From the argument stated in [25], the total computational error that arises in the use of the described algorithm can be estimated by the inequalities

$$R_\Phi \leqslant \frac{1}{\pi} R_{\widehat{f}_h} \ln(T_1\sqrt{T_1}) + \frac{C_6}{\pi}\frac{1}{\sqrt{T_1}} + 2e\frac{\ln T_1}{T_1} + \frac{C_7}{(K-1)^2}, \tag{39}$$

and

$$R_{\widehat{f}_h} \leqslant R_4 + R_5^{(r)} \leqslant \frac{1}{\pi}\frac{C_1}{(K-1)^2} r\ln(T\sqrt{T}) + \frac{C_3}{\pi}\frac{1}{\sqrt{T}} + 2e\frac{\ln T}{T} + \frac{C_4}{(K-1)^2} + \frac{C_5}{r+1}, \tag{40}$$

where C_i are positive constants, K is the number of dots in the Fourier transform, r is the number of elements in the row in the right part of (18), taken for the calculations, and T and T_1 are the scale parameters of *the Zolotarev inequality* [27]. Note that the main indicator of the calculation error is the index r defining the number of terms of the series (18), which we use for its estimate (neglecting the residual of the series).

We had an opportunity to test the algorithm for numerical estimation of the ruin probability for the random income process within the framework of the practical problem. In Section 5 we will outline that problem and present numerical results.

4. Cramér–Lundberg theory for models with random premiums

As we mentioned above, another approach to the problem of the investigation of the risk model with random premiums was made by A. Boykov in [5]. That work was devoted mainly to the estimation of upper bounds for the ruin probability for the random premiums risk process (as well as for its further generalizations), i.e., the analog of the classical Cramér–Lundberg theory.

4.1. Extension of Cramér–Lundberg theory to the random premium process

Now let us consider again the risk process with random premiums in its standard form

$$R(t) = z + \sum_{i=1}^{N_2(t)} Y_i - \sum_{j=1}^{N_1(t)} X_j, \quad t \geqslant 0. \tag{41}$$

Again F_X and G_Y are the distribution functions of the sizes of claims and premiums respectively; a_X and b_Y are the values of corresponding means.

Let us turn to the martingale theory again and consider the results that extend the Cramér-Lundberg theory to the case of the random premium process and its further generalizations.

Lemma 1. *Let the value ν exists, such that*

$$\mathsf{E}\exp\left(\nu(X_1-\mu_1)\right) = \int\limits_{-\infty}^{\infty} e^{\nu u} dW(u) = 1. \tag{42}$$

Then any of the two following conditions are equivalent to (42):

$$\lambda_1(\mathsf{E}e^{\nu X_1} - 1) + \lambda_2(\mathsf{E}e^{-\nu Y_1} - 1) = 0, \tag{43}$$

$$(1-q)\mathsf{E}\exp\left(\nu\cdot h_\tau \mid \tau<\infty\right) = (1-q)\int\limits_0^\infty e^{\nu u} dF_h(u) = 1. \tag{44}$$

In the theorem stated below, the basic result of [5] is formulated. We contribute to this result, so that the statements 1 and 2 of Theorem 1 are the results of [5], and the third one was obtained in [25].

Theorem 1. *For the random premium process* (41) *the following statements are valid:*

1. *The survival probability corresponding to* (41) *satisfies the integral equation*

$$(\lambda_1+\lambda_2)\Phi(z) = \lambda_1\int\limits_0^z \Phi(z-u)dF_X(u) + \lambda_2\int\limits_0^\infty \Phi(z+v)dG_Y(v). \tag{45}$$

2. *Let* ν *be a positive solution of the equation*

$$\int\limits_0^\infty e^{\nu u} dF_h(u) = \frac{1}{1-q}. \tag{46}$$

Then $\Psi(z) \leqslant e^{-\nu z}$.

3. *If, in addition,*

$$a_h^* = \int\limits_0^\infty x e^{\nu x} dF_h(x) < \infty. \tag{47}$$

then

$$\Psi(z) \sim e^{-\nu z} k_\nu, \quad z \to \infty, \tag{48}$$

where

$$k_\nu = \frac{q}{(1-q)\nu \cdot a_h^*}. \tag{49}$$

If $a_h^* = \infty$, *then* $\lim\limits_{z\to\infty} \Psi(z) e^{\nu z} = 0$.

Some corollaries immediately follow from this basic theorem; they are also presented in [5].

Corollary 1. *The following assertions are true:*

1. *If* $\mathsf{P}(Y_j = 1) = \mathsf{P}(X_i = 1) = 1$, *then*

$$\Psi(z) = \Psi([z]) = 1 - \left(\frac{\lambda_2}{\lambda_1}\right)^{[z]},$$

where $[z]$ *denotes the integer part of* z.

2. *If* $\mathsf{P}(X_i \leqslant x) = 1 - e^{a_X x}$ *and* $\mathsf{P}(Y_j \leqslant y) = 1 - e^{b_Y y}$, *then*

$$\Psi(z) = \frac{(a_X + b_Y)\lambda_1}{(\lambda_1 + \lambda_2) a_X} e^{\frac{\lambda_1 b_Y - \lambda_2 a_X}{\lambda_1 + \lambda_2} z}.$$

Corollary 2. *If* $F_X(x) = 1 - e^{-a_X x}$, *then Eq.* (45) *has at most one solution in the class of functions* $f(x)$ *bounded on the semiaxis* $[0, +\infty)$ *and such that*

$$\lim_{x \longrightarrow \infty} f(x) = 1.$$

Let, in addition, the adjustment coefficient ν *exists. Than the solution of Eq.* (45) *has the form*

$$\Psi(z) = \frac{a_X - \nu}{a_X} e^{-\nu z}.$$

4.2. Risk model with investment

Let us turn now to another interesting extension of the basic model with random premium: risk process with investment. The model of the classical risk process with investment in the stock market was investigated in [12] from the viewpoint of the Cramér–Lundberg theory. The basic idea of this extended model is taking into account the possibility for the insurance company to invest its earnings in a stock or market index. In the present paragraph we generalize the basic results of [12] with respect to the random premiums.

So we consider the famous Black–Scholes model and assume that the insurance company with random premiums described by (8) may invest in a stock or market index S_t described by the geometric Brownian motion

$$dS(t) = S(t)(\mu dt + \sigma dW(t)), \tag{50}$$

where $\mu, \sigma \in \mathbb{R}$ are fixed constants and $W(t)$ is a *standard Wiener process*.

First we consider the case where the model does not account for interest in the reserve, i.e. the insurance company may only invest in a bond with zero interest rate.

If at time t the company has wealth $\widetilde{R}(t)$ and invests an amount $K(t)$ of money in the stock and the remaining reserve $\widetilde{R}(t)-K(t)$ in the bond (which in the present model yields no interest), its wealth process $\widetilde{R}$ can be written as

$$\widetilde{R}(t,K)=z+\sum_{i=1}^{N_2(t)}Y_i-\sum_{j=1}^{N_1(t)}X_j+\left(\frac{K}{S}\cdot S\right)(t)=R(t)+(K\cdot W_{\mu,\sigma})(t), \tag{51}$$

where $W_{\mu,\sigma}(t)=\mu t+\sigma W(t)$ (*generalized Wiener process* with *drift* μ and *volatility* σ) and $(K\cdot W_{\mu,\sigma})$ denotes *stochastic integral* of the process K with respect to the process $W_{\mu,\sigma}$ (see, e.g. [22]).

The ruin probability and the time of ruin are defined as usual and they depend on the strategy in this model:

$$\widetilde{\Psi}(z,K)=\mathsf{P}\left(\inf_{t>0}\widetilde{R}(t,K)<0\,|\,R(0)=z\right),$$

$$T_z(K)=\inf\{t:\ \widetilde{R}(t,K)<0\}.$$

We will denote by $\mathbb{F}=(\mathcal{F}_t)_{t\geqslant 0}$ the filtration generated by processes $\widetilde{R}$ and S.

The set $\mathcal{K}$ of *admissible* strategies K is defined as

$$\mathcal{K}:=\left\{K=(K(t))_{t\geqslant 0}:\ K \text{ is predictable and adapted to } \mathbb{F} \text{ and } \mathsf{P}\left[\int_0^t K^2(s)ds<\infty\right]=1,\ t\in[0,\infty]\right\}. \tag{52}$$

Note that $K\in\mathcal{K}$ is necessary and sufficient for the stochastic integral $(K\cdot W_{\mu,\sigma})$ w.r.t. the generalized Wiener process appearing in (51) to exist.

Furthermore, the following value can be defined:

$$\Psi^*(z):=\inf_{K\in\mathcal{K}}\widetilde{\Psi}(z,K).$$

If this infimum is defined for a certain strategy K^*, it can be called an *optimal strategy* with respect to the initial capital z.

The following part of this paragraph briefly outlines the results of the extension of the Cramér–Lundberg theory on the model (51). We follow the results obtained in [12] for the case of a classical risk process with investment and we generalize these results to the random premiums model.

The main result is summarized in the following theorem.

Theorem 2. *For the model* (51), *assume that* $\sigma\neq 0$. *Then the minimal ruin probability* $\Psi^*(z)$, *investing in a stock market, can be bounded from above by*

$$\Psi^*(z)\leqslant e^{-\widetilde{\nu}z}, \tag{53}$$

where $0<\widetilde{\nu}<\infty$ *is the positive solution of the equation*

$$\lambda_1(\mathsf{E}e^{rX_1}-1)+\lambda_2(\mathsf{E}e^{-rY_1}-1)=\frac{\mu^2}{2\sigma^2}. \tag{54}$$

The statement of this theorem is actually a consequence of the following one.

Theorem 3. *Let* $\mu\neq 0$, $\sigma\neq 0$. *For the constant investment strategy* $\widehat{K}\equiv\mu/\widetilde{\nu}\sigma^2$, *the ruin probability can be bounded from above by* (*for all* $z\in\mathbb{R}_+$)

$$\widetilde{\Psi}(z,\widehat{K})\leqslant e^{-\widetilde{\nu}z}. \tag{55}$$

Remark 1. Statements of Theorems 2 and 3 stay true even when the PSL condition $a_X\lambda_1<b_Y\lambda_2$ does not hold. If $a_X\lambda_1<b_Y\lambda_2$, that is, if the Lundberg coefficient ν exists, we have that $\widetilde{\nu}>\nu$ if $\mu\neq 0$, so that one obtains a sharper bound for $\Psi^*(z)$ due to the Theorem 2. Dropping the assumption $a_X\lambda_1<b_Y\lambda_2$, for $\mu\neq 0$, we still obtain $\widetilde{\nu}>0$, that is, the exponential decay of the minimal ruin probability takes place.

Remark 2. It can be shown that a constant investment strategy obtained above possesses the properties of asymptotic optimality asymptotic uniqueness.

Proofs of the theorems and a detailed discussion can be found in [26].

5. Example of the practical application of random premium model: problem of estimation of debt losses risk

As we have already mentioned, the model with random premiums considered above, as well as its modifications, imply the possibility of application of the model not only to an insurance company, but also to other financial structures (moreover, such applications of random premiums models are even more natural). Let us consider an example, described by the following practical situation. Imagine a commercial company that provides services to its clients; for lucidity, let us suppose that these are telecommunication services. Due to the specificity of the business, financial settlement with clients is organized on a credit basis, i.e., invoices for the provided services are sent to the clients at the end of every account month. Invoices have to be paid off within a fixed period after their settlement (usually, several working days). As in every company with this kind of rules of payment, there should exist a system of debt control, which checks the timeliness of payment of invoices. Suppose that the basic rule that governs such a payment control system is that in cases where delay in payment is more than a predetermined number of working days, the company blocks the services for such a client until the moment of payment of the debt. Besides the amount of the debt, the client should pay also the service amount for the period when he was out of service (though the services were blocked, resources were maintained during the entire period, and the company bore expenses) with respect to the chosen tariff, and also some fixed penalty amount, which we denote by p.

With respect to the described scheme, it is natural to consider the following problem, which might be important for managers of the company: find the maximal possible loss of the company's profit due to the blocked services debt flow?

After a moment of reflection, one can come to the conclusion that the described situation and the task of the supremum of the cumulative debt loss flow estimation relate to the risk model with random premiums and to the problem of ruin probability estimation. Moreover, if certain assumptions concerning the blocked amounts and "unblocked amounts" hold, then the described problem reduces exactly to the ruin probability for the random premium process estimation.

These assumptions are:

- The loss amount values must be i.i.d.r.v's and the moments for each service to be blocked must be independent and exponentially distributed.
  The first condition seems realistic in practice. Usually, the distribution of the separate amounts paid to the company by its clients can be considered as a probabilistic distribution, not changing much from month to month in the case of large enough size of the company's business. So, if one suggests that the chance of a client missing the time of payment for the service, i.e., to become a debtor, on average does not depend on the amount paid to the service company, then the distribution of loss amount values must be close to the common distribution of the amounts paid for services. Specifically, in telecommunications it is natural to accept such a suggestion, that is, large enterprises using sufficient volume of services have approximately the same "chance to become a debtor" as small companies with not so large telecommunication services usage. The second condition seems quite a realistic approximation too; at least the flow of the "blockings", obviously, must be a homogeneous process in a usual situation.
- The amounts of the restored payments must also be i.i.d.r.v's and the moments for these payments must have an exponential distribution.
  This condition also appears quite realistic; the argument above concerning the process of loss amounts remains true for the restored payments flow as well.

As regards the positive relative safety loading condition essential for the risk models including risk processes with random premiums? Recall that in the described case there exists a positive addition to values of the recovered revenue — the penalty value p. The intensities of "blockings" and "unblockings" flows are approximately the same. Hence the PSL condition must be valid. We must take into account that not all the blocked revenue values are recovered since some clients never pay their debts and go out of business (here we deal with the so-called *bad debt*). But the relative number of such clients is obviously small, so that the penalty p covers these irretrievable losses on average.

Obviously, this practical task must be concentrated upon the estimation of the loss threshold on some limited interval of time. Usually, the directors of the company are interested in estimation of such kind of financial values for the nearest year, with respect to the budgetary planning. However, the estimate of such a value for an infinite interval of time gives the most pessimistic bound for it, which is usually important in planning.

Using the described theory for the ruin probability estimation, we investigated the outlined model of the debt losses from the point of the distribution of the debt flow infimum on the example of a real company. The graphs of the probability distribution densities of blocked revenue loss amounts, recovered revenue amounts, and the resulting estimation of the distribution density of the debt flow infimum are presented on Fig. 1. Since confidentiality prevents us from presenting real values, the absolute values of the initial data and the result are fictitious, but the relative magnitudes are real. The evaluation of debt losses distribution was made with the help of the calculation methods described above within the framework of the ruin probability estimation for a random premium process.

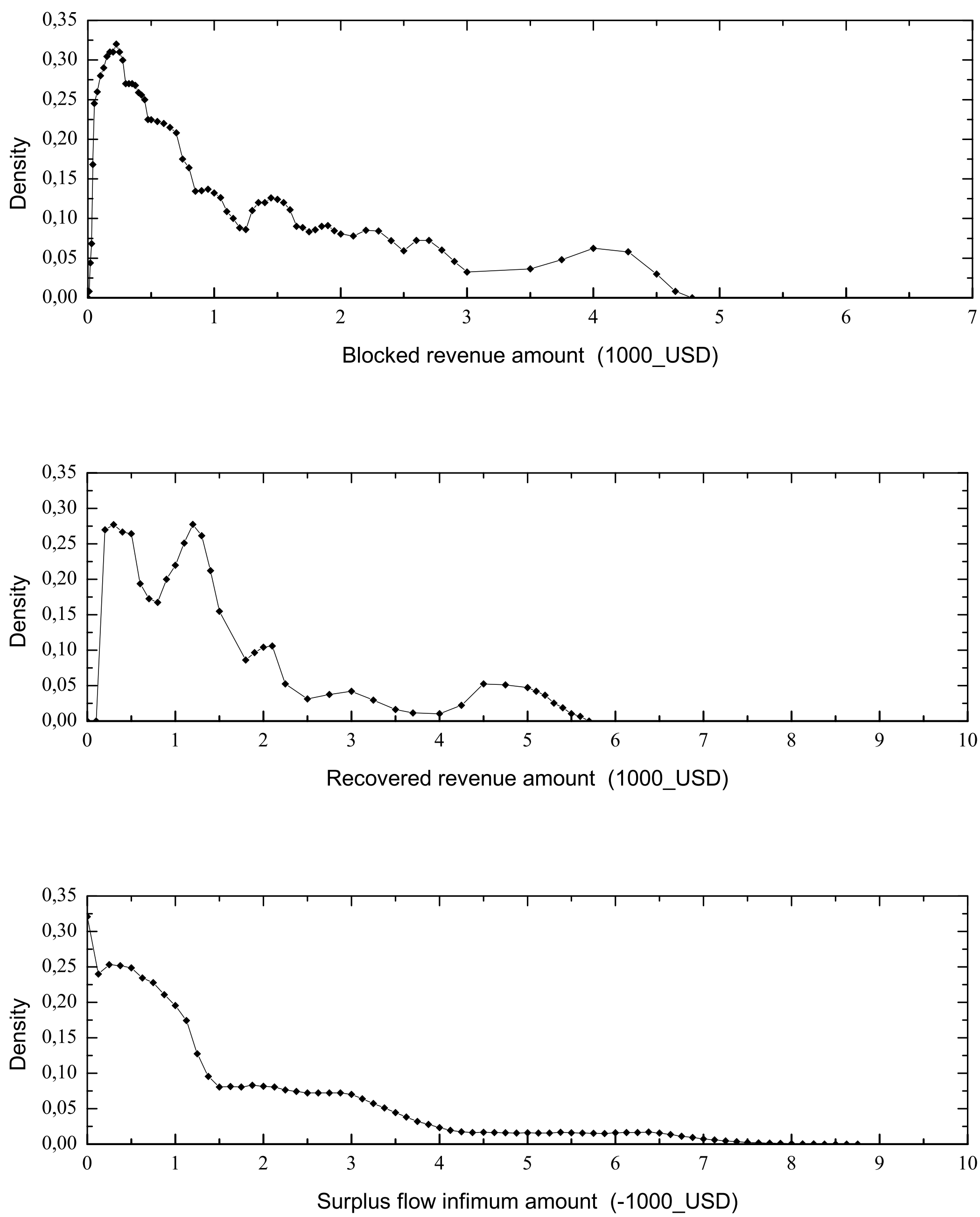

Fig. 1. Problem of maximum debt loss estimation.

Usually managers are interested in the concrete values characterizing the resulting distribution: the mean value, Value-at-Risk at a certain level, etc. Once the distribution itself is found, these values are easy to extract.

6. Conclusions

Let us summarize the basic results described in the present article.

The problem facing us in our investigation is to estimate the ruin probability for the random income process

$$R(t) = z + \sum_{i=1}^{N_2(t)} Y_i - \sum_{j=1}^{N_1(t)} X_j$$

and to further develop this model. The analytical representation of the ruin probability for this process is given by the chain of equations (17)–(19). At first sight, that representation might seem too complicated to be used in the practical tasks of ruin probability calculation. Nevertheless, we describe an algorithm that can be used for estimating the ruin probability in practice. The proposed scheme of calculation is based on the Fourier method, which allows us to reduce sufficiently the complexity of computation. We come to that scheme after considering the analogous problem of ruin probability estimation for the classical risk process. Estimations of computational errors produced by the algorithms are given by formulas (36) for the classical model and (39) for the random income.

Extension of the classical Cramer–Lundberg theory to the case of the random income model is also described. Here we review and complement the results of other authors.

Practical application of the random income process is discussed on the example of some specific problems of financial mathematics, apart from the common insurance model. Here we saw that the risk process with random premium, once appearing as an insurance model, and the theory of ruin probability estimation can be a useful tool for the solution of some problems of financial mathematics, not necessarily connected with insurance.

Let us also note at this point that models like the random premium process are used in investigations of other financial problems not connected with risk estimation, such as [1].

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